Solutions.

1. Let \( \Omega \) be the half plane \( \{ x_2 > 0 \} \) in \( \mathbb{R}^2 \). Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be harmonic:

\[
\Delta u = 0 \quad \text{in} \quad \Omega.
\]

Under the additional assumption that \( u(\cdot) \) is bounded from above in \( \Omega \), prove that

\[
\sup_{\Omega} u = \sup_{\partial \Omega} u.
\]

Note: The additional assumption is needed to exclude examples like \( u(x_1, x_2) = x_2 \).

Hint: Take for \( \epsilon > 0 \) the harmonic in \( \Omega \) function

\[
v(x_1, x_2) = u(x_1, x_2) - \epsilon \log \sqrt{x_1^2 + (x_2 + 1)^2}.
\]

Apply the maximum principle in an appropriate bounded region. Let \( \epsilon \to 0 \).

Solution

Denote \( \mathbb{R}^2_+ := \{ x_2 > 0 \} \), \( B_r^+ := B(0, r) \cap \mathbb{R}^2_+ \). Denote \( w(x, t) = \log \sqrt{x_1^2 + (x_2 + 1)^2} \), where \( x = (x_1, x_2) \). Then \( w \in C^\infty(\mathbb{R}^2_+) \), and, by explicit calculation, \( \Delta w = 0 \). Moreover, \( w > 0 \) in \( \mathbb{R}^2_+ \), and \( w(x) \to \infty \) as \( |x| \to \infty \) in \( \mathbb{R}^2_+ \). Also \( w(0) = 0 \).

Note that \( v = v_\epsilon = u - \epsilon w \). Thus we get \( \Delta v_\epsilon = 0 \) in \( \mathbb{R}^2_+ \). Then, for any \( r > 0 \), we apply the maximum principle in \( B_r^+ \) and get

\[
\max_{B_r^+} v_\epsilon = \max_{\partial B_r^+} v_\epsilon.
\]

Note that \( \partial B_r^+ = \Gamma_r^0 \cup \Gamma_r^1 \) where \( \Gamma_r^0 = B(0, r) \cap \{ x_2 = 0 \} \), and \( \Gamma_r^1 = \partial B(0, r) \cap \mathbb{R}^2_+ \).

Since \( u \) is bounded from above in \( \mathbb{R}^2_+ \), say \( \sup_{\mathbb{R}^2_+} u = M \), it follows that for any \( \epsilon > 0 \), there exists \( R(\epsilon) \) such that

\[
v_\epsilon < -M \quad \text{on} \quad \Gamma_r^1 \quad \text{for all} \quad r > R(\epsilon).
\]

On the other hand, \( w(0) = 0 \), thus \( v_\epsilon(0) = u(0) \geq -M \), and \( 0 \in \Gamma_r^0 \). Thus,

\[
\max_{\partial B_r^+} v_\epsilon = \max_{\Gamma_r^0} v_\epsilon \quad \text{for all} \quad r > R(\epsilon),
\]

which implies

\[
\max_{B_r^+} v_\epsilon = \max_{\Gamma_r^0} v_\epsilon \quad \text{for all} \quad r > R(\epsilon).
\]

Thus for each \( \epsilon > 0 \), we get

\[
\sup_{\mathbb{R}^2_+} v_\epsilon = \sup_{\partial \mathbb{R}^2_+} v_\epsilon. \tag{1}
\]

Since \( v(x_1, x_2) = u(x_1, x_2) - \epsilon w(x_1, x_2) \leq u(x_1, x_2) \) for each \( (x_1, x_2) \in \mathbb{R}^2_+ \), we get from (1)

\[
\sup_{\mathbb{R}^2_+} v_\epsilon \leq \sup_{\partial \mathbb{R}^2_+} u.
\]
Thus for each \((x_1, x_2) \in \mathbb{R}_+^2\)
\[ u(x_1, x_2) - \varepsilon w(x_1, x_2) \leq \sup_{\partial \mathbb{R}_+^2} u, \]
and, sending \(\varepsilon \to 0+\), we get
\[ u(x_1, x_2) \leq \sup_{\partial \mathbb{R}_+^2} u. \]
Thus
\[ \sup_{\mathbb{R}_+^2} u \leq \sup_{\partial \mathbb{R}_+^2} u \]
and since the opposite inequality is obviously true (since \(u\) is continuous in \(\mathbb{R}_+^2\)), we get
\[ \sup_{\mathbb{R}_+^2} u = \sup_{\partial \mathbb{R}_+^2} u. \]

2. Let \(U \subset \mathbb{R}^n\) be open and bounded, let \(U_T = U \times (0, T]\) be the parabolic cylinder, and \(\Gamma_T = \overline{U_T} \setminus U_T\) be the parabolic boundary. Let nonnegative \(u \in C^2(U_T) \cap C(\overline{U_T})\) satisfies
\[ u_t = \Delta u + cu \quad \text{in} \quad U_T, \]
where \(c(x, t)\) is continuous in \(\overline{U_T}\).

(a) Prove that if \(c(x, t) < 0\) in \(\overline{U_T}\), then
\[ \max_{\overline{U_T}} u = \max_{\Gamma_T} u. \]

\textit{Hint.} Show that \(\max_{\overline{U_T}} u\) cannot be assumed in \(U_T\) unless \(\max_{\overline{U_T}} u \leq 0\). In order to do that, consider the possible signs of \(u_t\) and \(\Delta u\) at a point of maximum \((x_0, t_0) \in U_T\). Note that it is possible that \(t_0 = T\).

(b) Show that more generally
\[ \max_{\overline{U_T}} u \leq e^{CT} \max_{\Gamma_T} u, \]
where
\[ C = \max_{\overline{U_T}}(0, \max_{\overline{U_T}} c) \]

\textit{Hint.} Substitute \(u(x, t) = e^{\gamma t} v(x, t)\), where \(\gamma > C\).

\textbf{Solution}

(a) We show (as hint suggests) that \(\max_{\overline{U_T}} u\) cannot be achieved in \(U_T\) unless \(\max_{\overline{U_T}} u \leq 0\). This proves assertion (a) since \(u \geq 0\) in \(U_T\), i.e. \(\max_{\Gamma_T} u \geq 0\).

Suppose \(\max_{\overline{U_T}} u\) is achieved at \((x_0, t_0) \in U_T = U \times (0, T]\). Then taking into account that \(t_0 = T\) is possible, we get \(u_t(x_0, t_0) \geq 0\). Also, since \(u(\cdot, t_0)\) achieves its maximum over \(U\) at \(x_0 \in U\), it follows that \(D^2u(x_0, t_0)\) is a nonpositive-definite matrix, thus \(\Delta u(x_0, t_0) \leq 0\). Thus we get at \((x_0, t_0)\)
\[ cu = \partial_t u - \Delta u \geq 0. \]
From this and since $c < 0$ in $U_T$, get $u(x_0, t_0) \leq 0$, as claimed.

(b) Let $u(x, t) = e^{\gamma t}v(x, t)$. Then $u_t = e^{\gamma t}(v_t + \gamma v)$, so
\[ 0 = u_t - \Delta u - cu = e^{\gamma t}[v_t - \Delta v - (c - \gamma)v]. \]
Choose $\gamma > \max(0, \max_{U_T} c)$, where we use continuity of $c$ in $U_T$. Then $d(x, t) := c(x, t) - \gamma < 0$ in $U_T$, and
\[ v_t = \Delta v + dv \quad \text{in} \quad U_T. \]
Thus, using part (a) and the fact that $0 \leq v \leq u \leq e^{\gamma T}v$ in $U_T$ from definition of $v$, get
\[ \max_{U_T} u \leq e^{\gamma T} \max_{U_T} v = e^{\gamma T} \max_{V_T} v \leq e^{\gamma T} \max_{V_T} u. \]
Thus
\[ \max_{U_T} u \leq e^{\gamma T} \max_{V_T} u \]
for any $\gamma > \max(0, \max_{U_T} c)$. Then the inequality also holds for $\gamma = \max(0, \max_{U_T} c)$.

3. (2.5 #17 from Evans book). Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one dimension:
\[ \begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \]
Suppose $g, h$ have compact support. The kinetic energy is $k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t)dx$, and the potential energy is $p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t)dx$. Prove:

(a) $k(t) + p(t)$ is constant in $t$;
(b) $k(t) = p(t)$ at all large enough times $t$.

Solution
We have $g(x), h(x) \equiv 0$ for all $|x| \geq M$. Let
\[ H(x) = \int_{-\infty}^{x} h(s)ds, \]
where the integral is finite for any $x$ since $h$ has compact support. Moreover, $H(x) = 0$ for all $x < -M$, and $H(x) = \text{const} = \int_{-M}^{M} h(s)ds$ for all $x > M$.

By uniqueness theorem $u(x, t)$ is defined by d’Alembert’s formula, which can be written as
\[ u(x, t) = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2}[H(x + t) - H(x - t)] \text{for } (x, t) \in \mathbb{R} \times [0, \infty). \]
From the properties of $g, H$, get

$$g(x + t) = g(x - t) = 0 \quad \text{and} \quad H(x + t) = H(x - t) \quad \text{if} \quad |x| > M + t, \quad \text{where} \quad t > 0.$$ 

Thus, $u(x, t) = 0$ if $|x| > M + t$. That is, for any $t > 0$ the function $u(\cdot, t)$ has compact support. Now we compute, integrating by parts with respect to $x$

$$\frac{d}{dt}[k(t) + p(t)] = \int_{-\infty}^{\infty} u_t(x, t)u_{tt}(x, t) + u_x(x, t)u_{xt}(x, t) \, dx$$

$$= \int_{-\infty}^{\infty} u_t(x, t)u_{tt}(x, t) - u_t(x, t)u_{xx}(x, t) \, dx = 0$$

since $u_{tt} = u_{xx}$. This implies assertion (a).

(b) Differentiating d’Alembert’s formula, get

$$u_x(x, t) = \frac{1}{2}[g'(x + t) + g'(x - t)] + \frac{1}{2}[h(x + t) - h(x - t)]$$

$$u_t(x, t) = \frac{1}{2}[g'(x + t) - g'(x - t)] + \frac{1}{2}[h(x + t) + h(x - t)].$$

Thus,

$$p(t) - k(t) = \frac{1}{2} \int_{-\infty}^{\infty} [u^2_r(x, t) - u^2_t(x, t)] \, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} [g'(x + t) + h(x + t)][g'(x - t) - h(x - t)] \, dx.$$ 

Now, if $t > M$, we have $(x + t) - (x - t) = 2t > 2M$, so either $|x + t| > M$ or $|x - t| > M$. Thus the last integral is zero for $t > M$, i.e. $p(t) = k(t)$ for $t > M$.

4. (2.5 #18 from Evans book). Let $u$ solve

$$\begin{cases}
\begin{aligned}
u_{tt} - \Delta u &= 0 & \text{in} \ & \mathbb{R}^3 \times (0, \infty), \\
u &= g, \ & u_t = h & \text{on} \ & \mathbb{R}^3 \times \{t = 0\},
\end{aligned}
\end{cases}$$

where $g, h$ are smooth and have have compact support. Show there exists constant $C$ such that

$$|u(x, t)| \leq C/t \quad (x \in \mathbb{R}^3, \ t > 0).$$

Solution

First we prove that $u(x, t)$ is given by Kirchhoff’s formula:

$$u(x, t) = \int_{\partial B(x, t)} th(y) + g(y) + Dg(y) \cdot (y - x) \, dS(y).$$

This will follow from the uniqueness of solution of the homogeneous wave equation in $\mathbb{R}^3 \times (0, \infty)$ when initial data have compact support:
Let $u(x,t), v(x,t)$ be two smooth solutions of the homogeneous wave equation in $\mathbb{R}^3 \times (0, \infty)$ with the same initial data $f, g$ with compact support. Suppose $R > 0$ is such that $\text{supp } g(\cdot), h(\cdot) \subset B(0, R)$. Then, by Theorem on finite propagation speed, $\text{supp } u(\cdot, t) \subset B(0, R + t)$ for any $t > 0$, and same for $v(\cdot, t)$. Now let $T > 0$, and $U = B(0, R + T)$, and $U_T = U \times (0, T]$, $\Gamma_T = \overline{U_T - U_T} = (\partial U \times [0, T]) \cup (U \times \{t = 0\})$. Since $u = v = 0$ on $\partial U \times [0, T]$ as we discussed above, we get $u(x, t) = v(x, t)$ on $\Gamma_T$. Then uniqueness Theorem in $U_T$ implies that $u = v$ in $U_T$. Since $u = v = 0$ on $(\mathbb{R}^3 \times [0, T]) \setminus U_T$, it follows that $u = v$ in $\mathbb{R}^3 \times [0, T]$. Since $T > 0$ is arbitrary, then $u = v$ in $\mathbb{R}^3 \times (0, \infty)$, thus solution in $\mathbb{R}^3 \times (0, \infty)$ is unique, thus given by Kirchhoff’s formula.

For any $x \in \mathbb{R}^3$, $t > 0$, the area of the part of the sphere $\partial B(x, t)$ which is inside the ball $B(0, R)$ is no greater than the area of the sphere $\partial B(0, R)$. Thus,

$$|\partial B(x, t) \cap B(0, R)| \leq \min(|\partial B(0, R)|, |\partial B(x, t)|) = 4\pi \min(R^2, t^2).$$

Then using Kirchhoff’s formula, and using the fact that $|f, g, Dg| \leq C$ on $\mathbb{R}^3$ for some $C$ since $f, g$ are smooth with compact support, and also noting that $|y - x| = t$ in the last term of Kirchhoff’s formula, we get:

$$|u(x, t)| \leq \frac{C}{t^2} \int_{\partial B(x, t) \cap B(0, R)} t|h(y)| + |g(y)| + t|Dg(y)| \, dS(y) \leq \frac{Ct + C}{t^2} |\partial B(x, t) \cap B(0, R)| \leq 4\pi \min(R^2, t^2) \frac{Ct + C}{t^2} \leq C_1/t.$$