On the average number of octahedral modular forms

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Let $N > 0$ be an integer, $\chi$ a Dirichlet character modulo $N$, and $k$ either 0 or 1. Let $f$ be a primitive eigenform, not necessarily holomorphic, of level $N$ and Nebentypus $\chi$, and let $\lambda_f(n)$ be the eigenvalue of $T_p$ on $f$. We say $f$ is associated to a Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C})$ if

$$\lambda_f(p) = \text{Tr}(\rho(Frob_p))$$
$$\chi(p) = \det(\rho(Frob_p))$$

for all $p$ not dividing $N$. Following [4], we define $S_{\text{Artin}}^{1/4,k}(N,\chi)$ to be the finite set of primitive weight $k$ cuspidal eigenforms which admit an associated Galois representation. If $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C})$ is a Galois representation, we define $\mathbb{P}\rho$ to be the composition of $\rho$ with the natural projection $\text{GL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C})$.

Two-dimensional complex Galois representations fall naturally into four types; we call $\rho$ dihedral, tetrahedral, octahedral, or icosahedral according as the projectivized image $\mathbb{P}\rho(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is isomorphic to a dihedral group, $A_4, S_4$, or $A_5$. Cusp forms associated to Galois representations are classified likewise.

The latter three types are called “exotic”; it is widely believed that the number of exotic cusp forms of level $N$ is at most $N^{\epsilon}$. The first results in this direction are due to Duke [3]. These results were later sharpened by Wong [6] and Michel and Venkatesh [4]. The latter authors proved that

$$n^{\text{tetr}}(N,\chi,k) \ll \epsilon N^{2/3+\epsilon}, n^{\text{oct}}(N,\chi,k) \ll \epsilon N^{4/5+\epsilon}, n^{\text{icos}}(N,\chi,k) \ll \epsilon N^{6/7+\epsilon}$$

where $n^T(N,\chi,k)$ is the number of weight $k$ cusp forms of level $N$, Dirichlet character $\chi$, and type $T$. (Note that $n^T(N,\chi,k) = 0$ unless $\chi$ sends complex conjugation to $(-1)^k$.)

The goal of this paper is to show that the Michel-Venkatesh bound on octahedral forms can be sharpened on average over squarefree levels $N$.

We begin by showing that, in the case of square-free level, one does not need to consider very many different Dirichlet characters $\chi$ when counting exotic cusp forms.

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Lemma 1. Let $N$ range over square-free integers. Then the number of Dirichlet characters $\chi$ of conductor $N$ such that there exists an exotic cusp form of level $N$ and Nebentypus $\chi$ is $O(N^\epsilon)$.

Proof. Let $\rho$ be the Galois representation attached to an exotic cusp form of level $N$ and conductor $\chi$. Let $p$ be a prime dividing $N$. Then, since $p | N$, the restriction $\rho : I_p \to GL_2(\mathbb{C})$ must decompose as $\chi \oplus 1$. In particular, the projection from $\rho(I_p)$ to $\mathbb{P}(\rho(I_p))$ is an isomorphism. So $\chi(I_p)$ is a cyclic subgroup of either $A_4, S_4$, or $S_5$; in particular, $\chi^{60}$ is unramified everywhere, whence trivial. Now the number of characters $\chi$ of level $N$ such that $\chi^{60} = 1$ is $O(N^\epsilon)$, which proves the lemma.

Suppose from now on that $N$ is squarefree. Let $n_{\mathrm{oct}}(N)$ be the number of octahedral cusp forms of level $N$. It follows from Lemma 1 and the theorem of Michel and Venkatesh that

$$n_{\mathrm{oct}}(N) \ll \epsilon N^{4/5 + \epsilon}.$$  

Let $p : S_4 \to S_3$ be the natural surjection. Then every homomorphism $\mathrm{Gal} \left( \overline{\mathbb{Q}} \Big/ \mathbb{Q} \right) \to S_4$ can be composed with $p$ to yield a homomorphism $\psi : \mathrm{Gal} \left( \overline{\mathbb{Q}} \Big/ \mathbb{Q} \right) \to S_3$. By combining arguments from [6] and [4], we obtain the following sharpening of Theorem 10 of [6]:

Proposition 2. Let $k$ be 0 or 1, and $N$ a positive squarefree integer. Let $\psi : \mathrm{Gal} \left( \overline{\mathbb{Q}} \Big/ \mathbb{Q} \right) \to S_3$ be a homomorphism, and let $n_{\psi \mathrm{oct}}(N)$ be the number of octahedral weight $k$ cusp forms associated to Galois representations $\rho$ such that $p \circ \mathbb{P}\rho = \psi$.

Then $n_{\psi \mathrm{oct}}(N) \ll \epsilon N^{2/3 + \epsilon}$.

Proof. Let $\chi$ be a character of level $N$. In the proof of [6, Thm. 10], Wong constructs an amplifier—that is, a set of complex numbers $\{c_n\}_{n \in \mathbb{N}}$ such that, for some absolute constants $C$ and $C'$,

- $\sum_{n \leq B} |c_n| \leq C B^{1/4}$.
- $\sum_{n \leq B} |c_n|^2 \leq C B^{1/4}$.
- If $\rho$ is an octahedral Galois representation such that $p \circ \mathbb{P}\rho = \psi$, and $f$ is a cusp form in $\mathcal{S}_{1/4,k}(N,\chi)$ associated to $\rho$, then $|\sum_{n \leq B} c_n \lambda_f(n)| \geq C' B^{1/4} / \log B$.

In [4, §3], Michel and Venkatesh use the Petersson-Kuznetzov formula and standard bounds on Kloosterman sums to obtain the following inequality:

$$\sum_{f \in \Sigma} (f,f)^{-1} \sum_{\substack{n \leq B \\mid (n,N) = 1}} c_n \lambda_f(n)^2 \ll \epsilon \sum_{\substack{n \leq B \\mid (n,N) = 1}} |c_n|^2 + (BN)^\epsilon B^{1/2} N^{-1} \left( \sum_{\substack{n \leq B \\mid (n,N) = 1}} |c_n| \right)^2$$  \hspace{1cm} (1)

where

$$\sum_{\substack{n \leq B \\mid (n,N) = 1}} |c_n|^2 + (BN)^\epsilon B^{1/2} N^{-1} \left( \sum_{\substack{n \leq B \\mid (n,N) = 1}} |c_n| \right)^2$$
• Σ is a set of eigenforms in $S_{1/4,k}^{\text{Artin}}(N,\chi)$;
• $(f,f)$ is the Petersson self-product of $f$;
• $\{c_n\}$ is an arbitrary sequence of complex numbers.

Speaking loosely, the idea of [4] and [3] is that, by the Petersson-Kuznetzov formula, the vectors $\{\lambda_f(n)\}_{f \in \Sigma}$ and $\{\lambda_f(m)\}_{f \in \Sigma}$ are “approximately orthogonal” when $m$ and $n$ are distinct integers. On the other hand, Wong shows that there exists a large set of $n$ such that the Fourier coefficients $\lambda_f(n)$ are real numbers of a fixed sign as $f$ ranges over $\Sigma$. The desired bound on $\Sigma$ will follow from the tension between these two constraints.

In the above inequality, take $\{c_n\}$ to be Wong’s amplifier and $\Sigma$ to be the set of octahedral forms in $S_{1/4,k}^{\text{Artin}}(N,\chi)$.

Note that $(f,f) = O(N \log^3 N)$ by [6, Lemma 6]. So the left hand side of (1) is bounded below by a constant multiple of

$$N^{-1} \log^{-3} N \sum_{f \in \Sigma} \left| \sum_{n \leq B} c_n \lambda_f(n) \right|^2 \geq N^{-1} \log^{-3}(N) |\Sigma| (C')^2 B^{1/2} \log^{-2}(B)$$

while the right hand side is bounded above by a constant multiple of

$$B^{1/4} + (BN)^\epsilon B^{1/2} N^{-1} B^{1/2}.$$ 

Combining these bounds, one has

$$|\Sigma| \ll \epsilon N \log^3(N) B^{-1/2} \log^2(B) (B^{1/4} + (BN)^\epsilon BN^{-1})$$

The bound is optimized when we take $B \sim N^{4/3}$, which yields

$$|\Sigma| \ll \epsilon N^{2/3+\epsilon}.$$ 

Combined with the fact that the number of $\chi$ under consideration is $O(N^\epsilon)$, this yields the desired result. \qed

Proposition 2, in combination with the theorem of Davenport and Heilbronn on cubic fields, allows us to improve Michel and Venkatesh’s bound on $n^\text{Oct}(N)$ in the average.

**Theorem 3.** For all $\epsilon > 0$ there exists a constant $C_\epsilon$ such that

$$(1/X) \sum_{n < X \atop N_{\text{sq.free}}} n^\text{Oct}(N) < C_\epsilon N^{2/3+\epsilon}$$

for all $X > 1$.  

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Proof. Let \( f \) be an octahedral form of level \( N \) associated to a representation \( \rho \). For each prime \( p | N \), the group \( \mathbb{P} \rho (I_p) \) is a cyclic subgroup of \( S_4 \). (Recall that \( N \) is squarefree.) Define

- \( N_1 \) to be the product of primes \( p \) such that \( \mathbb{P} \rho (I_p) \) is a nontrivial subgroup of the Klein four-group;
- \( N_2 \) to be the product of primes \( p \) such that \( \mathbb{P} \rho (I_p) \) is contained in \( A_4 \) but not in the Klein four-group;
- \( N_3 \) to be the product of primes \( p \) such that \( \mathbb{P} \rho (I_p) \) is not contained in \( A_4 \).

Then \( N_1 N_2 N_3 = N \). Let \( \psi : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to S_3 \) be the composition \( p \circ \mathbb{P} \rho \). Then the fixed field \( L \) of \( \ker \psi \) is a cyclic 3-cover of a quadratic number field \( K \), where \( K/\mathbb{Q} \) is the unique quadratic field ramified precisely at primes dividing \( N_3 \), and \( L/K \) is ramified only at primes dividing \( N_2 \).

Let \( b(N_2, N_3) \) be the number of such \( S_3 \)-extensions \( L \). (For notational convenience we take \( b(N_2, N_3) = 0 \) when either \( N_2 \) or \( N_3 \) is not square-free.) From Proposition 2, we have

\[
n_{\text{oct}}(N) \ll \epsilon \sum_{N_1 N_2 N_3 = N} b(N_2, N_3) N^{2/3+\epsilon}. \tag{2}
\]

Let \( T \) be the set of places of \( K \) dividing \( 3N_2 \infty \), and let \( G_T(K) \) be the Galois group of the maximal extension of \( K \) unramified away from \( T \). Each cubic field counted in \( b(N_2, N_3) \) is a cyclic 3-extension of \( K \) unramified away from \( T \), so

\[
b(N_2, N_3) \leq |\text{Hom}(G_T(K), \mathbb{Z}/3\mathbb{Z})|.
\]

The Galois cohomology group above fits in an exact sequence

\[
0 \to \text{Hom}(\text{Cl}_T(K), \mathbb{Z}/3\mathbb{Z}) \to \text{Hom}(G_T(K), \mathbb{Z}/3\mathbb{Z}) \to \prod_{v \in T} \text{Hom}(\text{Gal}(\bar{K}_v/K_v), \mathbb{Z}/3\mathbb{Z})
\]

where \( \text{Cl}_T(K) \) is the quotient of the class group of \( K \) by all primes in \( T \). (See [5, (8.6.3)]). Let \( h_3(N_3) \) be the order of the 3-torsion subgroup of the class group of \( K \). Since \( \dim_{\mathbb{F}_3} \text{Hom}(\text{Gal}(\bar{K}_v/K_v), \mathbb{Z}/3\mathbb{Z}) \) is at most 4 (see [5, (7.3.9)]), we have

\[
b(N_2, N_3) \leq h_3(N_3) 3^{4|T|} \ll \epsilon N_2^{4|T|} h_3(N_3). \tag{3}
\]

Combining (3) and (2) yields

\[
n_{\text{oct}}(N) \ll \epsilon \sum_{N_1 N_2 N_3 = N} h_3(N_3) N^{2/3+\epsilon}. \]


Since the sums over $N_1$ and $N_2$ have length at most $d(N) = O(N^\epsilon)$, we have

$$n^\text{oct}(N) \ll \sum_{N_3 | N} h_3(N_3) N^{2/3 + \epsilon}.$$ 

So

$$\sum_{N < X \text{ sq.free}} n^\text{oct}(N) \ll \sum_{N_3 = 0}^{X} h_3(N_3) \sum_{k=0}^{X/N_3} (kN_3)^{2/3 + \epsilon} \leq X^{5/3 + \epsilon} \sum_{N_3 = 0}^{X} h_3(N_3)(1/N_3). \quad (4)$$

The sum $\sum_{d=0}^{X} h_3(d)/d$ can be estimated as follows. Integration by parts yields

$$\sum_{d=0}^{X} h_3(d)/d = \left(1/X\right) \sum_{d=0}^{X} h_3(d) + \int_{1}^{X} (\sum_{d=1}^{t} h_3(d)) t^{-2} dt.$$ 

Now by the theorem of Davenport and Heilbronn [2, Theorem 3] we have $\sum_{d=0}^{t} h_3(d) = O(t)$. It follows that

$$\sum_{d=0}^{X} h_3(d)/d = O(\log X).$$

Substituting this bound into (4) gives

$$\sum_{N < X \text{ sq.free}} n^\text{oct}(N) \ll \epsilon X^{5/3 + \epsilon}$$

which yields the desired result. \qed

Theorem 3 can be thought of as a bound for the number of quartic extensions of $\mathbb{Q}$ whose Artin conductor, with respect to a certain 2-dimensional projective representation of $S_4$, is bounded by $X$. This is quite different from the problem, recently solved by Bhargava [1], of counting the number of quartic extensions of $\mathbb{Q}$ with discriminant less than $X$. For instance, quartic extensions attached to cusp forms of conductor $N$ might have discriminant as large as $N^3$.

References

