Asymptotics of coinvariants of Iwasawa modules under non-normal subgroups

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Let $G$ be a pro-$p$ $p$-adic analytic group, thought of as a closed subgroup of $GL_N(\mathbb{Z}_p)$, and let $\Sigma$ be a closed subgroup of $G$. Write $\Lambda$ for the completed group algebra $\mathbb{Z}_p[[G]]$ and let $M$ be a finitely generated $\Lambda$-module. Let $G = G^0 \supset G^1 \supset G^2 \supset \ldots$ be the descending sequence of principal congruence subgroups of $G$; write $G_n$ for the quotient $G/G^n$ and $\Sigma_n$ for the image of $\Sigma$ in $G_n$. Write $M_n$ for the coinvariant quotient of $M$ under $G^n$. Then $M_n$ is a module for the group algebra $\mathbb{Z}_p[G_n]$.

In Iwasawa theory, one often finds that the growth of arithmetic invariants of interest (e.g. class numbers, Mordell-Weil ranks) is controlled by a $\Lambda$-module $M$. In particular, the growth can be related to the $\mathbb{Z}_p$-ranks of the coinvariant quotients of $M$ by various subgroups of $G$. Understanding these quotients is a purely algebraic problem. For instance, Harris [3, Theorem 1.10] shows that, if $M$ is a $\Lambda$-torsion module,

$$\text{rank}_{\mathbb{Z}_p} M_n = O(p^n(\dim G - 1)).$$

Note that if $M$ is replaced by a free module of rank 1, we have

$$\text{rank}_{\mathbb{Z}_p} \Lambda G_n = |G_n| \sim p^n \dim G.$$ 

So one can read Harris’s result as saying “the coinvariants of a torsion $\Lambda$-module by congruence subgroups grow more slowly than do the coinvariants of a free $\Lambda$-module.” The goal of the present paper is to show a similar result for subgroups which are in some sense “far from normal.” As corollaries, we show that induced modules are often faithful in the sense of Venjakob [7] and we give an upper bound for the growth of Mordell-Weil ranks of elliptic curves over certain non-Galois towers of field extensions.

**Definition 1.** We say $\Sigma \subset G$ is **eccentric** if

$$\lim_{n \to \infty} \frac{|\Sigma_n \backslash G_n / \Sigma_n|}{|G_n||\Sigma_n|^{-2} p^n} = 0.$$ 

**Example 2.** Suppose $G = K \rtimes \Sigma$, where $K$ is isomorphic to $\mathbb{Z}_p^r$ and $\Sigma$ to $\mathbb{Z}_p$. Then $\Sigma$ is eccentric precisely when the action of $\Sigma$ on $K$ is nontrivial.

**Remark 3.** It seems likely that the limit

$$\lim_{n \to \infty} \frac{\log |\Sigma_n \backslash G_n / \Sigma_n|}{n \log p}$$

exists and is a non-negative integer, though this seems a bit complicated to prove. When this limit is an integer, it seems interesting to ask whether it is a “dimension” associated to the pair $(G, \Sigma)$ in any cohomological sense.
Remark 4. The condition of eccentricity, as we have written it, depends on the structure of \( G \) as a subgroup of \( GL_N(\mathbb{Z}_p) \); in fact, though we will not need this here, the condition is intrinsic to \( (G, \Sigma) \) and can be computed using the \( p \)-lower central series in place of the descending series of congruence subgroups.

We will prove the following theorem.

**Theorem 5.** Let \( G \) be a pro-\( p \) \( p \)-analytic group with no \( p \)-torsion, and let \( M \) be a finitely generated torsion module for \( \Lambda = \mathbb{Z}_p[[G]] \). Let \( \Sigma \) be an eccentric subgroup of \( G \). Then

\[
\lim_{n \to \infty} \frac{\text{rank}_{\mathbb{Z}_p}(M_n)_{\Sigma}}{\text{rank}_{\mathbb{Z}_p}(G)_{G^n \Sigma}} = 0.
\]

Recall that a \( \Lambda \)-module \( M \) is called *faithful* if \( \text{Ann}_\Lambda M = 0 \). When \( \Lambda \) is abelian, a torsion module cannot be faithful. By contrast, in the non-abelian cases, faithful torsion \( \Lambda \)-modules are quite prevalent; indeed they are ubiquitous among \( \Lambda \)-modules arising in arithmetic applications. Many examples of faithful torsion \( \Lambda \)-modules were constructed by Venjakob in [7]; for instance, he shows there that if \( G \) is a non-abelian semidirect product \( K \rtimes \Sigma \) with \( K \cong \Sigma \cong \mathbb{Z}_p \), then the induced module \( \text{Ind}^G_\Sigma \mathbb{Z}_p \) is a faithful \( \Lambda \)-module [7, Prop. 4.2]. In another example, he shows that if \( G \) is a pro-\( p \) subgroup of \( \text{SL}_2(\mathbb{Z}_p) \), and \( \Sigma \) is a maximal torus, then \( \text{Ind}^G_\Sigma \mathbb{Z}_p \) is again a faithful \( \Lambda \)-module.

The following corollary generalizes these examples.

**Corollary 6.** Let \( G \) be as above and let \( \Sigma \) be an eccentric subgroup. Then \( \text{Ann}_\Lambda \text{Ind}^G_\Sigma \mathbb{Z}_p \) is trivial.

**Proof.** Suppose \( A = \text{Ann}_\Lambda \text{Ind}^G_\Sigma \mathbb{Z}_p \) is nontrivial. Equivalently, the nonzero two-sided ideal \( A \) is contained in the left augmentation ideal \( I^L_\Sigma \). Since \( \Lambda \) is isomorphic to its opposite algebra, there is a nonzero two-sided ideal \( B \) contained in the right augmentation ideal \( I^R_\Sigma \). Now take \( M \) to be the torsion module \( \Lambda/B \). Then

\[
M_{G^n \Sigma} = \Lambda/(B + I^L_\Sigma \Lambda + I_G^n) = \Lambda/(I^L_\Sigma \Lambda + I_G^n) = \Lambda(G)_{G^n \Sigma}
\]

which contradicts Theorem 5.

**Remark 7.** Venjakob also proves that certain modules for the completed group algebra \( \mathbb{F}_p[[G]] \) have trivial annihilator. The method of the present paper does not work in characteristic \( p \); it is an interesting question whether the analogue of Theorem 5 still holds.

**Remark 8.** When \( \Sigma \) is trivial, Theorem 5 follows from the theorem of Harris cited above. Note also that some form of the eccentricity hypothesis on \( \Sigma \) is certainly necessary: if \( \Sigma \) is normal, for instance, then \( \mathbb{Z}_p[[G/\Sigma]] \) is a torsion \( \Lambda \)-module whose coinvariants are identical with those of the free module \( \Lambda \).

**Remark 9.** Eccentricity of \( \Sigma \) implies that \( \dim \Sigma \leq (1/2) \dim G \). If \( \dim \Sigma \) is any larger, it is not clear that any version of Theorem 5 can hold. Indeed, it is an interesting open question whether \( \text{Ind}^G_\Sigma \mathbb{Z}_p \) is faithful in this case. This question seems substantially harder; in particular, it does not seem likely that it can be resolved by consideration of representation theory in characteristic 0, as in the present paper.

We now prove Theorem 5.
Proof. We know $M$ is finitely generated, which is to say $M$ is a quotient of $\Lambda^C$ for some integer $C$; it follows that $M_p$ is a quotient of $\mathbb{Z}_p[G_n]^C$. Write $M^Q_n$ for $M_n \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$. Then $\text{Hom}_{\mathbb{Q}_p}(M^Q_n, M^Q_n)$ is a quotient of $\text{Hom}_{\mathbb{Q}_p}(M^Q_n, \mathbb{Q}_p[G_n]^C)$, which has dimension $C \dim_{\mathbb{Q}_p} M^Q_n$. Now by (1) we know

$$\dim_{\mathbb{Q}_p} \text{Hom}_{\mathbb{Q}_p}(M^Q_n, M^Q_n) \leq Cp^{n \dim G - n} \sim C|G_n|p^{-n}$$

(2)

On the other hand, if $[G_n/\Sigma_n]$ is the permutation representation of $G_n$ on the cosets of $\Sigma_n$ (with $\mathbb{Q}_p$-coefficients) then

$$\dim_{\mathbb{Q}_p} \text{Hom}_{\mathbb{Q}_p}([G_n/\Sigma_n], [G_n/\Sigma_n]) = |\Sigma_n|G_n/\Sigma_n|.$$ (3)

Now rank$\mathbb{Q}_p M^\Sigma_n$ is precisely $\dim_{\mathbb{Q}_p} \text{Hom}_{\mathbb{Q}_p}([G_n/\Sigma_n], M^Q_n)$. It follows from (2), (3), and the Cauchy-Schwarz inequality that

$$\dim_{\mathbb{Q}_p} \text{Hom}_{\mathbb{Q}_p}([G_n/\Sigma_n], M^Q_n) \leq (C|\Sigma_n|G_n/\Sigma_n||G_n|p^{-n})^{1/2}$$

and the hypothesis that $\Sigma$ is eccentric tells us exactly that the right hand side is $o(|G_n||\Sigma_n|^{-1}$. Since rank$\mathbb{Q}_p A(G)_{G=\Sigma}$ is precisely $|G_n||\Sigma_n|^{-1}$, we are done.

Remark 10. We do not expect the given upper bound on rank$\mathbb{Q}_p M^\Sigma_n$ to be sharp, because the inequality

$$\dim_{\mathbb{Q}_p} \text{Hom}_{\mathbb{Q}_p}(M^Q_n, M^Q_n) \leq \dim_{\mathbb{Q}_p} \text{Hom}_{\mathbb{Q}_p}(M^Q_n, \mathbb{Q}_p[G_n]^C)$$

is typically not sharp.

We conclude with an application to ranks of elliptic curves over towers of function fields. Let $p$ be a rational prime, $k$ a field of characteristic prime to 6, $C$ a smooth (but not necessarily proper) geometrically integral curve over $k$, and $\pi : E \to C$ a non-isotrivial elliptic surface with good reduction at all points of $C$. Suppose furthermore that the image of the absolute Galois group of $k(C)$ on $E[p^\infty]$ has image a pro-$p$ principal congruence subgroup $G$ of $\text{GL}_2(\mathbb{Z}_p)$. (This can be arranged by replacing $C$ with a finite cover, as long as $G_m[p^\infty](k)$ is finite.)

Now let $P$ be an element of the Tate module $T_pE$, and let $V$ be a pro-cyclic subgroup of $E$ not containing $P$. Then let $k(C_n)$ be the minimal extension of $k(C)$ over which the projections of $P$ and $V$ to $T_pE/p^nT_pE$ are defined, and let $C_n$ be the nonsingular curve with function field $k(C_n)$. Let $k(C_{\infty})$ be the union of all the $k(C_n)$. Then $k(C_{\infty})$ is an extension of $k(C)$, whose splitting field $k(C_{\infty}')$ has Galois group $G$. (Note that $k(C_{\infty}')$ is obtained from $k(C_{\infty})$ by extending the constant field that splits $C$. Write $G_n$ for the $n$th principal congruence subgroup of $G$, and $\Sigma \subset G$ for the subgroup whose fixed field is $k(C_{\infty})$. Then $\Sigma$ is the 1-dimensional subgroup of $G$ consisting of diagonal matrices fixing $P \in T_pE$. It is easy to check that $\Sigma$ is eccentric in $G$—indeed $[\Sigma_n]G_n/\Sigma_n| = O(p^{2n})$.

Now let $\pi_\mathcal{A} : \mathcal{A} \to C$ be a non-isotrivial elliptic surface over $k(C)$ with good reduction on $C$ (for instance, $\mathcal{A}$ might be $E$ itself.) A theorem of Shioda [4] shows that rank$\mathbb{Z} A(k(C_n))$ is $O(p^{3n})$. Several papers ([2],[3],[6]) have shown that in many pro-$p$ towers of curves, the Shioda bound can be substantially improved, but it does not seem that the methods there apply immediately to this case. However, Theorem 5 allows us to give a non-trivial upper bound for the growth of the rank of $\mathcal{A}$.

Corollary 11. The Mordell-Weil rank of $\mathcal{A}$ over $k(C_n)$ is $o(p^{3n})$.

Proof. Let $j : \eta \hookrightarrow C$ be the inclusion of the generic point, and write $\mathcal{F}$ for the sheaf $j_*j^*R^1(\pi_\mathcal{A}),Q_p/\mathbb{Z}_p$. Then we denote by $S(C,\mathcal{A}[p^\infty])$ the Selmer group $H^1(C \times_k k^s, \mathcal{F})$ of $\mathcal{A}/C$, as in [2, 82]. Then
rank\(_Z\) \(A(k(C)) \leq \text{corank}_{\mathbb{Z}} S(C, A[p^\infty])^{\text{Gal}(k'/k)}\). We also write \(S(C_{\infty}, A[p^\infty])\) for the direct limit of \(H^1(C_i \times_k k^s, F)\) as \(C_i\) ranges over the curves between \(C\) and \(C_{\infty}\). Write \(K\) for the kernel of the determinant map in \(G\), and \(K^n\) for \(K \cap G^n\). Then \(C_n \times_k k^s \rightarrow C_0 \times_k k^s\) is a Galois cover with group \(K/K^n\).

For each \(n\), we have a map
\[ S(C_n, A[p^\infty]) \rightarrow S(C_{\infty}, A[p^\infty])^{K^n} \]
whose kernel is \(H^1(K^n, A[p^\infty](k^s(C_{\infty})))\).

The coefficient module \(A[p^\infty](k^s(C_{\infty}))\) has \(\mathbb{Z}_p\)-corank at most 2, and the congruence subgroup \(K^n\), being a uniform group of rank 3, is generated by 3 elements. It follows that \(H^1(K^n, A[p^\infty](k^s(C_{\infty})))\) has \(\mathbb{Z}_p\)-corank at most 6. So the kernel of
\[ S(C_n, A[p^\infty])^{\text{Gal}(k'/k)} \rightarrow (S(C_{\infty}, A[p^\infty])^{K^n})^{\text{Gal}(k'/k)} \]
also has \(\mathbb{Z}_p\)-corank at most 6. Now \(N := S(C_{\infty}, A[p^\infty])^{\text{Gal}(k'/k(n, \infty))}\) is a module for \(\Lambda(G)\); it is cofinitely generated when considered as a \(\Lambda(K)\)-module by [2, Prop. 3.3], which immediately implies it is a cofinitely generated cotorsion \(\Lambda(G)\)-module – see for instance [1, Prop 2.3]. Now
\[
\text{rank}_{\mathbb{Z}} A(k(C_n)) \leq \text{corank}_{\mathbb{Z}_p} S(C_n, A[p^\infty])^{\text{Gal}(k'/k)} \leq \text{corank}_{\mathbb{Z}_p} N^{\Sigma K^n},
\]
Take \(M\) to be the finitely generated torsion \(\Lambda\)-module dual to \(N\). Now
\[
\text{corank}_{\mathbb{Z}_p} N^{\Sigma K^n} = \text{rank}_{\mathbb{Z}_p} M_{\Sigma K^n} = o(p^{3n})
\]
by Theorem 5, and we are done. \(\square\)

Indeed, the proof of Theorem 5 shows in this case that \(\text{rank}_{\mathbb{Z}}(A(k(C_n)))\) is bounded above by a constant multiple of \((|\Sigma_n| G_n/|\Sigma_n||G_n|p^{-n})^{1/2}\), which is \(O(p^{3n/2})\).

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References

