We say $v \in C^2(U)$ is subharmonic if 

$$-\Delta v \leq 0 \quad \text{in } U.$$ 

(a) Prove for subharmonic $v$ that 

$$v(x) \leq \int_{B(x,r)} v \, dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that $\max_{\partial U} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume $u$ is harmonic, $v := \phi(u)$. Prove $v$ is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic whenever $u$ is harmonic.

Solution

(a) Let 

$$\phi(r) = \int_{\partial B(x,r)} v \, dS(y) = \int_{\partial B(0,1)} v(x+rz) \, dS(z),$$

where $y = x + rz$. The outer unit normal at $y \in \partial B(x,r)$ is $\nu(y) = \frac{y-x}{r}$, thus 

$$\phi'(r) = \int_{\partial B(0,1)} z \cdot Dv(x+rz) \, dS(z) = \int_{\partial B(x,r)} \frac{y-x}{r} \cdot Dv(y) \, dS(y) = \int_{\partial B(x,r)} v(y) \cdot Dv(y) \, dS(y).$$

Since $-\Delta v \leq 0$ in $U$, we have 

$$0 \leq \int_{B(x,r)} \Delta v \, dy = \int_{\partial B(x,r)} v \, dS(y),$$

thus $\phi'(r) \geq 0$ for $r > 0$. Then we get for $r > 0$

$$v(x) = \phi(0) \leq \phi(r) = \int_{\partial B(x,r)} v(y) \, dS(y).$$

Thus, using polar coordinates, we get 

$$\int_{B(x,r)} v \, dy = \frac{1}{\alpha(n)r^n} \int_0^r d\rho \int_{\partial B(x,\rho)} v(y) \, dS(y) \geq \frac{1}{\alpha(n)r^n} \int_0^r n\alpha(n)\rho^{n-1}v(x) \, d\rho = \frac{v(x)}{\alpha(n)r^n} \int_0^r n\rho^{n-1} \, d\rho = v(x),$$

thus (i) is proved.

(b) Using (a), we can repeat the proof of Theorem 4 (maximum principle) in sect. 2.2.3.

(c) We compute: 

$$Dv = \phi'(u)Du,$$

$$\Delta v = \phi'(u)\Delta u + \phi''(u)|Du|^2 = \phi''(u)|Du|^2 \geq 0,$$

where we used harmonicity of $u$ and convexity of $\phi$.

(d) Since $u$ is harmonic, then $u_{x_i}$ is harmonic, for $i = 1, \ldots, n$. Then $(u_{x_i})^2$ is subharmonic by part (c). Sum of subharmonic functions is obviously subharmonic, so $|Du|^2 = \sum_{i=1}^n (u_{x_i})^2$ is subharmonic.