Calculus with Algebra and Trigonometry II
Lecture 1
Tangent lines, increasing and decreasing functions and some useful theorems

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The tangent line

The definition of the derivative of a function \( f(x) \) at a point \( x = a \) is

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

Geometrically it involves picking another point on the graph \((x, f(x))\) and calculating the slope of the line between the two points. The derivative is then the result of finding the limit of this slope as \( x \) moves close to \( a \). As \( x \) tends to \( a \) the line between them becomes closer to the line tangent to the graph at \( a \) i.e. the derivative is the slope of the tangent line at \( a \).

Using the point slope formula for a line through \((a, f(a))\) with slope \( f'(a) \), we obtain the equation of the tangent line as

\[
y = f(a) + f'(a)(x - a)
\]
An example

Find the tangent line to the function \( f(x) = x^3 - 3x^2 + x + 5 \) at the point \( x = 2 \)

Calculating the derivative

\[
f'(x) = 3x^2 - 6x + 1
\]

then

\[
f(2) = 2^3 - 3(2^2) + 2 + 5 = 3 \quad f'(2) = 3(2^2) - 6(2) + 1 = 1
\]

so the tangent line is

\[
y - 3 = 1(x - 2) \quad \Leftrightarrow \quad y = x + 1
\]
Application of the tangent line

The tangent line at a point is frequently used to approximate the function near the point, in other words near the point \( x = a \)

\[
f(x) \approx f(a) + f'(a)(x - a)
\]

For example, use the tangent line to the function \( f(x) = \sqrt{x} \) at \( x = 25 \) to approximate \( \sqrt{27} \).

\[
f(25) = \sqrt{25} = 5 \quad f'(25) = \frac{1}{2}(25)^{-1/2} = \frac{1}{10}
\]

The tangent line is

\[
y = 5 + \frac{1}{10}(x - 25)
\]

so

\[
\sqrt{27} \approx 5 + \frac{1}{10}(x - 25) = 5.2
\]

The actual value of \( \sqrt{27} = 5.196 \ldots \)
The normal line

The normal line to the graph of a function is the line through the point perpendicular to the tangent line (normal is this context is a synonym for perpendicular). Since the product of the slopes of two perpendicular lines is -1 the slope of the normal line to \( f(x) \) at \( a \) is

\[
\text{Slope} = -\frac{1}{f'(a)}
\]

and the equation of the normal line is

\[
y = f(a) - \frac{1}{f'(a)}(x - a)
\]

For example the normal line to \( f(x) = \cos x \) at \( x = \frac{\pi}{4} \) is

\[
y = \frac{\sqrt{2}}{2} + \sqrt{2} \left( x - \frac{\pi}{4} \right)
\]
The Intermediate Value Theorem

If \( f(x) \) is a continuous function on the interval \( a \leq x \leq b \) then for any value, \( y \), between \( f(a) \) and \( f(b) \) the exists a \( c \) in the interval i.e \( a \leq c \leq b \)

For example to estimate \( \tan^{-1}(0.8) \). First note

\[
\tan \left( \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}} = 0.577 \ldots \quad \tan \left( \frac{\pi}{4} \right) = 1
\]

Since \( 0.577 < 0.8 < 1 \) the IVT implies there is a \( c \) so that

\[
\tan c = 0.8 \quad c = \tan^{-1}(0, 8)
\]

where

\[
\frac{\pi}{6} = 0.524 \ldots < c < \frac{\pi}{4} = 0.785 \ldots
\]
Finding roots using the IVT

An application of this theorem is in the finding of roots of equations i.e finding values of $x$ that satisfy $f(x)$. If you can find two values $a$ and $b$ so that $f(a)$ and $f(b)$ have opposite signs there must be a root between $a$ and $b$

Find the roots of $f(x) = x^3 - 4x + 2$

$\begin{align*}
    f(-3) &= -13 \\
    f(-2) &= 2 \\
    f(0) &= 2 \\
    f(1) &= -1 \\
    f(2) &= 2
\end{align*}$

so the equations has three roots

$-3 \leq x_1 \leq -2 \quad 0 \leq x_2 \leq 1 \quad 1 \leq x_3 \leq 2$
Rolle’s Theorem

If $f(x)$ is differentiable on a closed interval $[a, b]$ and $f(a) = f(b) = 0$ then there is a $c \in [a, b]$ so that $f'(c) = 0$.

The idea behind the theorem is that either $f(x) = 0$ for all $x \in [a, b]$ or the graph must leave the $x$ axis. Suppose it has a positive slope, then since the graph must return to the $x$ axis its slope must become negative. Invoking the IVT implies there must be a point with zero slope.
An Example

The function

\[ f(x) = x \sin x \quad 0 \leq x \leq \pi \]

satisfies the conditions for Rolle’s theorem so there must be a \( c \) satisfying

\[ \sin c + c \cos c = 0 \]

The point is graphed below:

![Graph of the function showing a point satisfying Rolle's theorem.](image.png)
The Mean Value theorem

If $f(x)$ is differentiable on a closed interval $[a, b]$ then there exists $c \in [a, b]$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically the theorem states that there is a point in the interval, with the tangent line to the curve at that point parallel to the line joining $(a, f(a))$ and $(b, f(b))$
Increasing and decreasing functions

A function is increasing on an interval \((a, b)\) if for \(c, d \in (a, b)\)

\[
c < d \implies f(c) < f(d)
\]

A function is decreasing on an interval \((a, b)\) if for \(c, d \in (a, b)\)

\[
c < d \implies f(c) > f(d)
\]

In calculus terms a function is increasing at \(c\) if \(f'(c) > 0\).

and a function is decreasing at \(c\) if \(f'(c) < 0\).

We can use these criteria to determine the shape of a graph of a function
An Example

Consider the function

\[ f(x) = x^4 - 8x^2 + 15 = (x^2 - 3)(x^2 - 5) \]

It has zeros at \( \pm \sqrt{3}, \pm \sqrt{5} \).

the derivative is

\[ f'(x) = 4x^3 - 16x = 4x(x - 2)(x + 2) \]

then \( f'(x) = 0 \) at \( x = -2, 0, 2 \) and by the IVT it cannot change sign between the zero. To determine the sign in a given interval just pick a value in the interval and calculate \( f' \). A number line for \( f' \) is shown below.
A graph of the function is shown below

Note that the points with $f' = 0$ correspond to local maxima and minima of the function. You can tell them apart by looking at the slopes near the point in question. If the slope is positive, zero, then negative, the point is a maximum. If on the other hand the slope is negative, zero, then positive, the point is a minimum.