Section 11.6: Complex Numbers

First of all, I want to discuss the theorem about the quotients of power series. Observe that the power series

$$1 = 1 + 0x + 0x^2 + \dots$$

and

$$1 + x^2 = 1 + 0x + x^2 + 0x^3 + \dots$$

each converge for *all* x. So according to the theorem in the book, we can carry out long division and get

$$1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \dots$$

valid for all x. Is this correct? Ask.

Make a point about what's strange about this example. When we had $1/(1 - 2x)$, it made sense that the radius of convergence was 1/2. Draw the picture–we see that the function blows up there. Compare with functions like $e^x andcosx, whicharecontinuouseverywhere.Wemightformulatea$

FIRST GUESS: The radius of convergence is the distance from 0 to the nearest discontinuity.

But with $1/(1 + x^2)$, we see this guess is wrong. What's going on? Why can't we find a series which converges to $1/(1 + x^2)$ for all x?

To see the answer, we must pass to the wonderful world of complex numbers.

Recall that we define i to be the square root of -1. Don't worry too much about what this means, or whether this is really a "number." If you like, we are just expanding our concept of number.

Definition: a *complex number* is an expression $a + bi$.

Complex numbers can be added and multiplied. Examples:

$$(5 + i) + (6 + i) = 11 + 2i$$

and

$$(5 + i) * (6 + i) = 30 + 5i + 6i + i^2 = 29 + 11i.$$

The great thing about complex numbers is that they have geometry. Just as the real numbers describe a line, the complex numbers describe a plane. Draw this plane, plot $(5 + i)$ and $(6 + i)$. Observe that if $a + bi$ is our point, we get a distance from the origin $r = \sqrt{a^2 + b^2}$ and an angle $\theta = \arctan b/a$. So we can also describe a complex number as a pair (r, θ). And we can write

$$a = r\cos\theta, b = r\sin\theta.$$

Definition (absolute value for complex numbers) $|a + bi| = \sqrt{a^2 + b^2}$.

Example: $(1 + i)/2$. Let them work out that $r = \sqrt{2}/2$ and $\theta = \pi/4$. So we get

$$(1 + i)/2 = (\sqrt{2}/2)(\cos\pi/4 + i\sin\pi/4).$$

FACT: Let z_1 =$\mathrm{r}_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then

$$z_1z_2 = r_1r_2(\cos\theta + i\sin\theta)$$

Example: in the case above, $z_1 = 5+i$ and $z_2 = 6+i$. So $r_1 = |z_1| = \sqrt{26}$, and $r_2 = |z_2| = \sqrt{37}$. We check that, indeed, $|z_1z_2| = \sqrt{26 \cdot 37}$.

Split them up into groups, give 10-15 min (depending on time elapsed) for them to work out, if $z = (1 + i)/2$, what is z^2. What is z^3? What is z^4? What is z^8?

Then come back to my power series for $1/(1 + x^2)$ and ask, is this expansion still valid for x = (1+i)/2? We find that it is. Try it for some other complex numbers, suggested by the audience. Try to figure out for which complex numbers it converges.