Waves in Maxwell's Equations

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1 Light waves

Recall Maxwell's equations in a vacuum:

$$\partial_t B + c_1 \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\partial_t E - c_2 \nabla \times \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 0.$$

For SI units $c_1 = 1$ and $c_2 = c^2$; for Gaussian units, $c_1 = c$ and $c_2 = c$.

In one dimension this becomes:

$$\begin{pmatrix} B_2 \\ B_3 \\ E_2 \\ E_3 \end{pmatrix}_t + \begin{pmatrix} -c_1 E_3 \\ c_1 E_2 \\ c_2 B_3 \\ -c_2 B_2 \end{pmatrix}_x = 0$$

In matrix form this reads:

$$\begin{pmatrix} B_2 \\ B_3 \\ E_2 \\ E_3 \end{pmatrix}_t + \begin{pmatrix} 0 & 0 & 0 & -c_1 \\ 0 & 0 & c_1 & 0 \\ 0 & c_2 & 0 & 0 \\ -c_2 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} B_2 \\ B_3 \\ E_2 \\ E_3 \end{pmatrix}_x = 0.$$

To find the eigenstructure, we row reduce the system

$$\begin{pmatrix} c & 0 & 0 & c_1 \\ 0 & c & -c_1 & 0 \\ 0 & -c_2 & c & 0 \\ c_2 & 0 & 0 & c \end{pmatrix} \cdot \begin{pmatrix} B_2 \\ B_3 \\ E_2 \\ E_3 \end{pmatrix}' = 0.$$

This system decouples into two subsystems:

$$(1) \begin{pmatrix} c & c_1 \\ c_2 & c \end{pmatrix} \cdot \begin{pmatrix} B_2 \\ E_3 \end{pmatrix}' = 0, \text{ and}$$
$$(2) \begin{pmatrix} c & -c_1 \\ -c_2 & c \end{pmatrix} \cdot \begin{pmatrix} B_3 \\ E_2 \end{pmatrix}' = 0.$$

The nonzero eigenvalues are

$$c = \pm c_0, c_0 := \sqrt{c_1 c_2}.$$

Left and right eigenvectors for $c = \pm c_0$ are

$$\begin{pmatrix} B_2 \\ E_3 \end{pmatrix}'_{\text{right}} = \begin{pmatrix} \mp 1 \\ \sqrt{\frac{c_2}{c_1}} \end{pmatrix}, \quad \begin{pmatrix} B_2 \\ E_3 \end{pmatrix}'_{\text{left}} = \frac{1}{2} \begin{pmatrix} \mp 1 \\ \sqrt{\frac{c_1}{c_2}} \end{pmatrix},$$
$$\begin{pmatrix} B_3 \\ E_2 \end{pmatrix}'_{\text{right}} = \begin{pmatrix} \pm 1 \\ \sqrt{\frac{c_2}{c_1}} \end{pmatrix}, \quad \begin{pmatrix} B_3 \\ E_2 \end{pmatrix}'_{\text{left}} = \frac{1}{2} \begin{pmatrix} \pm 1 \\ \sqrt{\frac{c_1}{c_2}} \end{pmatrix}.$$

So we can write matrices of left and right eigenvectors ordered from slow to fast as:

$$L^T = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & \sqrt{\frac{c_1}{c_2}} & 0 & \sqrt{\frac{c_1}{c_2}} \\ \sqrt{\frac{c_1}{c_2}} & 0 & \sqrt{\frac{c_1}{c_2}} & 0 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & \sqrt{\frac{c_2}{c_1}} & 0 & \sqrt{\frac{c_2}{c_1}} \\ \sqrt{\frac{c_2}{c_1}} & 0 & \sqrt{\frac{c_2}{c_1}} & 0 \end{pmatrix},$$

with eigenvalue matrix

$$\Lambda = \begin{pmatrix} -c_0 & & & \\ & -c_0 & & \\ & & c_0 & \\ & & & c_0 \end{pmatrix}.$$