

General Moment Evolution

by E. Alec Johnson, June 2009

1 Boltzmann equation

Recall the Boltzmann equation,

$$\partial_t f_s + \nabla \cdot (\mathbf{v} f_s) + \nabla_{\tilde{\mathbf{v}}} \cdot (\mathbf{a} f_s) = C_s,$$

where $\nabla \cdot := \nabla_{\mathbf{x}} \cdot$, \mathbf{x} is position, \mathbf{v} is velocity, $\tilde{\mathbf{v}} = \gamma \mathbf{v}$ is proper velocity, where $\gamma := (1 + (\mathbf{v}/c)^2)^{-1/2}$, and $\mathbf{a} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is the rate of change of proper velocity with respect to time. (Ignore the blue text and wide tildes if you do not care about relativity.) Drop the subscript s .

2 Evolution of “conserved” moments

Let $\chi(\mathbf{v})$ be a generic moment. Multiply by χ and integrate by parts. Get the generic velocity moment evolution equation

$$\partial_t (\rho \langle \chi \rangle) + \nabla \cdot (\rho \langle \mathbf{v} \chi \rangle) = \rho \langle \mathbf{a} \cdot \nabla_{\tilde{\mathbf{v}}} \chi \rangle + \int_{\tilde{\mathbf{v}}} \chi C.$$

In this document products are by default tensor products and powers are by default tensor powers. Now choose $\chi^{[n]} = \mathbf{v}^n := \prod_{i=1}^n \mathbf{v}$. Let Sym be the map which takes a tensor and returns its symmetric part (obtained by summing over all permutations of the tensor subscripts and dividing by n -factorial, where n is the order of the tensor). But $\mathbf{a} \cdot \nabla_{\tilde{\mathbf{v}}} \mathbf{v}^n = \sum_j \mathbf{a}_j \partial_{v_j} \text{Sym}(\mathbf{v}^n) = n \text{Sym}(\mathbf{a} \mathbf{v}^{n-1}) = \frac{q}{m} n \text{Sym}(\mathbf{v}^{n-1} \mathbf{E} + \mathbf{v}^n \times \mathbf{B})$ (which is simply a sum over all distinguishable permutations of subscripts).

Define the generalized energy tensor $E^{[n]} := \int_{\mathbf{v}} f \mathbf{v}^n = \rho \langle \mathbf{v}^n \rangle$. Get the generalized conservative moment evolution equation

$$\partial_t E^{[n]} + \nabla \cdot E^{[n+1]} = \frac{q}{m} n \text{Sym}(E^{[n-1]} \mathbf{E} + E^{[n]} \times \mathbf{B}) + \int_{\tilde{\mathbf{v}}} \mathbf{v}^n C. \quad (2.1)$$

Setting $n = 0$ gives conservation of mass, setting $n = 1$ gives momentum evolution, and setting $n = 2$ gives energy tensor evolution, half of whose trace is energy evolution.

2.1 Primitive variables

Equation (2.1) is a coupled infinite system of evolution equations for moments of the Boltzmann equation. To provide finite closure we choose a maximum n and specify $E^{[n+1]}$ in terms of the lower moments. The problem of closure leads one naturally consider **primitive variables**. The closure relation should be invariant under change of inertial reference frame, so we are naturally lead to consider moments of the thermal speed $\mathbf{c} := \mathbf{v} - \mathbf{u}$, where $\mathbf{u} := \langle \mathbf{v} \rangle$ is the bulk fluid velocity.

For $n \geq 2$ the primitive moments are defined by

$$P^{[n]} := \rho \langle \mathbf{c}^n \rangle,$$

which we will refer to as the ***n*th order generalized pressure**. For tensor orders 0 and 1 the primitive variables are defined to be ρ and \mathbf{u} . The primitive variable corresponding to the scalar energy is the scalar pressure, defined to be one third the trace of $\mathbb{P} := P^{[2]}$.

To relate primitive and conserved variables we observe that

$$v^n = \text{Sym}[v^n] = \text{Sym}[(u + c)^n] = \text{Sym} \sum_{j=0}^n \binom{n}{j} u^j c^{n-j},$$

$$c^n = \text{Sym}[c^n] = \text{Sym}[(v - u)^n] = \text{Sym} \sum_{j=0}^n (-1)^j \binom{n}{j} u^j v^{n-j}.$$

Primitive and conserved variables are thus related by

$$E^{[n]} = \text{Sym} \sum_{j=0}^n \binom{n}{j} \mathbf{u}^j P^{[n-j]} = P^{[n]} + \text{Sym} \sum_{j=1}^{n-2} \binom{n}{j} \mathbf{u}^j P^{[n-j]} + \rho \mathbf{u}^n,$$

$$P^{[n]} = \text{Sym} \sum_{j=0}^n (-1)^j \binom{n}{j} \mathbf{u}^j E^{[n-j]} = E^{[n]} + \text{Sym} \sum_{j=1}^{n-2} (-1)^j \binom{n}{j} \mathbf{u}^j E^{[n-j]} + (-1)^n (1 - n) \rho \mathbf{u}^n.$$

Observe that for $n = 1$ and half the trace for $n = 2$ these formulas reduce to the familiar relations

$$\mathbf{M} := \rho \mathbf{u}, \quad \mathcal{E} = \frac{3}{2} p + \frac{1}{2} \rho u^2,$$

$$\mathbf{u} = \mathbf{M} / \rho, \quad p = \frac{2}{3} \mathcal{E} - \frac{1}{3} \rho u^2,$$

where \mathbf{M} is the momentum, $\mathcal{E} := \frac{1}{2} \rho \langle v^2 \rangle$ is the energy, and $p := \frac{1}{3} \rho \langle c^2 \rangle$ is the pressure. More generally, we have:

$$\begin{aligned} E^{[0]} &= \rho, & P^{[1]} &= \rho, \\ E^{[1]} &= \rho \mathbf{u}, & P^{[1]} &= \mathbf{0}, \\ E^{[2]} &= \rho \mathbf{u}^2 + P^{[2]}, & P^{[2]} &= E^{[2]} - \rho \mathbf{u}^2, \\ E^{[3]} &= \rho \mathbf{u}^3 + \text{Sym}(3\mathbf{u}P^{[2]}) + P^{[3]}, & P^{[3]} &= E^{[3]} - \text{Sym}(3\mathbf{u}E^{[2]}) + 2\rho \mathbf{u}^3, \\ E^{[4]} &= \rho \mathbf{u}^4 + \text{Sym}(6\mathbf{u}^2P^{[2]} + 4\mathbf{u}P^{[3]}) + P^{[4]}, & P^{[4]} &= E^{[4]} - \text{Sym}(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}) - 3\rho \mathbf{u}^4, \\ E^{[5]} &= \rho \mathbf{u}^5 + \text{Sym}(10\mathbf{u}^3P^{[2]} + 10\mathbf{u}^2P^{[3]} + 5\mathbf{u}P^{[4]}) + P^{[5]}, & P^{[5]} &= E^{[5]} - \text{Sym}(5\mathbf{u}E^{[4]} - 10\mathbf{u}^2E^{[3]} + 10\mathbf{u}^3E^{[2]}) + 4\rho \mathbf{u}^5 \\ &\vdots & &\vdots \end{aligned} \tag{2.2}$$

2.2 Temperature

The temperature is defined to be twice the average particle energy in a given direction (averaged over all three directions) in the reference frame of bulk flow: $T := \frac{1}{3} m \langle c^2 \rangle = \frac{p}{n}$, where $n := \rho/m$ is the number density of the species. By analogy, we define the generalized temperature tensor

$$T^{[n]} := P^{[n]} / n = m \langle \mathbf{c}^n \rangle.$$

Temperature is useful in positing closure relations; for example, one may posit that the heat flux is an isotropic linear function of the temperature gradient.

2.3 Closure.

To close a system of moment evolution equations up to $(n-1)$ th order, we need to specify $E^{[n]}$ in terms of lower-order moments. We do so by positing a constitutive relation for the “heat flux tensor” $P^{[n]}$. Probably the simplest generic closure is truncation, i.e., assuming $P^{[n]} = 0$. Then (for $n \geq 2$)

$$E^{[n]} = E^{[n]} - P^{[n]} = -\text{Sym} \sum_{j=1}^{n-2} (-1)^j \binom{n}{j} \mathbf{u}^j E^{[n-j]} + (-1)^n (n-1) \rho \mathbf{u}^n.$$

Specifically, we can use one of the closure approximations

$$\begin{aligned} E^{[2]} &= \rho \mathbf{u}^2 && \text{(cold plasma),} \\ E^{[3]} &= \text{Sym}(3\mathbf{u}E^{[2]}) - 2\rho \mathbf{u}^3 && \text{(10-moment closure)} \\ E^{[4]} &= \text{Sym}(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2 E^{[2]}) + 3\rho \mathbf{u}^4 && \text{(20-moment closure)} \\ E^{[5]} &= \text{Sym}(5\mathbf{u}E^{[4]} - 10\mathbf{u}^2 E^{[3]} + 10\mathbf{u}^3 E^{[2]}) - 4\rho \mathbf{u}^5 && \text{(35-moment closure)} \\ &\vdots && \end{aligned}$$

to truncate the moment hierarchy

$$\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{M} &= 0, \\ \partial_t \mathbf{M} + \nabla \cdot E^{[2]} &= \frac{q}{m} \text{Sym}(\rho \mathbf{E} + \mathbf{M} \times \mathbf{B}), \\ \partial_t E^{[2]} + \nabla \cdot E^{[3]} &= 2 \frac{q}{m} \text{Sym}(\mathbf{M} \mathbf{E} + E^{[2]} \times \mathbf{B}), \\ \partial_t E^{[3]} + \nabla \cdot E^{[4]} &= 3 \frac{q}{m} \text{Sym}(E^{[2]} \mathbf{E} + E^{[3]} \times \mathbf{B}). \\ \partial_t E^{[4]} + \nabla \cdot E^{[5]} &= 4 \frac{q}{m} \text{Sym}(E^{[3]} \mathbf{E} + E^{[4]} \times \mathbf{B}). \\ &\vdots \end{aligned}$$

A prohibitive problem with truncation closure is that it does not seem to give a hyperbolic system in case the highest moment has order greater than 2. The derivative of flux with respect to state has non-real eigenvalues, resulting in unbounded growth, i.e., an ill-posed system.

2.4 Contracted moments

The number of independent entries in a totally symmetric n th order tensor in three spacial dimensions is $\binom{n+2}{2} = \frac{1}{2}(n+1)(n+2)$, and the number of moments up to n th order is $\binom{n+3}{3} = \frac{1}{6}(n+1)(n+2)(n+3)$:

n	$\binom{n+2}{2}$	$\binom{n+3}{3}$
0	1	1
1	3	4
2	6	10
3	10	20
4	15	35

To avoid the expense of evolving a high number of moments yet still retain higher-order information, one can replace the evolution equation for a higher moment with a contracted evolution equation for a contracted moment. To contract a tensor you set two indices equal and sum. For example, $\sum_i \alpha_{i,i,j}$ represent the contraction of the tensor α over its first two indices. Observe that the expectation of any power of particle velocity is a totally symmetric tensor, that any contraction of a totally symmetric tensor is a totally symmetric tensor, and that for a totally symmetric tensor it is irrelevant over which two indices you contract. We are thus motivated to define the trace tr of a totally symmetric tensor α to be its contraction over (any) two of its indices: $\text{tr } \alpha := \mathbb{I} : \alpha = \alpha : \mathbb{I}$, where \mathbb{I} is the identity tensor.

For example, the 14-moment system contracts the evolution equation for the 10 independent moments $\rho \langle \mathbf{v} \mathbf{v} \mathbf{v} \rangle = E^{[3]}$ to get an evolution equation for the 3 moments $\rho \langle \mathbf{v} \mathbf{v} \cdot \mathbf{v} \rangle =: \text{tr } E^{[3]}$, and twice contracts the evolution equation for the 15 independent moments $\rho \langle \mathbf{v} \mathbf{v} \mathbf{v} \mathbf{v} \rangle = E^{[4]}$ to get an evolution equation for the scalar $\rho \langle \mathbf{v} \cdot \mathbf{v} \mathbf{v} \cdot \mathbf{v} \rangle =: \text{tr tr } E^{[4]}$.

no. moments	evolved quantities					truncation
5	$\rho,$ 1	$\rho \mathbf{u},$ +3	$\text{tr } E^{[2]}$ +1			$P^{[3]} = 0$
10	$\rho,$ 1	$\rho \mathbf{u},$ +3	$E^{[2]}$ +6			$P^{[3]} = 0$
14	$\rho,$ 1	$\rho \mathbf{u},$ +3	$E^{[2]},$ +6	$\text{tr } E^{[3]},$ +3	$\text{tr tr } E^{[4]}$ +1	$P^{[5]} = 0$
21	$\rho,$ 1	$\rho \mathbf{u},$ +3	$E^{[2]},$ +6	$E^{[3]},$ +10	$\text{tr tr } E^{[4]}$ +1	$P^{[5]} = 0$
26	$\rho,$ 1	$\rho \mathbf{u},$ +3	$E^{[2]},$ +6	$E^{[3]},$ +10	$\text{tr } E^{[4]}$ +6	$P^{[5]} = 0$
35	$\rho,$ 1	$\rho \mathbf{u},$ +3	$E^{[2]},$ +6	$E^{[3]},$ +10	$E^{[4]}$ +15	$P^{[5]} = 0$

Replacing a moment with a contracted moment poses a closure problem. To provide for closure one posits that the uncontracted moment $P^{[n]}$ is a linear isotropic function of its contracted moment.

In the case of the 5-moment system, positing that the pressure tensor is an isotropic linear function of a scalar gives the constitutive relation

$$P^{[2]} = \mathbb{I} p = \mathbb{I} \text{tr } P^{[2]} / 3. \quad (2.3)$$

In the case of the 14-moment system, this leads to the constitutive relations

$$P_{ijk}^{[3]} = \frac{1}{5} \sum_m \left(\delta_{ij} P_{kmm}^{[3]} + \delta_{ik} P_{jmm}^{[3]} + \delta_{jk} P_{imm}^{[3]} \right) = \frac{3}{5} \text{Sym}(\mathbb{I} \otimes \text{tr } P^{[3]})$$

and

$$P_{ijkl}^{[4]} = \frac{1}{15} \sum_m \sum_n P_{mmnn}^{[4]} \left(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} \right) = \frac{3}{15} \text{tr tr } P^{[4]} \text{Sym}(\mathbb{I} \otimes \mathbb{I}), \quad (2.4)$$

and in the case of the 26-moment system one gets the constitutive relation [NEED TO FINISH THIS]

where \mathbb{I} is the identity tensor, δ_{ij} denotes Kronecker delta, and Sym is the map which takes a tensor and returns its symmetric part (obtained e.g. by averaging over all permutations of subscripts). The truncation closure remains $P^{[5]} = 0$.

2.5 10-moment system

In conserved variables the ten-moment system is

$$\begin{aligned}\partial_t \rho + \nabla \cdot \mathbf{M} &= 0, \\ \partial_t \mathbf{M} + \nabla \cdot E^{[2]} &= \frac{q}{m} \text{Sym}(\rho \mathbf{E} + \mathbf{M} \times \mathbf{B}), \\ \partial_t E^{[2]} + \nabla \cdot E^{[3]} &= 2 \frac{q}{m} \text{Sym}(\mathbf{M} \mathbf{E} + E^{[2]} \times \mathbf{B}), \\ E^{[3]} &= \text{Sym}(3\mathbf{u}E^{[2]}) - 2\rho \mathbf{u}^3.\end{aligned}$$

That is,

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + \mathbb{P}) &= \frac{q}{m} \rho (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \\ \partial_t (\rho \mathbf{u} \mathbf{u} + \mathbb{P}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u} \mathbf{u} + 3 \text{Sym}(\mathbf{u} \mathbb{P})) &= \frac{q}{m} 2 \text{Sym}(\rho \mathbf{u} \mathbf{E} + (\mathbb{P} + \rho \mathbf{u} \mathbf{u}) \times \mathbf{B}).\end{aligned}$$

2.6 5-moment system

The 5-moment system replaces the evolution equation for the second moment with half its trace and uses the isotropic pressure constitutive relation (2.3),

$$P^{[2]} = \mathbb{I} p = \mathbb{I} \text{tr} P^{[2]} / 3.$$

To obtain a corresponding constitutive relation in conserved variables for $E^{[2]}$, recall that (2.2) expresses $P^{[2]}$ in terms of conserved variables:

$$P^{[2]} = E^{[2]} - \rho \mathbf{u} \mathbf{u}. \quad (2.5)$$

(We remark that taking half the trace of this equation gives the familiar relation $\mathcal{E} = \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{3}{2} p$, where we define the scalar energy by $\mathcal{E} := \frac{1}{2} \text{tr} E^{[2]}$.) Substituting this equation and its trace into the primitive-variables constitutive relation gives the conservative-variables constitutive relation

$$\begin{aligned}E^{[2]} &= \rho \mathbf{u} \mathbf{u} + \mathbb{I} p \\ &= \rho \mathbf{u} \mathbf{u} + \mathbb{I} \text{tr} \left(\frac{1}{3} E^{[2]} - \frac{1}{3} \rho \mathbf{u} \mathbf{u} \right) \\ &= \rho \mathbf{u} \mathbf{u} + \mathbb{I} \left(\frac{2}{3} \mathcal{E} - \frac{1}{3} \rho |\mathbf{u}|^2 \right).\end{aligned}$$

Substituting this into the momentum evolution equation and taking the trace of the energy equation gives a closed system. Since

$$\begin{aligned}\frac{1}{2} \text{tr} E^{[3]} &= \frac{1}{2} \text{tr} [\text{Sym}(3\mathbf{u}E^{[2]}) - 2\rho \mathbf{u}^3], \\ &= E^{[2]} \cdot \mathbf{u} + \mathcal{E} \mathbf{u} - \rho |\mathbf{u}|^2 \mathbf{u}, \\ &= \left(\frac{5}{3} \mathcal{E} - \frac{1}{3} \rho |\mathbf{u}|^2 \right) \mathbf{u},\end{aligned}$$

the 5-moment system in conserved variables is

$$\begin{aligned}\partial_t \rho + \nabla \cdot \mathbf{M} &= 0, \\ \partial_t \mathbf{M} + \nabla \cdot \rho \mathbf{u} \mathbf{u} + \nabla \cdot \left(\frac{2}{3} \mathcal{E} - \frac{1}{3} \rho |\mathbf{u}|^2 \right) &= \frac{q}{m} \text{Sym}(\rho \mathbf{E} + \mathbf{M} \times \mathbf{B}), \\ \partial_t \mathcal{E} + \nabla \cdot \left(\frac{5}{3} \mathcal{E} \mathbf{u} - \frac{1}{3} \rho |\mathbf{u}|^2 \mathbf{u} \right) &= \frac{q}{m} \mathbf{M} \cdot \mathbf{E}.\end{aligned}$$

2.7 21-moment (and 14-moment) closure for $E^{[4]}$

The 21-moment system replaces the evolution equation for $E^{[4]}$ with the trace of its trace.

Equation (2.2) gives $E^{[4]}$ in terms of $P^{[4]}$ and lower conserved moments:

$$E^{[4]} = P^{[4]} + \text{Sym}(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}) + 3\rho\mathbf{u}^4,$$

But $P^{[4]}$ is given from twice its trace by (2.4),

$$P_{ijkl}^{[4]} = \frac{3}{15} \text{tr} \text{tr} P^{[4]} \text{Sym}(\mathbb{I} \otimes \mathbb{I}).$$

Twice taking the trace of (2.2) for $P^{[4]}$ in terms of $E^{[4]}$ and substituting into (2.4) gives a constitutive relation for $E^{[4]}$ in terms of its trace and lower moments,

$$E^{[4]} = \text{tr} \text{tr} (E^{[4]} - \text{Sym}(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}) - 3\rho\mathbf{u}^4) \frac{3}{15} \text{Sym}(\mathbb{I} \otimes \mathbb{I}) + \text{Sym}(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}) + 3\rho\mathbf{u}^4.$$

2.8 14-moment closure for $E^{[3]}$

The 14-moment system is obtained from the 21-moment system by taking the trace of the evolution equation for $E^{[3]}$. Equation (2.2) gives $E^{[3]}$ in terms of $P^{[3]}$ and lower conserved moments:

$$E^{[3]} = P^{[3]} + \text{Sym}(3\mathbf{u}E^{[2]}) - 2\rho\mathbf{u}^4,$$

But $P^{[3]}$ is given from its trace by (2.4),

$$P_{ijk}^{[3]} = \frac{3}{5} \text{Sym}(\mathbb{I} \otimes \text{tr} P^{[3]}).$$

Taking the trace of (2.2) for $P^{[3]}$ in terms of $E^{[3]}$ and substituting into (2.4) gives a constitutive relation for $E^{[3]}$ in terms of its trace and lower moments:

$$E^{[3]} = \frac{3}{5} \text{Sym}(\mathbb{I} \otimes \text{tr} (E^{[3]} - \text{Sym}(3\mathbf{u}E^{[2]}) + 2\rho\mathbf{u}^4)) + \text{Sym}(3\mathbf{u}E^{[2]}) - 2\rho\mathbf{u}^4.$$

3 Evolution of conserved moments

3.1 Momentum evolution

Let $\chi = \tilde{\mathbf{v}}$. Define the average velocity $\mathbf{u} := \langle \mathbf{v} \rangle$ and the average proper velocity $\tilde{\mathbf{u}} := \langle \tilde{\mathbf{v}} \rangle$. Define the thermal velocity $\mathbf{c} := \mathbf{v} - \langle \mathbf{v} \rangle$ and the thermal proper velocity $\tilde{\mathbf{c}} := \tilde{\mathbf{v}} - \langle \tilde{\mathbf{v}} \rangle$. The momentum is $\mathbf{M} := \rho\tilde{\mathbf{u}}$. Then the velocity moment evolution becomes the the balance law for momentum,

$$\partial_t(\rho\tilde{\mathbf{u}}) + \nabla \cdot (\rho\mathbf{u}\tilde{\mathbf{u}} + \tilde{\mathbb{P}}) = \frac{q}{m}\rho(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \int_{\tilde{\mathbf{v}}} \tilde{\mathbf{v}}C,$$

where $\tilde{\mathbb{P}} := \rho\langle \mathbf{c}\tilde{\mathbf{c}} \rangle$ is the pressure tensor.

3.2 “Energy” tensor evolution

[From here on results hold only for the non-relativistic domain.] Let $\chi = \mathbf{v}\mathbf{v}$. So $\langle\chi\rangle = \mathbf{u}\mathbf{u} + \langle\mathbf{c}\mathbf{c}\rangle$. Define the pressure tensor $\mathbb{P} := \rho\langle\mathbf{c}\mathbf{c}\rangle$ and the “energy tensor” $\mathbb{E} := \rho\langle\mathbf{v}\mathbf{v}\rangle$ (whose trace is twice the gas-dynamic energy). So $\mathbb{E} := \rho\mathbf{u}\mathbf{u} + \mathbb{P}$, where $\rho\mathbf{u}\mathbf{u}$ is the “kinetic energy tensor”.

We calculate the terms of the velocity moment evolution equation.

$$\rho\langle\mathbf{v}\mathbf{v}\mathbf{v}\rangle = \rho(\mathbf{u}\mathbf{u}\mathbf{u} + \langle\mathbf{c}\mathbf{c}\mathbf{u}\rangle + \langle\mathbf{c}\mathbf{u}\mathbf{c}\rangle + \langle\mathbf{u}\mathbf{c}\mathbf{c}\rangle + \langle\mathbf{c}\mathbf{c}\mathbf{c}\rangle) = \rho(\mathbf{u}\mathbf{u}\mathbf{u} + 3\text{Sym}\langle\mathbf{u}\mathbf{c}\mathbf{c}\rangle + \langle\mathbf{c}\mathbf{c}\mathbf{c}\rangle)$$

and

$$\begin{aligned} \rho\langle\mathbf{a} \cdot \nabla_{\mathbf{v}} \cdot \mathbf{v}\mathbf{v}\rangle &= \rho\langle\mathbf{a}\mathbf{v} + \mathbf{v}\mathbf{a}\rangle = 2\rho\text{Sym}\langle\mathbf{a}\mathbf{v}\rangle = 2\rho\text{Sym}(\langle\mathbf{a}\rangle\mathbf{u} + \langle\mathbf{a}\mathbf{c}\rangle) \\ &= 2\frac{q}{m}\rho\text{Sym}((\mathbf{E} + \mathbf{u} \times \mathbf{B})\mathbf{u} + \langle\mathbf{c} \times \mathbf{B}\mathbf{c}\rangle) = 2\frac{q}{m}\text{Sym}(\rho\mathbf{u}\mathbf{E} + (\rho\mathbf{u}\mathbf{u} + \mathbb{P}) \times \mathbf{B}). \end{aligned}$$

The velocity moment evolution equation becomes the energy tensor evolution equation

$$\partial_t(\rho\mathbf{u}\mathbf{u} + \mathbb{P}) + \nabla \cdot (\rho\mathbf{u}\mathbf{u}\mathbf{u} + 3\text{Sym}(\mathbf{u}\mathbb{P}) + \mathbb{P}^{[3]}) = \frac{q}{m}2\text{Sym}(\rho\mathbf{u}\mathbf{E} + (\mathbb{P} + \rho\mathbf{u}\mathbf{u}) \times \mathbf{B}) + \int_{\mathbf{v}} \mathbf{v}\mathbf{v}C. \quad (3.1)$$

Taking half the trace of this gives the energy evolution equation,

$$\partial_t\mathcal{E} + \nabla \cdot (\mathbf{u}\mathcal{E} + \mathbf{u} \cdot \mathbb{P}) + \rho\langle\mathbf{c}\mathbf{c}^2\rangle = \frac{q}{m}\rho\mathbf{u} \cdot \mathbf{E} + \int_{\mathbf{v}} Cv^2/2. \quad (3.2)$$

4 Evolution of primitive moments

4.1 Evolution of generalized moment

Let $\chi(t, \mathbf{x}, \mathbf{v})$ be a generic generalized moment. (We will later impose $\chi(\mathbf{c})$. Note that $\mathbf{c}(t, \mathbf{x}, \mathbf{v}) = \mathbf{v} - \mathbf{u}(t, \mathbf{x})$.) Multiply the Boltzmann equation by χ and integrate by parts. Get the generic moment evolution equation

$$\partial_t(\rho\langle\chi\rangle) + \nabla \cdot (\rho\langle\mathbf{v}\chi\rangle) = \rho\langle(d_t^{\mathbf{v}} + \mathbf{a} \cdot \nabla_{\mathbf{v}})\chi\rangle + \int_{\mathbf{v}} \chi C,$$

where $d_t^{\mathbf{v}} := \partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}}$.

4.2 Evolution of generic thermal velocity moment

Impose $\chi(\mathbf{c})$, where $\mathbf{c} = \mathbf{v} - \mathbf{u}(t, \mathbf{x})$. So $\nabla_{\mathbf{v}} = \nabla_{\mathbf{c}}$; also, $d_t^{\mathbf{v}}\chi = (d_t^{\mathbf{v}}\mathbf{c}) \cdot \nabla_{\mathbf{c}}\chi = -(d_t^{\mathbf{v}}\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi$. So the generic moment evolution equation becomes

$$\partial_t(\rho\langle\chi\rangle) + \nabla \cdot (\rho\mathbf{u}\langle\chi\rangle) + \nabla \cdot (\rho\langle\mathbf{c}\chi\rangle) = \rho\langle(\mathbf{a} - d_t^{\mathbf{v}}\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi\rangle + \int_{\mathbf{v}} \chi C.$$

But momentum conservation says that $\langle\mathbf{a}\rangle - d_t^{\mathbf{u}}\mathbf{u} = (\nabla \cdot \mathbb{P} - \mathbf{R})/\rho$, where $\mathbf{R} = \int_{\mathbf{v}} \mathbf{v}C$ is collisional resistance. So $\mathbf{a} - (d_t^{\mathbf{v}}\mathbf{u}) = \mathbf{a}' + \langle\mathbf{a}\rangle - d_t^{\mathbf{u}}\mathbf{u} - \mathbf{c} \cdot \nabla\mathbf{u} = (\nabla \cdot \mathbb{P} - \mathbf{R})/\rho + \mathbf{a}' - \mathbf{c} \cdot \nabla\mathbf{u}$ where $\mathbf{a}' := \mathbf{a} - \langle\mathbf{a}\rangle = \frac{q}{m}\mathbf{c} \times \mathbf{B}$. So the generic thermal velocity moment evolution equation is

$$\partial_t(\rho\langle\chi\rangle) + \nabla \cdot (\rho\mathbf{u}\langle\chi\rangle) + \nabla \cdot (\rho\langle\mathbf{c}\chi\rangle) = (\nabla \cdot \mathbb{P} - \mathbf{R}) \cdot \langle\nabla_{\mathbf{c}}\chi\rangle + \rho\langle(\mathbf{a}' - \mathbf{c} \cdot \nabla\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi\rangle + \int_{\mathbf{v}} \chi C. \quad (4.1)$$

4.3 Evolution of generalized pressure tensor

Choose $\chi = \chi^{[n]} := \prod_{i=1}^n \mathbf{c}$. Then (4.1) gives an evolution equation for the generalized pressure $\mathbb{P}^{[n]} := \rho \langle \chi^{[n]} \rangle$. We seek to express the rest of the equation in terms of generalized pressures. Note that $\chi = \text{Sym}(\chi)$ and $\mathbb{P}^{[n]} = \text{Sym}(\mathbb{P}^{[n]})$. For a generic $\underline{\alpha}$,

$$\underline{\alpha} \cdot \nabla_{\mathbf{c}} \chi^{[n]} = \sum_j \alpha_j \partial_{c_j} \text{Sym}(\chi^{[n]}) = n \text{Sym}(\underline{\alpha} \chi^{[n-1]}).$$

So

$$\begin{aligned} \rho \langle (\mathbf{a}' - \mathbf{c} \cdot \nabla \mathbf{u}) \cdot \nabla_{\mathbf{c}} \chi^{[n]} \rangle &= n \rho \text{Sym}(\langle (\mathbf{a}' - \mathbf{c} \cdot \nabla \mathbf{u}) \chi^{[n-1]} \rangle) \\ &= n \text{Sym}\left(\frac{q}{m} \mathbb{P}^{[n]} \times \mathbf{B} - \mathbb{P}^{[n]} \cdot \nabla \mathbf{u}\right) \end{aligned}$$

The generic thermal velocity moment evolution equation becomes the following generalized pressure tensor evolution equation,

$$\delta_t(\mathbb{P}^{[n]}) + \nabla \cdot (\mathbb{P}^{[n+1]}) + n \text{Sym}\left(\mathbb{P}^{[n-1]}(\mathbf{R} - \nabla \cdot \mathbb{P}^{[2]})/\rho + \mathbb{P}^{[n]} \cdot \nabla \mathbf{u}\right) = n \text{Sym}\left(\frac{q}{m} \mathbb{P}^{[n]} \times \mathbf{B}\right) + \int_{\mathbf{c}} C \prod_{i=1}^n \mathbf{c}.$$

4.4 Evolution of pressure tensor

In case $n = 2$, write $\mathbb{P} := \mathbb{P}^{[2]} = \rho \langle \mathbf{c} \mathbf{c} \rangle$. $\mathbb{P}^{[1]} = \rho \langle \mathbf{c} \rangle = 0$. So the pressure tensor evolution equation becomes

$$\delta_t(\mathbb{P}) + \nabla \cdot (\mathbb{P}^{[3]}) + 2 \text{Sym}(\mathbb{P} \cdot \nabla \mathbf{u}) = 2 \text{Sym}\left(\frac{q}{m} \mathbb{P} \times \mathbf{B}\right) + \int_{\mathbf{c}} C \mathbf{c} \mathbf{c},$$

i.e.,

$$\partial_t(\mathbb{P}) + \nabla \cdot (\mathbb{P}^{[3]} + 3 \text{Sym}(\mathbf{u} \mathbb{P})) = 2 \text{Sym}\left(\frac{q}{m} \mathbb{P} \times \mathbf{B}\right) + \int_{\mathbf{c}} C \mathbf{c} \mathbf{c}.$$

Subtracting this from the evolution equation (3.2) for the energy tensor gives a kinetic energy tensor evolution equation,

$$\partial_t(\rho \mathbf{u} \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u} \mathbf{u}) = \frac{q}{m} 2 \text{Sym}(\rho \mathbf{u} \mathbf{E} + (\rho \mathbf{u} \mathbf{u}) \times \mathbf{B}) + \int_{\mathbf{v}} C(\mathbf{v} \mathbf{v} - \mathbf{c} \mathbf{c}),$$

which can be obtained by multiplying the momentum equation $\rho d_t \mathbf{u} = \rho \frac{q}{m} (\mathbf{E} + \mathbf{u} \times \mathbf{B})$ by \mathbf{u} and taking the symmetric part. For closure we neglect $\mathbb{P}^{[3]}$ (which is equivalent to assuming that the pressure tensor is an anisotropic Gaussian), and assume that the collision operator C is zero. This will close the system, giving us the ten-moment collisionless plasma model.

4.5 Evolution of pressure

To get an evolution equation for the pressure we take the trace of the evolution equation for the pressure tensor.