Waves in MHD Plasmas

by E. Alec Johnson, May 2008

1 Ideal MHD eigenstructure

1.1 Linearization

Recall the equations of smooth MHD in primitive variables:

$$
\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \mu_0^{-1}(\nabla\times\mathbf{B})\times\mathbf{B},\\
&p_t + \mathbf{u}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{u} = 0,\\
&\mathbf{B}_t + \nabla\times(\mathbf{B}\times\mathbf{u}) = 0.
\end{aligned}
$$

We remark that the pressure evolution equation is a form of the thermal energy evolution equation, which is obtained from energy conservation by subtracting kinetic energy balance (obtained by dotting $\mathbf{u}$ with the momentum equation) and subtracting magnetic field energy balance (obtained by dotting $\mathbf{B}$ with the magnetic field evolution equation). It implies that entropy is invariant along particle paths.

To facilitate linearization, we apply the product rule and rewrite these equations in a form that contains no derivatives of products. (We vertically align according to differentiated variable to prepare to put the equations in matrix form.)

$$
\begin{aligned}
0 &= \rho_t + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u},\\
0 &= \mathbf{u}_t \quad + \mathbf{u}\cdot\nabla\mathbf{u} \quad + \frac{1}{\rho}\nabla p \quad + \frac{1}{\mu_0\rho}\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{\mu_0\rho}(\nabla\mathbf{B})\cdot\mathbf{B},\\
0 &= p_t \quad + \gamma p\nabla\cdot\mathbf{u} \quad + \mathbf{u}\cdot\nabla p,\\
0 &= \mathbf{B}_t \quad + \mathbf{B}\nabla\cdot\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{u} \quad + \mathbf{u}\cdot\nabla\mathbf{B}.
\end{aligned}
$$

Assuming $\partial_2 = 0 = \partial_3$ implies that B_1 is constant (in space by the divergence condition $\nabla\cdot\mathbf{B} = 0$ and in time by Faraday's law $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$) and thus gives the 1-dimensional MHD system

$$
\begin{pmatrix}\rho\\ u_1\\ u_2\\ u_3\\ p\\ B_2\\ B_3\end{pmatrix}_t
+
\begin{bmatrix}
u_1 & \rho & 0 & 0 & 0 & 0 & 0\\
0 & u_1 & 0 & 0 & \frac{1}{\rho} & \frac{B_2}{\mu_0\rho} & \frac{B_3}{\mu_0\rho}\\
0 & 0 & u_1 & 0 & 0 & \frac{-B_1}{\mu_0\rho} & 0\\
0 & 0 & 0 & u_1 & 0 & 0 & \frac{-B_1}{\mu_0\rho}\\
0 & \gamma p & 0 & 0 & u_1 & 0 & 0\\
0 & B_2 & -B_1 & 0 & 0 & u_1 & 0\\
0 & B_3 & 0 & -B_1 & 0 & 0 & u_1
\end{bmatrix}
\cdot
\begin{pmatrix}\rho\\ u_1\\ u_2\\ u_3\\ p\\ B_2\\ B_3\end{pmatrix}_x
= 0
$$

This is in quasilinear form. To linearize, we merely freeze the matrix entries at a background state (denoted with a zero subscript) and regard the state variables as perturbations (denoted with a prime).

1.2 Eigenstructure

To simplify notation and to facilitate finding left eigenvectors later, we generalize the matrix (and make it look closer to a "generically self-adjoint" matrix) by making the definitions

$$g := \gamma p,$$
$$g^* := \frac{1}{\rho},$$
$$\mathbf{B}^* := \frac{\mathbf{B}}{\mu_0 \rho}.$$

Observe that $gg^* = \frac{\gamma p}{\rho} =: v_s$, the acoustic sound speed. We remark that we could assume without loss of generality that $\rho = 1 = g^*$ by replacing $\mathbf{B}$ with $\sqrt{\mu_0^{-1}}\mathbf{B}$ and by choice of units of mass. Furthermore, if we are willing to rescale time (or space), we could also assume that $v_s = 1$ by choice of units of velocity.

Transforming into a frame of reference convected with the fluid shifts the diagonal entries to zero. By choosing units of mass appropriately, we can force $\rho_0 = 1 = g^*$. By choosing units of time properly, we can also force the sound speed to be one, i.e., $v_s = 1 = g$. The result of such a redimensionalization is to make the lower right 6×6 matrix symmetric. It easily follows that the matrix as a whole has real eigenvalues and a full set of eigenvectors. Thus the system is hyperbolic, and its general solution is a superposition of "eigenperturbations" propagating at a speed equal to the corresponding eigenvalue.

So our formally near-self-adjoint system is

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}_t' + \begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & g^* & B_2^* & B_3^* \\ 0 & 0 & u_1 & 0 & 0 & -B_1^* & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & -B_1^* \\ 0 & g & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & u_1 \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}_x' = 0.$$

By rotation of coordinates in dimensions 2 and 3, we may assume without loss of generality that $B_3 = 0$ and $B_2 \geq 0$.

So to find the eigenstructure, we row-reduce the system

$$\begin{bmatrix} -c & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & g^* & B_2^* & 0 \\ 0 & 0 & -c & 0 & 0 & -B_1^* & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & -B_1^* \\ 0 & g & 0 & 0 & -c & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -B_1 & 0 & 0 & -c \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}' = 0;$$

avoiding division will give eigenvectors with polynomial components and a polynomial dispersion relation for the eigenvalues. The dispersion relation can then be used to rewrite any polynomial in c in terms of polynomials of lesser order than the order of the dispersion relation. This technique is useful in simplifing expressions for the norms of the eigenvectors.

This system decouples into the following subsystems:

$$\begin{bmatrix} c & B_1^* \\ B_1 & c \end{bmatrix}_0 \cdot \begin{pmatrix} u_3 \\ B_3 \end{pmatrix}' = 0 \text{ and } \begin{bmatrix} -c & 1 & 0 & 0 & 0 \\ 0 & -c & 0 & g^* & B_2^* \\ 0 & 0 & -c & 0 & -B_1^* \\ 0 & g & 0 & -c & 0 \\ 0 & B_2 & -B_1 & 0 & -c \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0.$$

The Alfvèn system on the left gives rise to a pair of "oblique" Alfvèn waves. The magnetosonic system on the right gives rise to an entropy wave and pairs of fast and slow magnetosonic waves.

The velocities of the oblique Alfvèn waves are given by $c_A = \pm\sqrt{B_1 B_1^*} = \pm\frac{B_1}{\sqrt{\mu_0 \rho}}$, and the corresponding eigenvectors are

$$\begin{pmatrix} u_3 \\ B_3 \end{pmatrix}' \propto \begin{pmatrix} c \\ \mp B_1 \end{pmatrix} \propto \begin{pmatrix} 1 \\ \mp\sqrt{\mu_0 \rho} \end{pmatrix}$$

Notice that the pertubations are in the plane perpendicular to the direction of propagation.

To find the characteristic polynomial and right eigenvectors of the magnetosonic system, we row-reduce to upper triangular form. We leave out the first row and column for now, since nothing needs to be done there.

Our system is

$$\begin{bmatrix} -c & 0 & g^* & B_2^* \\ 0 & -c & 0 & -B_1^* \\ g & 0 & -c & 0 \\ B_2 & -B_1 & 0 & -c \end{bmatrix}_0 \cdot \begin{pmatrix} u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0,$$

Notice that the pertubations are in the plane spanned by the direction of propagation and the magnetic field. Notice also that just as for Alfvèn waves, B_2' and u_2' (which are in the direction perpendicular to wave propagation) are in a ratio equal to the ratio of the wave speed, and that just as for sound waves, p' and u_1' (which are related to perturbations parallel to the direction of motion) are in a ratio equal to the ratio of g to the sound speed.

This system is similar to

$$\begin{bmatrix} g & 0 & -c & 0 \\ 0 & c & 0 & B_1^* \\ 0 & gB_1 & -cB_2 & cg \\ 0 & 0 & gg^* - c^2 & gB_2^* \end{bmatrix}_0 \cdot \begin{pmatrix} u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0,$$

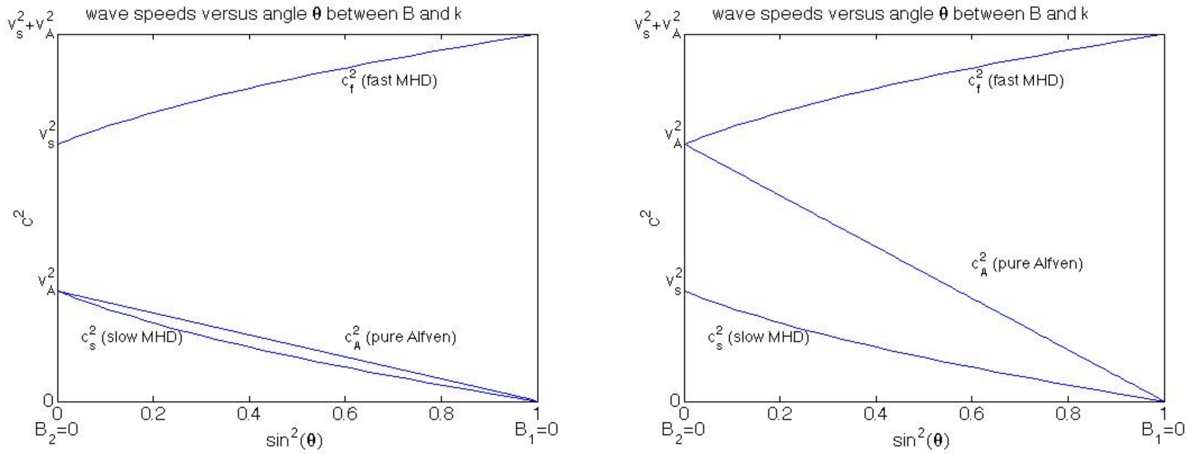
which is similar to

$$\begin{bmatrix} g & 0 & -c & 0 \\ 0 & c & 0 & B_1 \\ 0 & 0 & c^2 B_2 & g(B_1 B_1^* - c^2) \\ 0 & 0 & gg^* - c^2 & gB_2^* \end{bmatrix}_0 \cdot \begin{pmatrix} u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0.$$

For the determinant to vanish, the determinant of the lower right 2-by-2 matrix must vanish, i.e., the last two equations must be redundant.

Taking the determinant of the lower right 2-by-2 system yields the dispersion relation

$$(c^2 - B_1 B_1^*)(c^2 - gg^*) - c^2 B_2 B_2^* = 0, \text{ i.e.,}$$
$$c^4 - c^2(gg^* + \mathbf{B} \cdot \mathbf{B}^*) + gg^* B_1 B_1^*,$$



(a) $v_A^2 < v_s^2$ (b) $v_A^2 > v_s^2$

Figure 1: MHD wave speeds as a function of the angle θ between the magnetic field **B** and the direction of propagation **k**.

i.e.,

$$c^4 - c^2(v_s^2 + v_A^2) + v_s^2 c_A^2.$$

where

$$v_A^2 := \mathbf{B}\cdot\mathbf{B}^* = \frac{B^2}{\mu_0\rho} = \text{ (Alfvèn speed),}$$
$$c_A^2 := B_1 B_1^* = \frac{B_1^2}{\mu_0\rho} = \text{ (oblique Alfvèn wave speed), and}$$
$$v_s^2 := gg^* = \frac{\gamma p}{\rho} = \text{ (sound speed).}$$

Note that $c_A^2 = v_A^2 \cos^2\theta$, where θ is the angle between **B** and the positive x-axis.

The roots of the quadratic dispersion relation in c^2 define the fast and slow magnetosonic speeds:

$$(c^2 - c_f^2)(c^2 - c_s^2) = 0.$$

So $c_f^2, c_s^2 = (1/2)\left[(v_s^2 + v_A^2) \pm \sqrt{(v_s^2 + v_A^2)^2 - 4v_s^2 c_A^2}\right]$, an expression of which we make little direct use, preferring to work directly with the polynomial dispersion relation.

Observe that we can rewrite the magnetosonic dispersion relation as

$$(c^2 - v_s^2)(c^2 - v_A^2) = v_s^2 v_A^2 \sin^2\theta.$$

Recall also the formula for the Alfvèn speed,

$$c_A^2 = v_A^2 \cos^2\theta.$$

Graphing these relations between wave speed and $\sin^2\theta$ for the generic cases $0 < v_A^2 < v_s^2$ and $v_A^2 > v_s^2 > 0$ characterizes the general relationship among MHD plasma wave speeds (see Figure 1).

Recall the eigenvector system

$$\begin{bmatrix} c & -1 & 0 & 0 & 0 \\ 0 & g & 0 & -c & 0 \\ 0 & 0 & c & 0 & B_1 \\ 0 & 0 & 0 & c^2 B_2 & g(B_1 B_1^* - c^2) \\ 0 & 0 & 0 & gg^* - c^2 & gB_2^* \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = \begin{bmatrix} c & -1 & 0 & 0 & 0 \\ 0 & \gamma p & 0 & -c & 0 \\ 0 & 0 & c & 0 & B_1 \\ 0 & 0 & 0 & c^2 B_2 & \gamma p(c_a^2 - c^2) \\ 0 & 0 & 0 & v_s^2 - c^2 & v_s^2 B_2 \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0.$$

One's choice among the bottom two equations gives two possible ways to express the right eigenvectors:

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}'_{\text{right}} \propto \begin{pmatrix} cB_2^* \\ c^2 B_2^* \\ B_1(gg^* - c^2) \\ cgB_2^* \\ c(c^2 - gg^*) \end{pmatrix} \propto \begin{pmatrix} c^2 - B_1 B_1^* \\ c(c^2 - B_1 B_1^*) \\ -cB_1 B_2 \\ gg^*(c^2 - B_1 B_1^*) \\ c^2 B_2 \end{pmatrix} = \begin{pmatrix} c^2 - c_A^2 \\ c(c^2 - c_A^2) \\ -cB_1 B_2 \\ v_s^2(c^2 - c_A^2) \\ c^2 B_2 \end{pmatrix}.$$

For the left eigenvectors, transposing the system matrix and using c to kill 1 in the first column (which assumes $c \neq 0$) shows that to get the left eigenvalues we just zero out the density perturbation and swap the starred and unstarred variables:

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}'_{\text{left}} \propto \begin{pmatrix} 0 \\ c^2 B_2 \\ B_1^*(gg^* - c^2) \\ cgB_2 \\ c(c^2 - gg^*) \end{pmatrix} \propto \begin{pmatrix} 0 \\ c(c^2 - B_1 B_1^*) \\ -cB_1^* B_2^* \\ gg^*(c^2 - B_1 B_1^*) \\ c^2 B_2^* \end{pmatrix}.$$

1.3 Case $B_1 = 0$

1.4 Case $B_2 = 0$

1.5 Alfvèn waves

In the case of Alfvèn waves, the fact that the wave speed is independent of the perturbation variables suggests that we should seek a finite-amplitude solution of the nonlinearized MHD equations.

Recall 1-dimensional MHD in quasilinear form:

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}_t + \begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & g^* & B_2^* & B_3^* \\ 0 & 0 & u_1 & 0 & 0 & -B_1^* & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & -B_1^* \\ 0 & g & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & u_1 \end{bmatrix} \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}_x = 0.$$

Without linearizing, we seek a solution for which ρ and p are constant. The evolution equation for ρ then implies that u_1 is constant in space, and hence constant in time by global momentum conservation. The evolution equation for u_1 in turn simplifies to $\partial_t u_1 + B_2^* \partial_x B_2 + B_3^* \partial_x B_3 = 0$, i.e., $0 = \partial_x (B_2^2 + B_3^2)$, which, since u_1 is constant in space, says that $\partial_x (B_2^2 + B_3^2)$ must be constant in space. Transforming into the frame of reference of the fluid, we may say without loss of generality that $u_1 = 0$.

So our system simplifies to

$$\begin{pmatrix} u_2 \\ B_2 \end{pmatrix}_t + \begin{bmatrix} 0 & -B_1^* \\ -B_1 & 0 \end{bmatrix} \cdot \begin{pmatrix} u_2 \\ B_2 \end{pmatrix}_x = 0 \qquad \text{and} \qquad \begin{pmatrix} u_3 \\ B_3 \end{pmatrix}_t + \begin{bmatrix} 0 & -B_1^* \\ -B_1 & 0 \end{bmatrix} \cdot \begin{pmatrix} u_3 \\ B_3 \end{pmatrix}_x = 0.$$

This is just a pair of wave equations with speeds $c = \pm\sqrt{B_1 B_1^*} = \pm\frac{B_1}{\sqrt{\mu_0 \rho}}$ and corresponding right- and left-traveling waves

$$\begin{pmatrix} u_i \\ B_i \end{pmatrix} = f_i^{\pm}(x - ct) \begin{pmatrix} \mp 1 \\ \sqrt{\mu_0 \rho} \end{pmatrix},$$

where the $f_i^{\pm}$ need to satisfy the requirement that $(f_2^+)^2 + (f_3^+)^2$ and $(f_2^-)^2 + (f_3^-)^2$ are constant.

For a ready such pair of functions, choose $f_2(x) = u_0 \cos(kx)$ and $f_3(x) = u_0 \sin(kx)$. This gives the solution

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \mp u_0 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = B_0 \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix},$$

where

$$u_0 = \frac{B_0}{\sqrt{\mu_0 \rho}}, \qquad \theta := kx - \omega t, \qquad \text{and} \qquad \frac{\omega}{k} = c = \frac{\pm B_1}{\sqrt{\mu_0 \rho}}.$$

For this solution the components of **u** and **B** perpendicular to the x-axis are rotationally polarized and aligned or anti-aligned depending on whether the wave is propagating in the negative or positive direction.

1.5.1 Two-fluid quantities

An Alfvèn wave is an exact solution of ideal MHD. Ideal MHD is derived from two-fluid plasma equations by summing over the constituent species the equations that govern the evolution of mass, momentum, and energy density.

We wish to know what the two-fluid quantitities are doing in the case of an Alfvén wave. Specifically, we wish to determine the drift velocities of each of the two species. MHD assumes quasineutrality. Assuming two species of equal and opposite charge,

$$\mathbf{w}_i = \frac{m_e}{m_i + m_e} \frac{\mathbf{J}}{en} = \frac{\tilde{m}_e \mathbf{J}}{\rho}, \quad \text{where } \tilde{m}_e := \frac{m_e}{e}, \text{ and}$$
$$\mathbf{w}_e = \frac{-m_i}{m_i + m_e} \frac{\mathbf{J}}{en} = -\frac{\tilde{m}_i \mathbf{J}}{\rho}, \text{ where } \tilde{m}_i := \frac{m_i}{e},$$

where $\mathbf{J}$ is the current, n is the number density (of each species), e is the charge on an electron, and for each species s, $\mathbf{w}_s$ denotes the drift velocity, m_s denotes particle mass.

MHD assumes that displacement current is negligible. So in one space dimension, assuming

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \mp u_0 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = B_0 \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix} =: \mathbf{B}_\perp + \mathbf{B}_\parallel,$$

we find that $\mathbf{J}$ is antiparallel to $\mathbf{B}_\perp$:

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B} = \mu_0^{-1} \partial_x \begin{pmatrix} 0 \\ -B_3 \\ B_2 \end{pmatrix} = \mu_0^{-1} B_0 k \begin{pmatrix} 0 \\ -\cos\theta \\ -\sin\theta \end{pmatrix}.$$

Hence (using $B_0 = u_0\sqrt{\mu_0\rho}$) the species velocities are

$$\mathbf{u}_i = u_0\Big(\mp 1 - \frac{k\tilde{m}_e}{\sqrt{\mu_0\rho}}\Big)\begin{pmatrix}0\\ \cos\theta\\ \sin\theta\end{pmatrix},$$
$$\mathbf{u}_e = u_0\Big(\mp 1 + \frac{k\tilde{m}_i}{\sqrt{\mu_0\rho}}\Big)\begin{pmatrix}0\\ \cos\theta\\ \sin\theta\end{pmatrix}.$$

1.5.2 Species velocities for finite gyroradius

In case of nonvanishing gyroradius, the assumptions of MHD are not fully satisfied, the Alfvèn wave is no longer an exact solution, and the velocities of the constituent fluids depart from the MHD prediction.

Recall the nondimensionalized two-fluid plasma equations

$$\partial_t \begin{bmatrix}\rho_i\\ \rho_e\\ \rho_i\mathbf{u}_i\\ \rho_e\mathbf{u}_e\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho_i\mathbf{u}_i\\ \rho_e\mathbf{u}_e\\ \rho_i\mathbf{u}_i\mathbf{u}_i+\mathbb{P}_i\\ \rho_e\mathbf{u}_e\mathbf{u}_e+\mathbb{P}_e\end{bmatrix} = \frac{1}{r}\begin{bmatrix}0\\ 0\\ \sigma_i\mathbf{E}+\mathbf{J}_i\times\mathbf{B}\\ \sigma_e\mathbf{E}+\mathbf{J}_e\times\mathbf{B}\end{bmatrix}$$
$$\partial_t\begin{bmatrix}(c\mathbf{B})\\ \mathbf{E}\end{bmatrix} + c\nabla\times\begin{bmatrix}\mathbf{E}\\ -(c\mathbf{B})\end{bmatrix} = \frac{1}{r}\begin{bmatrix}0\\ -\mathbf{J}/\lambda^2\end{bmatrix},\ \text{and}\ \nabla\cdot\begin{bmatrix}(c\mathbf{B})\\ \mathbf{E}\end{bmatrix} = \frac{1}{r}\begin{bmatrix}0\\ \sigma/\lambda^2\end{bmatrix}.$$

Here $r := \frac{v_0 m_0}{x_0 q_0 B_0} = r_L/x_0$ is the nondimensionalized gyroradius and $\varepsilon = \frac{\varepsilon_0 v_0 B_0}{q_0 n_0 x_0}$ is a fake permittivity; we can write $\varepsilon = r\lambda^2$, where $\lambda^2 := \frac{\varepsilon_0 B_0 v_0}{q_0 n_0 x_0}$ defines the ratio of the Debye length $\lambda_D := \sqrt{\left(\frac{\varepsilon_0 m_0 v_0^2}{n_0 q_0^2}\right)}$ to the gyroradius.

We remark that in primitive variables we have

$$\begin{aligned}
&\partial_t\rho_i + \nabla\cdot(\rho_i\mathbf{u}_i) = 0,\\
&\partial_t\rho_e + \nabla\cdot(\rho_e\mathbf{u}_e) = 0,\\
&\rho_i(\partial_t\mathbf{u}_i + \mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \nabla p_i = (1/r)\big(\sigma_i\mathbf{E} + \mathbf{J}_i\times\mathbf{B}\big),\\
&\rho_e(\partial_t\mathbf{u}_e + \mathbf{u}_i\cdot\nabla\mathbf{u}_e) + \nabla p_e = (1/r)\big(\sigma_e\mathbf{E} + \mathbf{J}_e\times\mathbf{B}\big),\\
&\partial_t p_i + \mathbf{u}_i\cdot\nabla p_i + \gamma p_i\nabla\cdot\mathbf{u}_i = 0,\\
&\partial_t p_e + \mathbf{u}_e\cdot\nabla p_e + \gamma p_e\nabla\cdot\mathbf{u}_e = 0,\\
&\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0,\\
&\partial_t\mathbf{E} - c^2\nabla\times\mathbf{B} = -\mathbf{J}/\varepsilon,\\
&\mathbf{B}_t + \nabla\times(\mathbf{B}\times\mathbf{u}) = 0,\\
&\nabla\cdot\mathbf{B} = 0,\\
&\nabla\cdot\mathbf{E} = \sigma/\varepsilon.
\end{aligned}$$

We seek a solution that approximates an Alfvèn wave. So we assume that densities and pressures are constant. Then

$$
\begin{aligned}
&\nabla\cdot\mathbf{u}_i = 0, \nabla\cdot\mathbf{u}_e = 0,\\
&\partial_t\mathbf{u}_i + \mathbf{u}_i\cdot\nabla\mathbf{u}_i = \frac{e}{rm_i}(\mathbf{E}+\mathbf{u}_i\times\mathbf{B})\\
&\partial_t\mathbf{u}_e + \mathbf{u}_e\cdot\nabla\mathbf{u}_e = \frac{e}{rm_e}(\mathbf{E}+\mathbf{u}_e\times\mathbf{B})\\
&\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0,\\
&\partial_t\mathbf{E} - c^2\nabla\times\mathbf{B} = -\mathbf{J}/\varepsilon,\\
&\mathbf{B}_t + \nabla\times(\mathbf{B}\times\mathbf{u}) = 0,\\
&\nabla\cdot\mathbf{B} = 0,\\
&\nabla\cdot\mathbf{E} = \sigma/\varepsilon.
\end{aligned}
$$

Numerical simulations lead us to seek Again, we will quasilinearize.