Derivation of the Boussinesq Approximation

by Alec Johnson, April 26, 2007

1 Definition of Quantities.

$\mathbf{u} :=$ velocity field
$\rho :=$ mass density
$p :=$ pressure
$\mathbf{g} :=$ gravity
$\underline{\underline{\tau}} :=$ total stress
$\underline{\underline{\sigma}} :=$ viscous stress
$e :=$ heat energy per mass
$T :=$ temperature
$\mathbf{q} :=$ heat flux
$\kappa :=$ heat conductivity
$R :=$ gas constant
$c_v :=$ specific heat at constant volume
$c_p :=$ specific heat at constant pressure
$\gamma := c_p/c_v$

2 Overview

The Boussinesq equations for stratified flow (e.g. of the atmosphere or ocean) assume that fluid flow is incompressible yet convects a diffusive quantity that endows the fluid with positive or negative buoyancy. This buoyancy quantity is identified with a linear function of the deviation of temperature or density from adiabatic hydrostatic balance.

3 Balance laws.

To derive these equations we begin by writing balance equations for mass, momentum, and thermal energy.

3.1 Conservation of mass.

$d_t\rho + \rho\nabla\cdot\mathbf{u} = 0$, i.e., $d_t \ln\rho = -\nabla\cdot\mathbf{u}$. The Boussinesq approximation assumes that $\boxed{\nabla\cdot\mathbf{u}\simeq 0}$.

3.2 Conservation of momentum.

$$\rho d_t\mathbf{u} = \rho\mathbf{g} - \nabla p + \nabla\cdot\underline{\underline{\sigma}}$$

where the viscous stress tensor is given by $\underline{\underline{\sigma}} = \lambda\nabla\cdot\mathbf{u}\underline{\underline{\delta}} + \mu(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{tr}})$. (With generality $\lambda = \frac{-2}{3}\mu$.) Assuming that $\nabla\lambda\simeq 0$, $\nabla\mu\simeq 0$, and $\nabla\cdot\mathbf{u}\simeq 0$, this simplifies to:

$$\rho d_t\mathbf{u} \simeq \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u}$$

3.3 Thermal energy.

The general balance of thermal energy is:

$$\rho d_t e = \underline{\underline{\tau}} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q},$$

where $\underline{\underline{\tau}} = \underline{\underline{\sigma}} + p\underline{\underline{\delta}}$. To obtain this equation, write the balance of energy,

$$\rho d_t(e + u^2/2) = \nabla\cdot(\underline{\underline{\tau}}\cdot\mathbf{u}) - \nabla\cdot\mathbf{q}$$

and subtract the balance of kinetic energy (i.e. $\mathbf{u}$ dot momentum balance):

$$\rho d_t(u^2/2) = (\nabla\cdot\underline{\underline{\tau}})\cdot\mathbf{u}.$$

In the thermal energy balance, we neglect the term representing the contribution of the viscous stress $\underline{\underline{\sigma}}$ to thermal energy production. Then

$$\rho d_t e + p\nabla\cdot\mathbf{u} = -\nabla\cdot\mathbf{q}.$$

We now simplify each term. We simplify the terms in the left hand side using the ideal gas relations $e = c_v T$ and $p = \rho RT$ respectively:

$$\rho d_t e = \rho c_v d_t T,$$

and using $\ln p = \ln\rho + \ln T$,

$$\begin{aligned} p\nabla\cdot\mathbf{u} &= -p d_t\ln\rho \\ &= p d_t(\ln T - \ln p) \\ &= p d_t\ln T - d_t p \\ &= \rho R d_t T - d_t p. \end{aligned}$$

So the left hand side is $\rho\underbrace{(c_v + R)}_{\text{Call } c_p} d_t T - d_t p.$

To simplify the right hand side, assume $\mathbf{q} = -\tilde{\kappa}\nabla T$. Assume that $\nabla\tilde{\kappa}\simeq 0$. We get:

$$\rho c_p d_t T \simeq d_t p + \tilde{\kappa}\nabla^2 T, \text{ i.e.,}$$
$$d_t T \simeq \frac{1}{\rho c_p} d_t p + \kappa\nabla^2 T, \text{ where } \kappa := \frac{\tilde{\kappa}}{c_p\rho}.$$

We will use that $\nabla\kappa\simeq 0$.

4 Hydrostatic balance.

A stratified fluid is said to be in *hydrostatic equilibrium* if it is at rest ($\mathbf{u} = 0$) and the fluid state variables are simply functions of height z. Let $\rho_0(z)$, $p_0(z)$, $T_0(z)$, $\mathbf{u} = 0$ be the state variables of an atmosphere in hydrostatic equilibrium. Conservation of momentum just reduces to the requirement that such an atmosphere in hydrostatic balance must satisfy static force balance of pressure and gravitational forces:

$$d_z p_0 = -\rho_0 g.$$

5 Adiabatic hydrostatic equilibrium.

A hydrostatic equilibrium is said to be *stable* if for any test volume selected from any level of the column of fluid, if we transport it to another fluid level and adiabatically change its pressure to match the pressure at the new level, the test volume will experience a buoyancy force in a direction that pushes it toward its original level. (Recall that a test volume experiences a buoyancy force when its density differs from the density of the surrounding fuid.) If such an adiabatically transported test volume never experiences a buoyancy force, the fluid column is said to be in *neutral equilibrium.* Such a neutrally stable atmosphere is called an *isentropic* or *adiabatic atmosphere*. The entropy of such an atmosphere is constant with height. This holds because (1) entropy is an invariant of an adiabatic process, (2) entropy is a function of pressure and density, and (3) in a neutrally stable atmosphere the pressure and density of an adiabatically transported test volume always match the surrounding fluid.

6 Aside: ideal gas hydrostatic equilibrium.

6.1 Entropy.

To determine expressions for isentropic equilibrium, we write expressions for the differential of entropy.

$$\begin{aligned} ds &= \tfrac{dq}{T} = \tfrac{de + p\,dv}{T} = c_v \tfrac{dT}{T} + R\tfrac{dv}{v} = c_v\, d\ln T + R\, d\ln v \\ &= d(c_v \ln T - R\ln\rho) = c_v\, d\ln\left(T\rho^{\left(-\frac{R}{c_v}=1-\gamma\right)}\right) \\ &= d(c_v \ln p - c_p \ln\rho) = c_v\, d\ln\left(p\rho^{\left(-\frac{c_p}{c_v}=-\gamma\right)}\right) \\ &= d(c_p \ln T - R\ln p) = c_p\, d\ln\left(Tp^{\left(-\frac{R}{c_p}=\frac{1-\gamma}{\gamma}\right)}\right) \end{aligned}$$

6.2 Ideal gas isentropic relations.

For an isentropic atmosphere, $d_z s(z) = 0$, so the entropy differential formulas yield isentropic relations between state variables at heights z_0 and z:

$$\frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \quad \left(\frac{\rho}{\rho_0}\right) = \left(\frac{p}{p_0}\right)^{1/\gamma},$$
$$\left(\frac{T}{T_0}\right)^{\gamma} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \text{ and lastly, } \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}.$$

This last relation provides us an explicit formula for the *potential temperature*. The potential temperature is defined to be the temperature that a parcel of air would have if it were brought adiabatically to a reference temperature. Thus, if we take p_0 as the reference temperature (typically 1000 mbars), the potential temperature is

$$\theta := T_0 = T\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}.$$

6.3 Aside: ideal gas adiabatic lapse rate.

Recall isentropy: $d\ln T = \frac{R}{c_p} d\ln p$.
Recall hydrostatic force balance: $d_z p = -\rho g$.
Dividing by p and using $p = \rho RT$ gives $d_z \ln p = \frac{-g}{TR}$.
Multiplying by $\frac{R}{c_p}$ gives $d_z \ln T = \frac{-g}{Tc_p}$, i.e., $\boxed{d_z T = \frac{-g}{c_p}}$.

7 Perturbation from hydrostatic balance.

For each state variable q let $q_0(z)$ represent a stratified hydrostatic balance. Write the state variables as perturbations from this hydrostatic balance:

$$q = q_0 + q'.$$

7.1 Perturbation from hydrostatic momentum balance.

Recall the balance law that we derived for momentum,

$$\rho d_t \mathbf{u} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u},$$

and subtract the hydrostatic balance relation

$$0 = \rho_0\mathbf{g} - \nabla p_0.$$

We get

$$(\rho_0 + \rho')d_t\mathbf{u} = \rho'\mathbf{g} - \nabla p' + \mu\nabla^2\mathbf{u}.$$

Dividing by ρ_0 gives

$$\left(1 + \frac{\rho'}{\rho_0}\right)d_t\mathbf{u} = \frac{\rho'}{\rho_0}\mathbf{g} - \frac{1}{\rho_0}\nabla p' + \nu\nabla^2\mathbf{u},$$

where $\nu := \frac{\mu}{\rho_0}$ is the *kinematic viscosity*. Invoking the Boussinesq assumption $\rho' \ll \rho_0$ gives:

$$d_t\mathbf{u} \simeq \frac{\rho'}{\rho_0}\mathbf{g} - \frac{1}{\rho_0}\nabla p' + \nu\nabla^2\mathbf{u}.$$

7.2 Perturbation from hydrostatic thermal energy balance.

Substitute the perturbation expansions into the balance law that we derived for thermal energy:

$$d_t(T_0 + T') = \frac{d_t(p_0 + p')}{(\rho_0 + \rho')c_p} + \kappa\nabla^2(T_0 + T').$$

and subtract the hydrostatic balance relation,

$$d_t T_0 = \frac{d_t p_0}{\rho_0 c_p} + \kappa\nabla^2 T_0.$$

Invoking the Boussinesq assumption $\rho' \ll \rho_0$, we get:

$$d_t T' = \frac{d_t p'}{\rho_0 c_p} + \kappa\nabla^2 T'.$$

7.3 Perturbation from hydrostatic isentropic thermal energy balance.

We wish to choose a reference hydrostatic equilibrium for which we can neglect the term $\frac{d_t\tilde{p}}{\rho_0 c_p}$.

I claim that we can neglect this term if we assume that the hydrostatic equilibrium is isentropic. In this case, if the state of a convected volume element agrees with the hydrostatic equilibrium, this agreement will persist, and if the state differs slightly from hydrostatic equilibrium, then the difference will likewise tend to persist (assuming no heat diffusion).

This term essentially represents the contribution to the rate of change of the temperature perturbation *due to the deviation of the hydrostatic equilibrium from isentropy.*

7.4 Perturbation from hydrostatic equilibrium as a perturbation from isentropic hydrostatic equilibrium.

In general the atmosphere is close to a hydrostatic equilibrium, but that equilibrium is not typically isentropic.

Expand each state variable q as $q = q_a(z) + q_1(z) + q'(z)$, where $q_a + q_1 =: q_0$ represents the actual hydrostatic equilibrium of the atmosphere, and where q_a represents some isentropic hydrostatic equilibrium. Let $\tilde{q} = q_1(z) + q'(z)$, the perturbation from isentropic equilibrium.

Recall that $d \ln p = d \ln \rho + d \ln T$. Assuming that perturbations from isentropy are small, this means that $\frac{\tilde{p}}{p_0} \simeq \frac{\tilde{\rho}}{\rho_0} + \frac{\tilde{T}}{T_0}$. That is, $\tilde{T} \simeq \left(\frac{-T_0}{\rho_0}\right)\left(\tilde{\rho} - \frac{\rho_0}{p_0}\tilde{p}\right)$. We use this to eliminate T from the heat diffusion equation, $d_t\tilde{T} = +\kappa\nabla^2\tilde{T}$:

$$d_t\left(\tilde{\rho} - \frac{\rho_0}{p_0}\tilde{p}\right) = +\kappa\nabla^2\left(\tilde{\rho} - \frac{\rho_0}{p_0}\tilde{p}\right)$$

Using $\tilde{q} = q_1 + q'$ and $\nabla^2 q_1 \simeq 0$,

$$d_t\left(\rho' - \frac{\rho_0}{p_0}p'\right) + \mathbf{u}\cdot\hat{\mathbf{z}}\,\underbrace{\partial_z\left(\rho_1 - \frac{\rho_0}{p_0}p_1\right)}_{\text{Call } -b} = +\kappa\nabla^2\left(\rho' - \frac{\rho_0}{p_0}p'\right).$$

We neglect the derivatives of the pressure deviation, $d_t p'$ and $\nabla^2 p'$. (It seems that this is justified by the quasi-geostrophic and quasihydrostatic balance assumptions.)

This gives an evolution equation for the perturbation in the density:

$$d_t\rho' - b\mathbf{u}\cdot\hat{\mathbf{z}} = +\kappa\nabla^2\rho'$$

8 Boussinesq system.

The full set of Boussinesq equations is thus:

$$\nabla\cdot\mathbf{u} = 0,$$
$$d_t\mathbf{u} = -\frac{\rho'}{\rho_0}g\hat{\mathbf{z}} - \frac{1}{\rho_0}\nabla p' + \nu\nabla^2\mathbf{u},$$
$$d_t\rho' = b\mathbf{u}\cdot\hat{\mathbf{z}} + \kappa\nabla^2\rho'.$$

In order to reduce the number of parameters by one, we define the "buoyancy frequency" N and the rescaled temperature perturbation θ and pressure p by

$$N^2 := \frac{gb}{\rho_0}$$
$$\rho' = \sqrt{\frac{b\rho_0}{g}}\theta$$
$$p = \frac{p'}{\rho_0}$$

This gives a system with a minimal number of free parameters:

$$\nabla\cdot\mathbf{u} = 0,$$
$$d_t\mathbf{u} = -N\theta\hat{\mathbf{z}} - \nabla p + \nu\nabla^2\mathbf{u},$$
$$d_t\theta = N\mathbf{u}\cdot\hat{\mathbf{z}} + \kappa\nabla^2\theta$$

Ertel's Potential Vorticity Theorem: a Derivation.

by Alec Johnson, April 26, 2007

9 Definition of Quantities.

$\mathbf{u}$ = velocity field
$\omega = \nabla\times\mathbf{u}$ = vorticity
ρ = mass density
p = pressure
$\mathbf{g} = \nabla\phi$ = gravity
$\mathcal{F}$ = viscous drag
d_t = convective derivative
$\mathbf{\Omega}$ = angular speed of reference frame
$\omega_a = \omega + 2\mathbf{\Omega}$ = "absolute vorticity"

10 Balance laws.

10.1 Conservation of mass.

$d_t\rho + \rho\nabla\cdot\mathbf{u} = 0$, i.e., $d_t \ln\rho = -\nabla\cdot\mathbf{u}$

10.2 Conservation of momentum.

$$\rho d_t\mathbf{u} = -\nabla p + \rho\nabla\phi + \mathcal{F}, \text{ i.e.,}$$
$$d_t\mathbf{u} = \frac{-\nabla p}{\rho} + \nabla\phi + \frac{\mathcal{F}}{\rho}$$

11 Rotating coordinate frame.

Let d_t continue to denote the convective derivative with respect to an inertial (i.e. "fixed" or nonrotating) reference frame. Let d_t' denote the convective derivative in a frame that is rotating with angular velocity $\mathbf{\Omega}$. Then:

$$\begin{aligned} d_t &= (d_t' + \mathbf{\Omega}\times) \\ d_t{}^2 &= (d_t' + \mathbf{\Omega}\times)(d_t' + \mathbf{\Omega}\times) \\ &= d_t'^2 + 2\mathbf{\Omega}\times d_t' + \mathbf{\Omega}\times\mathbf{\Omega}\times + \cancel{d_t'\mathbf{\Omega}}\times) \end{aligned}$$

Applying these operator identities to a moving position vector $\mathbf{r}(t)$ (e.g. of a convected fluid element) gives the relations

$$\begin{aligned} \mathbf{u} &= d_t\mathbf{r} = \mathbf{u}' + \mathbf{\Omega}\times\mathbf{r} \\ d_t\mathbf{u} &= (d_t' + \mathbf{\Omega}\times)(\mathbf{u}' + \mathbf{\Omega}\times\mathbf{r}) \\ &= d_t'\mathbf{u}' + 2\mathbf{\Omega}\times\mathbf{u}' + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r} \\ &= d_t'\mathbf{u}' + 2\mathbf{\Omega}\times\mathbf{u}' - \Omega^2\mathbf{r}_\perp \\ &= d_t'\mathbf{u}' + 2\mathbf{\Omega}\times\mathbf{u}' - \nabla\phi_c \end{aligned}$$

where $\mathbf{r}_\perp$ denotes the projection of $\mathbf{r}$ onto the place perpendicular to $\mathbf{\Omega}$ and $\phi_c := \Omega^2\cdot|\mathbf{r}_\perp|^2$ is a potential for the centripetal acceleration.

Since $\nabla\cdot(\mathbf{\Omega}\times\mathbf{r}) = 0$, $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{u}'$, so mass conservation looks the same: $d_t\rho + \rho\nabla\cdot\mathbf{u}' = 0$, i.e., $d_t\ln\rho = -\nabla\cdot\mathbf{u}'$.

The momentum equation now writes:

$$d_t'\mathbf{u}' + 2\mathbf{\Omega}\times\mathbf{u}' = \frac{-\nabla p}{\rho} + \nabla\underbrace{(\phi + \phi_c)}_{\text{Call }\Phi} + \frac{\mathcal{F}}{\rho}$$

Henceforth we concern ourselves only with the rotating reference frame and dispense with primes.

12 The vorticity equation.

This section follows [1] section 2.4. Define the vorticity to be the curl of the fluid velocity: $\omega := \nabla \times \mathbf{u}$. We wish to obtain an evolution equation for the vorticity from the momentum equation. Recall that $d_t\mathbf{u} = \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}$. To recast this in terms of the vorticity, consider the identity

$$\omega \times \mathbf{u} = \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\Big(\frac{u^2}{2}\Big)$$

So the momentum equation writes:

$$\partial_t\mathbf{u} + \underbrace{(2\mathbf{\Omega} + \omega)}_{\text{Call } \omega_a} \times \mathbf{u} = \frac{-\nabla p}{\rho} + \nabla\Big(\Phi - \frac{u^2}{2}\Big) + \frac{\mathcal{F}}{\rho}$$

To obtain an equation for vorticity, we take the curl of both sides. This involves:

$$\begin{aligned}
\nabla \times (\omega_a \times \mathbf{u}) &= \nabla\cdot(\mathbf{u}\omega_a - \omega_a\mathbf{u}) \\
&= \omega_a\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\omega_a - \cancel{\mathbf{u}\nabla\cdot\omega_a} - \omega_a\cdot\nabla\mathbf{u}, \\
\nabla \times \frac{-\nabla p}{\rho} &= \frac{\nabla\rho\times\nabla p}{\rho^2}.
\end{aligned}$$

Hence we get the **vorticity equation**:

$$d_t\omega_a = \omega_a\cdot\nabla\mathbf{u} - \omega_a\nabla\cdot\mathbf{u} + \frac{\nabla\rho\times\nabla p}{\rho^2} + \nabla\times\frac{\mathcal{F}}{\rho},$$

where, recall, $\omega_a := \omega + 2\mathbf{\Omega}$

13 Potential vorticity.

This section follows [1] section 2.5. Let λ be some scalar fluid property.

Divide the vorticity equation by ρ and use $\nabla\cdot\mathbf{u} = \frac{-1}{\rho}d_t\rho$ to eliminate $\nabla\cdot\mathbf{u}$:

$$\underbrace{\frac{d_t\omega_a}{\rho} - \omega_a\frac{d_t\rho}{\rho^2}}_{d_t\left(\frac{\omega_a}{\rho}\right)} = \frac{\omega_a}{\rho}\cdot\nabla\mathbf{u} + \frac{\nabla\rho\times\nabla p}{\rho^3} + \frac{1}{\rho}\nabla\times\frac{\mathcal{F}}{\rho}$$

Take the dot product of $\nabla\lambda$ with this equation and get:

$$\begin{aligned}
d_t\big(\frac{\omega_a}{\rho}\big)\cdot\nabla\lambda &= \frac{\omega_a}{\rho}\cdot(\nabla\mathbf{u})\cdot\nabla\lambda \\
&\quad + \nabla\lambda\cdot\frac{\nabla\rho\times\nabla p}{\rho^3} + \frac{\nabla\lambda}{\rho}\cdot\nabla\times\frac{\mathcal{F}}{\rho}
\end{aligned}$$

Try to rewrite the left hand side as a derivative:

$$d_t\big(\frac{\omega_a}{\rho}\big)\cdot\nabla\lambda = d_t\Big(\frac{\omega_a}{\rho}\cdot\nabla\lambda\Big) - \frac{\omega_a}{\rho}\cdot d_t\nabla\lambda$$

But

$$\begin{aligned}
d_t\nabla\lambda &= \partial_t\nabla\lambda + \mathbf{u}\cdot\nabla\nabla\lambda \\
&= \nabla\partial_t\lambda + \nabla(\mathbf{u}\cdot\nabla\lambda) + \mathbf{u}\cdot\nabla\nabla\lambda - \nabla(\mathbf{u}\cdot\nabla\lambda) \\
&= \nabla d_t\lambda - \nabla\mathbf{u}\cdot\nabla\lambda
\end{aligned}$$

So the left hand side becomes

$$d_t\Big(\frac{\omega_a}{\rho}\cdot\nabla\lambda\Big) - \frac{\omega_a}{\rho}\cdot\nabla d_t\lambda + \frac{\omega_a}{\rho}\cdot\nabla\mathbf{u}\cdot\nabla\lambda,$$

giving us the **potential vorticity equation**

$$d_t\Big(\frac{\omega_a}{\rho}\cdot\nabla\lambda\Big) = \frac{\omega_a}{\rho}\cdot\nabla d_t\lambda + \nabla\lambda\cdot\frac{\nabla\rho\times\nabla p}{\rho^3} + \frac{\nabla\lambda}{\rho}\cdot\nabla\times\frac{\mathcal{F}}{\rho}$$

We define the **potential vorticity** to be $\Pi := \frac{\omega_a}{\rho}\cdot\nabla\lambda$. The potential vorticity equation tells us that:

If

1. λ is a conserved quantity for each fluid element, i.e., $d_t\lambda = 0$,
2. the frictional force is negligible, i.e., $\mathcal{F} \simeq 0$,
3. and *either*
   (a) the fluid is barotropic, i.e., $\nabla\rho\times\nabla p = 0$, *or*
   (b) λ is a function only of p and ρ,

then the potential vorticity $\Pi = \frac{(\omega+2\mathbf{\Omega})}{\rho}\cdot\nabla\lambda$ is conserved, i.e., $d_t\Pi = 0$.

Examples of such conserved scalar fluid properties include the entropy, the potential temperature, or the density/temperature in the Boussinesq approximation.

References

[1] Joseph Pedlosky, *Geophysical Fluid Dynamics, 2nd Ed.,* Springer ©1987.

[2] Pijush Kundu, *Fluid Mechanics,* Academic Press, ©1990.

[3] Ertel H., 1942, *Ein neuer hydrodynamischer Wirbelsatz.* Meteor. Z, 59, 277-281.