Translation between Heaviside-Lorentz (HL), Gaussian (cgs), and SI (mks or "rationalized") systems of units

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1 Definitions

variable	**meaning**
n	particle density
q	charge per particle
m	mass per particle
$\mathbf{v}(\mathbf{x}, t)$	velocity of particles
$d_t = \partial_t + \mathbf{v} \cdot \nabla$	convective derivative

variable	**meaning**
$\mathbf{B}$	magnetic field
$\mathbf{E}$	electric field
c	speed of light
ϵ_0	permittivity of free space

One way to determine how to relate quantities in different units is to make the equations of the one system look like the equations of the corresponding system. In this note we take this approach to relate Heaviside-Lorentz, Gaussian, SI units. For simplicity we derive dimension transformations using Maxwell's equations with the momentum equation for a cold one-species plasma, rather than with the momentum equation for individual particles or with the Boltzmann equation.

2 Heaviside-Lorentz (HL) $\longleftrightarrow$ Gaussian

Heaviside-Lorentz	Gaussian-looking
$\nabla \cdot \mathbf{E} = qn$	$\nabla \cdot (\sqrt{4\pi}\mathbf{E}) = 4\pi\left(\frac{q}{\sqrt{4\pi}}\right)n$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot (\sqrt{4\pi}\mathbf{B}) = 0$
$\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$	$\partial_t(\sqrt{4\pi}\mathbf{B}) + c\nabla \times (\sqrt{4\pi}\mathbf{E}) = 0$
$\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -qn\mathbf{v}$	$\partial_t(\sqrt{4\pi}\mathbf{E}) - c\nabla \times (\sqrt{4\pi}\mathbf{B}) = -4\pi\left(\frac{q}{\sqrt{4\pi}}\right)n\mathbf{v}$
$mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$mnd_t\mathbf{v} = \left(\frac{q}{\sqrt{4\pi}}\right)n((\sqrt{4\pi}\mathbf{E}) + \frac{\mathbf{v}}{c} \times (\sqrt{4\pi}\mathbf{B}))$

Gaussian	Heaviside-Lorentz-looking
$\nabla \cdot \mathbf{E} = 4\pi qn$	$\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) = (\sqrt{4\pi}q)n$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right) = 0$
$\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$	$\partial_t\left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right) + c\nabla \times \left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) = 0$
$\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -4\pi qn\mathbf{v}$	$\partial_t\left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) - c\nabla \times \left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right) = -(\sqrt{4\pi}q)n\mathbf{v}$
$mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$mnd_t\mathbf{v} = (\sqrt{4\pi}q)n(\left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) + \frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right))$

$$\mathbf{E}_{\text{gauss}} = \sqrt{4\pi}\mathbf{E}_{\text{HL}}, \qquad \mathbf{E}_{\text{HL}} = \frac{\mathbf{E}_{\text{gauss}}}{\sqrt{4\pi}},$$

$$\mathbf{B}_{\text{gauss}} = \sqrt{4\pi}\mathbf{B}_{\text{HL}}, \qquad \mathbf{B}_{\text{HL}} = \frac{\mathbf{B}_{\text{gauss}}}{\sqrt{4\pi}},$$

$$q_{\text{gauss}} = \frac{q_{\text{HL}}}{\sqrt{4\pi}}, \qquad q_{\text{HL}} = \sqrt{4\pi}q_{\text{gauss}}.$$

3 Heaviside-Lorentz (HL) $\longleftrightarrow$ SI

Heaviside-Lorentz	**SI-looking**
$\nabla \cdot \mathbf{E} = qn$	$\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) = \frac{\left(\sqrt{\epsilon_0} q\right) n}{\epsilon_0}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right) = 0$
$\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$	$\partial_t \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right) + c\nabla \times \left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) = 0$
$\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -qn\mathbf{v}$	$\partial_t \left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) - c\nabla \times \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right) = -\frac{\left(\sqrt{\epsilon_0} q\right) n\mathbf{v}}{\epsilon_0}$
$mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$mnd_t\mathbf{v} = \left(\sqrt{\epsilon_0} q\right) n(\left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) + \mathbf{v} \times \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right))$

SI	**Heaviside-Lorentz-looking**
$\nabla \cdot \mathbf{E} = \frac{qn}{\epsilon_0}$	$\nabla \cdot (\sqrt{\epsilon_0}\mathbf{E}) = \left(\frac{q}{\sqrt{\epsilon_0}}\right) n$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot (c\sqrt{\epsilon_0}\mathbf{B}) = 0$
$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$	$\partial_t (c\sqrt{\epsilon_0}\mathbf{B}) + c\nabla \times (\sqrt{\epsilon_0}\mathbf{E}) = 0$
$\partial_t \mathbf{E} - c^2\nabla \times \mathbf{B} = -\frac{qn\mathbf{v}}{\epsilon_0}$	$\partial_t (\sqrt{\epsilon_0}\mathbf{E}) - c\nabla \times (c\sqrt{\epsilon_0}\mathbf{B}) = -\left(\frac{q}{\sqrt{\epsilon_0}}\right) n\mathbf{v}$
$mnd_t\mathbf{v} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$mnd_t\mathbf{v} = \left(\frac{q}{\sqrt{\epsilon_0}}\right) n((\sqrt{\epsilon_0}\mathbf{E}) + \frac{\mathbf{v}}{c} \times (c\sqrt{\epsilon_0}\mathbf{B}))$

$$\mathbf{E}_{\text{HL}} = \sqrt{\epsilon_0}\mathbf{E}_{\text{SI}}, \qquad \mathbf{E}_{\text{SI}} = \frac{\mathbf{E}_{\text{HL}}}{\sqrt{\epsilon_0}},$$

$$\mathbf{B}_{\text{HL}} = \sqrt{\epsilon_0}\mathbf{B}_{\text{SI}}, \qquad \mathbf{B}_{\text{SI}} = \frac{\mathbf{B}_{\text{HL}}}{\sqrt{\epsilon_0}},$$

$$q_{\text{HL}} = \frac{q_{\text{SI}}}{\sqrt{\epsilon_0}}. \qquad q_{\text{SI}} = \sqrt{\epsilon_0}q_{\text{HL}}.$$

4 SI $\longleftrightarrow$ Gaussian

SI	Gaussian-looking
$\nabla \cdot \mathbf{E} = \frac{qn}{\epsilon_0}$	$\nabla \cdot (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 4\pi\left(\frac{q}{\sqrt{4\pi\epsilon_0}}\right)n$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = 0$
$\partial_t\mathbf{B} + \nabla \times \mathbf{E} = 0$	$\partial_t(c\sqrt{4\pi\epsilon_0}\mathbf{B}) + c\nabla \times (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 0$
$\partial_t\mathbf{E} - c^2\nabla \times \mathbf{B} = -\frac{qn\mathbf{v}}{\epsilon_0}$	$\partial_t(\sqrt{4\pi\epsilon_0}\mathbf{E}) - c\nabla \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = -4\pi\left(\frac{q}{\sqrt{4\pi\epsilon_0}}\right)n\mathbf{v}$
$mnd_t\mathbf{v} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$mnd_t\mathbf{v} = \left(\frac{q}{\sqrt{4\pi\epsilon_0}}\right)n((\sqrt{4\pi\epsilon_0}\mathbf{E}) + \frac{\mathbf{v}}{c} \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}))$

Gaussian	SI-looking
$\nabla \cdot \mathbf{E} = 4\pi qn$	$\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = \frac{(\sqrt{4\pi\epsilon_0}q)n}{\epsilon_0}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = 0$
$\partial_t\mathbf{B} + c\nabla \times \mathbf{E} = 0$	$\partial_t\left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) + \nabla \times \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = 0$
$\partial_t\mathbf{E} - c\nabla \times \mathbf{B} = -4\pi qn\mathbf{v}$	$\partial_t\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) - c^2\nabla \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = -\frac{(\sqrt{4\pi\epsilon_0}q)n\mathbf{v}}{\epsilon_0}$
$mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$mnd_t\mathbf{v} = (\sqrt{4\pi\epsilon_0}q)n(\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) + \mathbf{v} \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right))$

$$\mathbf{E}_{\rm SI} = \frac{\mathbf{E}_{\rm gauss}}{\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{E}_{\rm gauss} = \sqrt{4\pi\epsilon_0}\mathbf{E}_{\rm SI},$$

$$\mathbf{B}_{\rm SI} = \frac{\mathbf{B}_{\rm gauss}}{c\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{B}_{\rm gauss} = c\sqrt{4\pi\epsilon_0}\mathbf{B}_{\rm SI},$$

$$q_{\rm SI} = \sqrt{4\pi\epsilon_0}q_{\rm gauss}. \qquad q_{\rm gauss} = \frac{q_{\rm SI}}{\sqrt{4\pi\epsilon_0}}.$$

5 Plasma

We now consider what happens to the plasma equations under transformation between SI and Gaussian units.

6 Boltzmann

SI	Gaussian-looking
$\nabla \cdot \mathbf{E} = \frac{\sigma}{\epsilon_0}$	$\nabla \cdot (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 4\pi\left(\frac{\sigma}{\sqrt{4\pi\epsilon_0}}\right)$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = 0$
$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$	$\partial_t(c\sqrt{4\pi\epsilon_0}\mathbf{B}) + c\nabla \times (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 0$
$\partial_t \mathbf{E} - c^2\nabla \times \mathbf{B} = -\frac{\mathbf{J}}{\epsilon_0}$	$\partial_t(\sqrt{4\pi\epsilon_0}\mathbf{E}) - c\nabla \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = -4\pi\left(\frac{\mathbf{J}}{\sqrt{4\pi\epsilon_0}}\right)$
$m d_t \mathbf{v}_p = q_p(\mathbf{E} + \mathbf{v}_p \times \mathbf{B})$	$m d_t \mathbf{v}_p = \left(\frac{q_p}{\sqrt{4\pi\epsilon_0}}\right)((\sqrt{4\pi\epsilon_0}\mathbf{E}) + \frac{\mathbf{v}_p}{c} \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}))$

Gaussian	SI-looking
$\nabla \cdot \mathbf{E} = 4\pi\sigma$	$\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = \frac{(\sqrt{4\pi\epsilon_0}\sigma)}{\epsilon_0}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = 0$
$\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$	$\partial_t\left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) + \nabla \times \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = 0$
$\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -4\pi\mathbf{J}$	$\partial_t\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) - c^2\nabla \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = -\frac{(\sqrt{4\pi\epsilon_0}\mathbf{J})}{\epsilon_0}$
$m_p d_t \mathbf{v}_p = q_p(\mathbf{E} + \frac{\mathbf{v}_p}{c} \times \mathbf{B})$	$m_p d_t \mathbf{v}_p = (\sqrt{4\pi\epsilon_0}q_p)(\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) + \mathbf{v}_p \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right))$

$$\mathbf{E}_{\mathrm{SI}} = \frac{\mathbf{E}_{\mathrm{gauss}}}{\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{E}_{\mathrm{gauss}} = \sqrt{4\pi\epsilon_0}\mathbf{E}_{\mathrm{SI}},$$

$$\mathbf{B}_{\mathrm{SI}} = \frac{\mathbf{B}_{\mathrm{gauss}}}{\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{B}_{\mathrm{gauss}} = \sqrt{4\pi\epsilon_0}\mathbf{B}_{\mathrm{SI}},$$

$$\sigma_{\mathrm{SI}} = \sqrt{4\pi\epsilon_0}\sigma_{\mathrm{gauss}}. \qquad \sigma_{\mathrm{gauss}} = \frac{\sigma_{\mathrm{SI}}}{\sqrt{4\pi\epsilon_0}}.$$