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Normalization of complex-valued planar vector fields which degenerate along a real curve ☆

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Abstract

Taking as a start point the recent article of Meziani [7], we present several results concerning the normalization of a class of complex vector fields in the plane which degenerate along a real curve. We mainly deal with operators with finite regularity and analyze both the local situation as well as the case of normalization near a circle. Some related questions (e.g., on semi-global solvability and on the normalization of a class of generalized Mizohata operators) are also discussed.

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1. Introduction

Consider a complex-valued vector field with no singularities defined on the real plane given by

$$\mathscr{L} = a(x,t)\frac{\partial}{\partial t} + b(x,t)\frac{\partial}{\partial x},$$

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where a, b are $\mathbb{C}$-valued $\mathscr{C}^s$ functions ($0 < s \leqslant \infty$, or $s = \omega$). It is a classical result that if $\mathscr{L}$ and $\overline{\mathscr{L}}$ are $\mathbb{C}$-linearly independent, then $\mathscr{L}$ is locally equivalent to a multiple of $\frac{\partial}{\partial \bar{z}}$ by a $\mathscr{C}^{s'}$ diffeomorphism, where s' is any non-integer $\leqslant s+1$ (see [16, p. 93]). In this paper we shall consider vector fields $\mathscr{L}$ which are tangent to a real curve γ of constant finite order $m < s$, while, off the real curve, $\mathscr{L}$ and $\overline{\mathscr{L}}$ are $\mathbb{C}$-linearly independent. In suitable local coordinates, such a vector field takes the form

$$\frac{\partial}{\partial t} - ix^m a(x,t)\frac{\partial}{\partial x}, \quad \Re a(0) \neq 0, \tag{1.1}$$

where m is a positive integer and $(x,t) \mapsto x^m a(x,t) \in \mathscr{C}^s$. We have

Theorem 1.1. *Let* $\mathscr{L}$ *be a* $\mathscr{C}^s$ *vector field given by* (1.1) *with* s *a real number and* m *a positive integer satisfying* $s \geqslant m^2 + 2m$. *Then there is a* $\mathscr{C}^{[s/m]-m-1+\varepsilon}$ *diffeomorphism for all* $0 < \varepsilon < 1$, *defined near the origin and sending* $\mathscr{L}$ *into a multiple of* $\frac{\partial}{\partial t} - ix^m \frac{\partial}{\partial x}$.

When (1.1) is real-analytic ($s = \omega$) then Theorem 1.1 is an easy consequence of the Cauchy–Kowalevsky theorem: the diffeomorphism can be obtained in the $\mathscr{C}^\omega$ class. In the $\mathscr{C}^\infty$ case then Theorem 1.1 also follows easily from general results concerning the local solvability of a single vector field satisfying the Nirenberg–Treves condition (P). The proof of Theorem 1.1, for the finite smoothness case, is motivated by recent results of Meziani [7]. We should also make clear that the solvability of vector fields, in general terms, does not have immediate consequences to the problem of normalizing the corresponding vector fields. In fact the generalized Mizohata operators

$$\frac{\partial}{\partial t} - it^m a(x,t)\frac{\partial}{\partial x}, \quad \Re a(0) \neq 0 \tag{1.2}$$

with $a \in \mathscr{C}^\infty$, are always $\mathscr{C}^\infty$-solvable if m is even. Moreover, there exist $\mathscr{C}^\infty$ coordinates on which the vector field becomes a multiple of $\frac{\partial}{\partial t} - i(t^m + O(|t|^\infty))\frac{\partial}{\partial x}$ [6, Chapter 2]. However, we have

Theorem 1.2. *For each even integer* m, *there exists a* $\mathscr{C}^\infty$ *vector field* (1.2) *that cannot be transformed into a multiple of* $\frac{\partial}{\partial t} - it^m \frac{\partial}{\partial x}$ *by any* $\mathscr{C}^\infty$ *diffeomorphism.*

Nirenberg [9, p. 8] showed the existence of a $\mathscr{C}^\infty$ vector field (1.2) with $m = 1$ that is not equivalent to a multiple of $\frac{\partial}{\partial t} - it\frac{\partial}{\partial x}$. Such phenomenon was later studied in detail by Treves [14]. Afterwards Sjöstrand [12] gave a complete classification for the standard $\mathscr{C}^\infty$ Mizohata structure ($m = 1$) in the plane (see also [15, Section VII.3]).

We shall also be interested in complex-valued vector fields $\mathscr{L}$ in the real plane which are tangent to a closed real curve γ to the first order, while off the real curve $\mathscr{L}$ and $\overline{\mathscr{L}}$ are $\mathbb{C}$-linearly independent. By applying a diffeomorphism, which is defined near the real curve and sends it to the unit circle $r \equiv |z| - 1 = 0$ and by parametrizing the unit circle by $z = e^{i\theta}$, we arrive at the vector fields

$$\mathscr{L} = \frac{\partial}{\partial\theta} - ira(r,\theta)\frac{\partial}{\partial r}, \quad \Re a(0,\theta) > 0. \tag{1.3}$$

Meziani proved that

$$\lambda \doteq \frac{1}{2\pi}\int_0^{2\pi} a(0,\theta)\, d\theta \tag{1.4}$$

is an invariant attached to $\mathscr{L}$ and also that when $\lambda \in \mathbb{C}\backslash\mathbb{R}$ then $\mathscr{L}$ can be transformed, by a finitely smooth diffeomorphism defined near the circle, into a multiple of the vector field

$$\mathscr{L}_\lambda = \frac{\partial}{\partial\theta} - i\lambda r\frac{\partial}{\partial r}.$$

The case when $\lambda \in \mathbb{C}\backslash\{0\}$ for (1.3) will be treated in the present article:

Theorem 1.3. *Let* $\mathscr{L}$ *be a* $\mathscr{C}^\infty$ *vector field* (1.3) *with* λ *as in* (1.4). *If* $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ *then for each finite k there exists a* $\mathscr{C}^k$ *diffeomorphism, defined near the unit circle, sending* $\mathscr{L}$ *into a multiple of* $\mathscr{L}_\lambda$. *On the other hand, if* $\lambda \in \mathbb{C}\backslash\{0\}$ *there exists a* $\mathscr{C}^\infty$ *vector field* (1.3) *that is not equivalent to* $\mathscr{L}_\lambda$ *by any* $\mathscr{C}^\infty$ *diffeomorphism.*

For an irrational λ, define

$$\omega_n(\lambda) = \min\{|k\lambda - l|; k, l \in \mathbb{Z}, 0 < |k| < 2^n\}.$$

One says that λ satisfies the *Bruno condition*, if $-\sum 2^{-n}\log\omega_n(\lambda) < \infty$.

Theorem 1.4. *Let* $\mathscr{L}$ *be a real analytic vector field given by* (1.3). *If* λ *is irrational and satisfies the Bruno condition, there exists a real analytic diffeomorphism, defined near the unit circle, sending* $\mathscr{L}$ *into a multiple of* $\mathscr{L}_\lambda$. *Conversely, if* λ *is either rational, or is irrational and violates the Bruno condition, there exists a real analytic vector field of the form* (1.3) *that is not equivalent to a multiple of* $\mathscr{L}_\lambda$ *by any real analytic diffeomorphism defined near* $r = 0$.

In the proof of normalization due to Meziani [7] the holomorphic holonomy of the truncated vector fields (1.3) played a certain role. Our approach is based on a preliminary normalization of (1.3) up to any finite order in the r variable. However, the holonomy transformation, used by Meziani [7], turns out to be crucial to the real analytic classification. The proof of Theorem 1.3 is based on identifying the classification of germs of holomorphic functions $f(z) = e^{2\pi\lambda i}z + O(|z|^2)$ with that of analytic vector fields (1.3) (cf. Lemma 3.1).

The paper is organized as follows.

In Section 2 we shall normalize vector fields (1.1) and prove Theorem 1.1. We shall see how suitable first integrals of (1.1) can be used to normalize the vector fields (see

Lemma 2.2). In the $\mathscr{C}^\infty$ case it is known that one can construct first integrals from $\mathscr{C}^\infty$ solvability. In the case of finitely smooth vector fields (2.2) solvability is not available and then we construct first integrals of (1.1) through Beltrami vector fields, as in case of Meziani [7]. The normalization of $\mathscr{C}^\infty$ vector fields (1.2) (for m even), including the proof of Theorem 1.2, is also given in Section 2. When m is even the vector fields (1.2) are indeed hypo-elliptic and this property turns out to be crucial to the normalization of (1.2). Hypo-ellipticity gives us a non-trivial first integral that is unique to a certain degree and we then completely classify the vector fields (1.2) by associating a real curve to the non-trivial first integrals (see Theorem 2.4).

In Section 3 we shall normalize analytic vector fields (1.3) along the circle on which they degenerate. Section 3 contains the proof of Theorem 1.4.

In Section 4 we shall normalize finitely smooth vector fields (1.3) along the circle. We shall first obtain a result analogous to Theorem 1.1, which is formulated in Theorem 4.4. Theorem 1.3 follows from Theorems 4.4 and 4.5. The latter are proved in Section 4.

2. Local normalizations

Let L be given by

$$L = a(x,t)\frac{\partial}{\partial t} + b(x,t)\frac{\partial}{\partial x},$$

where a, b are $\mathbb{C}$-valued $\mathscr{C}^s$ function defined near the origin. Recall that two such vector fields L and $\tilde{L}$ are *equivalent* up to a multiple under $\mathscr{C}^k$ diffeomorphisms if there is a $\mathscr{C}^k$ diffeomorphism φ, defined near the origin with $\varphi(0) = 0$, such that $\varphi_* L = \mu\tilde{L}$ for some non-vanishing complex-valued function μ. We assume that L does not vanish at the origin. Interchanging x with t if necessary, one may assume that $a(0) \neq 0$. Thus L is equivalent to

$$\frac{\partial}{\partial t} + ib_1(x,t)\frac{\partial}{\partial x}.$$

By a $\mathscr{C}^{s+1}$ diffeomorphism of the form $(x,t) \to (X(x,t), t)$, $\Re L$ is equivalent to $\frac{\partial}{\partial t}$. Thus we may assume that b_1 is real valued. Denote by γ the set of points (x,t) at which L and $\bar{L}$ are linearly dependent over $\mathbb{C}$. Obviously, γ is the zero set of b_1. We assume that γ is a $\mathscr{C}^1$ real curve passing through the origin. Let γ be defined by $p(x,t) = 0$ with $dp \neq 0$. Assume further that L is tangent to γ. Then $\partial p/\partial t$ vanishes on γ, while $\partial p/\partial x \neq 0$; in particular, p is a multiple of the function $(x,t) \mapsto -x + p_0(t)$ with $p_0 \in \mathscr{C}^1$. One readily sees that p_0 must vanish identically and thus γ is given by $x = 0$; in particular, γ is actually $\mathscr{C}^s$. At a point $(x_0, t_0) \in \gamma$, we define the order of tangency of L to the real curve γ to be the maximum integer m such that $Lp(x,t) = o((|x - x_0| + |t - t_0|)^{m-1})$ holds for some $\mathscr{C}^s$ defining function p of γ ($dp \neq 0$). Finally, we assume that L is tangent to γ at each point of γ with a constant finite order $m \leqslant s$.

The above reduction shows that L is equivalent to a multiple of

$$\frac{\partial}{\partial t} - ix^m a(x,t)\frac{\partial}{\partial x}, \tag{2.1}$$

where $m < s$ is a positive integer, and $a(x,t)$ is a non-vanishing real-valued function with $(x,t) \mapsto x^m a(x,t) \in \mathscr{C}^s$.

Lemma 2.1. *Let L be a $\mathscr{C}^s$ vector field given by* (2.1). *There exists a $\mathscr{C}^{s+1-m}$ diffeomorphism sending L into a multiple of*

$$\frac{\partial}{\partial t} - ix^m(1+b(x,t))\frac{\partial}{\partial x} \tag{2.2}$$

with $b \in \mathscr{C}^{s-m}$, real-valued and $b(x,t) = o(|x|^{[s]-m})$.

Proof. Since the function $t \mapsto a(0,t)$ is $\mathscr{C}^{s-m}$, the map

$$(x,t) \mapsto (\tilde{x},\tilde{t}) = \left(x, \int_0^t a(0,s)\,ds\right)$$

is a $\mathscr{C}^{s-m+1}$ diffeomorphism which transforms (2.1) into

$$a(0,t(\tilde{t}))\frac{\partial}{\partial \tilde{t}} - i\tilde{x}^m a(\tilde{x},t(\tilde{t}))\frac{\partial}{\partial \tilde{x}} = a(0,t(\tilde{t}))\left\{\frac{\partial}{\partial \tilde{t}} - i\tilde{x}^m \frac{a(\tilde{x},t(\tilde{t}))}{a(0,t(\tilde{t}))}\frac{\partial}{\partial \tilde{x}}\right\}.$$

This allows one to assume from beginning that $a(0,t) = 1$ for all t. Consider then a transformation F of the form

$$x^* = X(x,t) = x\tilde{X}(x,t), \quad t^* = T(x,t)$$

with $X \in \mathscr{C}^{s+1}$, $T \in \mathscr{C}^{s+1-m}$, and

$$\tilde{X}(x,0) = 1, \quad T(x,0) = 0. \tag{2.3}$$

Up to a multiple, F_*L is then of the form

$$\frac{\partial}{\partial t^*} - ix^{*m}(1+b(x^*,t^*))\frac{\partial}{\partial x^*}$$

with $b \in \mathscr{C}^{s-m}$. Explicitly, we have

$$x^m a(x,t)X_x + iX_t = (1+b(X,T))X^m(T_t - ix^m a(x,t)T_x).$$

In order then to obtain $b(x^*,t^*) = o(|x^*|^{[s]-m})$ it suffices to show that is possible to achieve

$$x^m a(x,t)X_x + iX_t = X^m(T_t - ix^m a(x,t)T_x) + o(|x^*|^{[s]}). \tag{2.4}$$

Recall that for a $\mathscr{C}^s$ function $f(x,t)$, we have the Taylor formula

$$f(x,t) = \sum_{j=0}^{[s]} f_j(t)x^j + f_{[s]}(x,t), \quad f_{[s]}(x,t) = o(|x|^{[s]}), \tag{2.5}$$

in which $f_j \in \mathscr{C}^{s-j}$. Note also that we actually have $f_{[s]}(x,t) = O(|x|^s)$ if s is not an integer. Conversely, the Whitney extension says that if $f_j(t) \in \mathscr{C}^{s-j}$ for $j = 0, \ldots, [s]$, there exists a $\mathscr{C}^s$ function $f(x,t)$ of the form (2.5). (See Lemma 4.1 also.)

Let $\tilde{X}_n(t), T_n(t)$ and $b_n(t)$ be the coefficients of Taylor expansion of $\tilde{X}, T$ and b in the variable x, respectively. Now (2.3) reads

$$\tilde{X}_0(t) = 1, \quad \tilde{X}_j(0) = 0, \quad T_{j-1}(0) = 0, \quad j > 0. \tag{2.6}$$

Expand both sides of (2.4) as formal power series in x. Comparing the coefficients of x^j for $j \leqslant m$ on both sides of (2.4) yields (recall that $a(0,t) = 1$)

$$\tilde{X}'_j(t) = 0 \ (1 \leqslant j < m), \quad T'_0(t) = 1.$$

With (2.6), we obtain

$$\tilde{X}_j(t) = 0 \ (1 \leqslant j < m), \quad T_0(t) = t.$$

Assume for the sake of induction that by comparing coefficients of $x^0, \ldots, x^k$ on both sides of (2.4), we have determined $\tilde{X}_j(t), T_{j+1-m}(t) \in \mathscr{C}^{s-j}$ for $j < k$ and for $k < s$. Comparing coefficients of x^{k+1} of (2.4) yields

$$T'_{k+1-m} - i(k+2-2m)T_{k+2-2m} - i\tilde{X}'_k - (k+2-2m)\tilde{X}_{k+1-m} = b_{k+1-m} + \ldots,$$

in which the omitted terms form a polynomial of constant coefficients in $\tilde{X}_j$ with $j < k+1-m$, T_j with $j < k+2-2m$, T'_j with $j < k+1-m$, and b_j with $j < k+1-m$. Note that $b_j \in \mathscr{C}^{s-m-j}$.

We first consider the case $m \geqslant 2$. We have

$$T'_{k+1-m} - i\tilde{X}'_k = b_{k+1-m} + \ldots,$$

in which the omitted terms are $\mathscr{C}^{s-k-1}$. Therefore, the real functions $\tilde{X}_k, T_{k+1-m} \in \mathscr{C}^{s-k}$ are uniquely determined by initial values $\tilde{X}_k(0) = T_{k+1-m}(0) = 0$. For $m = 1$, we have

$$T'_k - ikT_k - k\tilde{X}_k - i\tilde{X}'_k = b_k + \ldots,$$

where the omitted terms are $\mathscr{C}^{s-k-1}$. The above is a system of first-order differential equations for $\tilde{X}_k$ and T_k with constant coefficients, and it admits a unique solution $\tilde{X}_k, T_k \in \mathscr{C}^{s-k}$ with $\tilde{X}_k(0) = T_k(0) = 0$.

Summing up, we determine $\tilde{X}_k, T_{k+1-m} \in \mathscr{C}^{s-k}$ through systems of ordinary differential equations of constant coefficients; in particular, $\tilde{X}_k, T_{k+1-m}$ are $\mathscr{C}^{s-k}$ functions defined on a fixed neighborhood of $t = 0$ for all $k \leqslant [s]$. Applying the Whitney extension theorem, there exist real functions $x\tilde{X}(x,t) \in \mathscr{C}^{s+1}$, $T(x,t) \in \mathscr{C}^{s+1-m}$ defined near the origin, of which the formal power series expansions in x have coefficients $x\tilde{X}_k, T_k$, respectively. Now $(x,t) \to (x\tilde{X}(x,t), T(x,t))$ transforms L into the desired form. □

From now on we shall avail ourselves to the vector field given by (2.2). We introduce the "model operator"

$$L^\bullet = \frac{\partial}{\partial t} - ix^m \frac{\partial}{\partial x} \tag{2.7}$$

and the corresponding first integral

$$Z^\bullet(x,t) = \begin{cases} xe^{it} & \text{if } m = 1, \\ x(1 - i(m-1)tx^{m-1})^{-1/(m-1)} & \text{if } m \geqslant 2. \end{cases}$$

This of course means that $L^\bullet Z^\bullet = 0$ and $Z^\bullet_x \neq 0$. Moreover $(\partial Z^\bullet/\partial t) = iZ^{\bullet m}$, from which we obtain

$$x^m(\partial Z^\bullet/\partial x) = Z^{\bullet m}. \tag{2.8}$$

In order to study the reduction of (2.2) to the normal form (2.7) it is convenient first to perform a new change of variables. We can write

$$Z^\bullet(x,t) = xA(x,t) + ix^m tB(x,t), \tag{2.9}$$

where A and B are real-valued, smooth (indeed real-analytic) and do not vanish. We then consider the smooth diffeomorphism $\vartheta : (x,t) \mapsto (X,T)$, where

$$X = xA(x,t), \quad T = tC(x,t), \tag{2.10}$$

and $C = B/A^m$. The relevance of this diffeomorphism is that in these new variables the function $Z^\bullet$ can be written in one single formula

$$Z^\bullet = X + iX^m T. \tag{2.11}$$

Thus we have

$$\vartheta_* L^\bullet = \alpha(X,T) L^{\bullet\bullet}$$

with

$$L^{\bullet\bullet} = \frac{\partial}{\partial T} - i\frac{X^m}{1 + imX^{m-1}T}\frac{\partial}{\partial X}. \tag{2.12}$$

Also $\alpha \neq 0$ is smooth. Hence

$$L^{\sharp} \doteq \alpha^{-1}\vartheta_* L = L^{\bullet\bullet} - ib^{\sharp}(X,T)\vartheta_*\left\{x^m \frac{\partial}{\partial x}\right\}, \tag{2.13}$$

where we have written

$$b^{\sharp}(X,T) = \alpha(X,T)^{-1}(b \circ \vartheta^{-1})(X,T). \tag{2.14}$$

Since $x(X,T) = Xg(X,T)$ with $g \neq 0$ we can finally write

$$L^{\sharp} = L^{\bullet\bullet} - iX^m b^{\sharp}(X,T)\mathscr{X}, \tag{2.15}$$

where $\mathscr{X} \doteq g^m \vartheta_*(\partial/\partial x)$ is a smooth real vector field with no singularities.

Remark. Notice that, thanks to (2.8), we have

$$X^m \mathscr{X} Z^{\bullet} = Z^{\bullet m}.$$

To sum up we can assume from now on that our vector field L is given by

$$L = L_0 - ix^m b(x,t)\mathscr{X}, \quad b(0,0) = 0, \tag{2.2$'$}$$

where $m \in \mathbb{Z}_+$, b belongs to $\mathscr{C}^{s-m}$, and $\mathscr{X}$ is a $\mathscr{C}^\infty$ real vector field with no singularities. Here we are writing

$$L_0 = \frac{\partial}{\partial t} - i\frac{x^m}{1 + imx^{m-1}t}\frac{\partial}{\partial x}. \tag{2.7$'$}$$

If we set

$$Z^{\bullet}(x,t) = x + ix^m t \tag{2.11$'$}$$

then $L_0 Z^{\bullet} = 0$, $Z^{\bullet}_x \neq 0$ and also

$$x^m \mathscr{X} Z^{\bullet} = Z^{\bullet m}. \tag{2.8$'$}$$

Lemma 2.2. *Let $m + 1 \leqslant \sigma \leqslant s - m + 1$ and suppose there is $Z \in \mathscr{C}^\sigma$ such that $LZ = 0$, $Z_x \neq 0$. There is a $\mathscr{C}^{\sigma-m}$ local diffeomorphism ψ near the origin such that $\psi_* L$ is a multiple of L_0.*

Proof. We can assume that $Z(0,0) = 0$. Using the fact that $m + 1 \leqslant \sigma$ we can find constants $c_2, \dots, c_m$ such that

$$Z^{\bigstar} \doteq Z + \sum_{j=2}^{m} c_j Z^j$$

satisfies

$$(\partial_x^j Z^{\bigstar})(0,0) = 0, \quad j = 2, \dots, m. \tag{2.16}$$

If we set $\zeta(x,t) = Z^\star(x,t)/Z^\star_x(0,0)$ then $L\zeta = 0$, $\zeta_x(0,0) = 1$, and furthermore (2.16) holds when $Z^\star$ is replaced by ζ (we set $Z^\star = Z$ when $m = 1$).

The lemma will follow if we can show the existence of a local $\mathscr{C}^{\sigma-m}$ diffeomorphism $(x,t) \mapsto (X(x,t), T(x,t))$ such that

$$\zeta(x,t) = X(x,t) + iX(x,t)^m T(x,t).$$

Thus we must show that

$$X(x,t) = \Re\zeta(x,t), T(x,t) = \frac{\Im\zeta(x,t)}{(\Re\zeta(x,t))^m} \tag{2.17}$$

defines a $\mathscr{C}^{\sigma-m}$ diffeomorphism. To this end it suffices to show that the function T defined in (2.17) is $\mathscr{C}^{\sigma-m}$ and $T_t(0,0) \neq 0$.

Since $\zeta_t(0,t) = 0$ we obtain $\zeta(0,t) = 0$. Thus we can write $\zeta(x,t) = x\tilde{\zeta}(x,t)$ from which we derive $\Re\tilde{\zeta}(0,0) = 1$. Hence we can write

$$(\Re\zeta(x,t))^m = x^m\lambda(x,t),$$

where $\lambda \in \mathscr{C}^{\sigma-1}$ is not zero.

On the other hand from $L\zeta = 0$ we obtain

$$\zeta_t(x,t) = x^m\theta(x,t),$$

where $\theta \in \mathscr{C}^{\sigma-1}$ for $s \leqslant \sigma - 1$, and $\Im\theta(0,0) \neq 0$ since $b(0,0) = 0$. Then

$$\Im\zeta(x,t) = x^m \int_0^t \Im\theta(x,s)\,ds + \Im\zeta(x,0).$$

By (2.16) we have $\Im\zeta(x,0) = O(x^{m+1})$ and thus

$$\Im\zeta(x,t) = x^m\Theta(x,t),$$

where now $\Theta \in \mathscr{C}^{\sigma-m}$, $\Theta(0,0) = 0$, and $\Theta_t(0,0) = \Im\theta(0,0) \neq 0$.

We have thus concluded that

$$T(x,t) = \frac{\Theta(x,t)}{\lambda(x,t)};$$

then $T \in \mathscr{C}^{\sigma-m}$ and $T_t(0,0) = \Theta_t(0,0)/\lambda(0,0) \neq 0$. □

In other words, Lemma 2.2 states that we can reduce the vector field (2.2′) to model (2.7′) if and only if (2.2′) is integrable. In order to prove Theorem 1.1 we must show that this is always the case.

When $s = \infty$ then integrability is standard since (2.2′) satisfies the Nirenberg–Treves condition (P). To apply solvability, write L in the form

$$L = \frac{\partial}{\partial t} + \lambda(x,t)\frac{\partial}{\partial x}.$$

By solvability (cf. [4, Section 26.11]) we can find a smooth function v such that

$$Lv = -\lambda_x$$

near the origin. Set

$$Z(x,t) = \int_0^x e^{v(x',t)}\,dx'.$$

Using that $\lambda(0,t) = 0$, a direct computation gives $LZ = 0$ and $Z_x = \exp\{v\} \neq 0$.

When $1 \leqslant s < \infty$ we must change the argument, since it does not seem that a solvability result within the class of Hölder functions is available. However, see [5] for L^2-solvability of a first-order operator with Lipschitz coefficients satisfying condition (P).

We recall that we are availing ourselves to (2.2′), (2.7′) (2.8′) and (2.11′). Also observe that the argument that reduces (2.2) and (2.7) to (2.2′) and (2.7′) shows that the vanishing order of b stated in Lemma 2.1 remains unchanged. Thus we have

$$\frac{\partial^{j+k}}{\partial x^j \partial t^k} b(x,t) = o(|x|^{[s]-m-j}),\ j+k \leqslant [s]-m, \tag{2.18}$$

$$\frac{\partial^{j+k}}{\partial x^j \partial t^k} b(x,t) = O(|x|^{s-m-j}),\ j+k \leqslant [s]-m. \tag{2.18′}$$

After multiplying by a cut-off function which is identically equal to one in a neighborhood of the origin we can assume that $b \in \mathscr{C}^{s-m}(\mathbb{R}^2)$ and that b is supported in the region $|x| < \delta$, $|t| < \delta$.

The vector field L is elliptic for $x \neq 0$. Let

$$U \doteq \{(x,t) : x \neq 0\}$$

and consider the smooth diffeomorphism $Z^\bullet : U \to Z^\bullet(U) \doteq W$ given by (2.11′). We have

$$W = \{w = \xi + i\eta : \xi \neq 0\}. \tag{2.19}$$

Now $Z^\bullet_* L^\sharp$ is a multiple of the Beltrami vector field

$$\mathscr{B} = \frac{\partial}{\partial \bar{w}} + \mu(w)\frac{\partial}{\partial w}. \tag{2.20}$$

Of course we have $\mu \in \mathscr{C}^{s-m}(W)$; more precisely,

$$\mu = (LZ^\bullet / L\overline{Z^\bullet}) \circ (Z^\bullet)^{-1}. \tag{2.21}$$

Now we have [cf. (2.8′)]

$$LZ^\bullet = -ix^m b(x,t)\mathscr{X} Z^\bullet = -ib(x,t) Z^{\bullet m}. \tag{2.22}$$

On the other hand,

$$L_0\overline{Z^\bullet} = L_0(Z^\bullet + \overline{Z^\bullet}) = L_0 x = -i\frac{x^m}{1+imx^{m-1}t}$$

and thus [again cf. (2.8′)]

$$L\overline{Z^\bullet} = -i\frac{x^m}{1+imx^{m-1}t} - ib(x,t)\overline{Z^\bullet}^m. \tag{2.23}$$

Introducing a further notation

$$b^\bullet(w) \doteq (b\circ(Z^\bullet)^{-1})(w) = b\left(\xi, \frac{\eta}{\xi^m}\right)$$

from (2.21)–(2.23) we finally obtain

$$\mu(w) = \frac{w^m b^\bullet(w)}{\bar{w}^m b^\bullet(w) + \frac{\xi^{m+1}}{\xi+im\eta}} = \hat{b}\left(\xi, \frac{\eta}{\xi^m}\right) \tag{2.24}$$

with

$$\hat{b}(x,t) = \frac{(1+ix^{m-1}t)^m b(x,t)}{\frac{1}{1+imx^{m-1}t} + (1-ix^{m-1}t)^m b(x,t)}.$$

Notice that μ is supported in the region

$$(\xi,\eta)\in W, \quad |\xi|<\delta, \quad |\eta|<\delta|\xi|^m. \tag{2.25}$$

In particular, we can trivially extend μ to $\mathbb{C}\backslash\{0\}$ as a $\mathscr{C}^{s-m}$ function, vanishing identically for $|w|>\delta$.

From (2.18′) we obtain:

$$\frac{\partial^{j+k}}{\partial x^j \partial t^k}\hat{b}(x,t) = O(|x|^{s-m-j}),\ j+k \leqslant [s]-m.$$

Hence

$$\begin{aligned}\frac{\partial^{j+k}}{\partial \xi^j \partial \eta^k}\left\{\hat{b}\left(\xi,\frac{\eta}{\xi^m}\right)\right\} &= \frac{\partial^j}{\partial \xi^j}\left\{\left(\frac{\partial^k \hat{b}}{\partial t^k}\right)\left(\xi,\frac{\eta}{\xi^m}\right)\cdot\frac{1}{\xi^{mk}}\right\}\\ &= \sum_{j_1+j_2+j_3=j} c_{j_1j_2j_3}\left(\frac{\partial^{j_1+j_2+k}\hat{b}}{\partial x^{j_1}\partial t^{k+j_2}}\right)\left(\xi,\frac{\eta}{\xi^m}\right)\cdot\left(\frac{\eta}{\xi^{m+1}}\right)^{j_2}\cdot\frac{1}{\xi^{mk+j_3}}\\ &= O(|w|^{s-km-j}),\end{aligned}$$

where the second identity is obtained by applying induction on j. Thus

$$\left|\frac{\partial^{j+k}\mu}{\partial w^j \partial \bar{w}^k}(w)\right| = O(|w|^{s-(j+k+1)m}), \quad j+k \leqslant s-m. \tag{2.26}$$

End of proof of Theorem 1.1: It follows from (2.26) that μ defines an element in $W^{[s/m]-1,\infty}(\mathbb{C})$ since $m<s$ implies $[s/m]-1\leqslant s-m$. Since we also have $|\mu(w)|=O(|w|^{s-m})$ we may assume that $|\mu|\leqslant\mu_0<1$ (choosing δ appropriately). Moreover

$$s\geqslant m^2+m\Rightarrow m+1\leqslant[s/m]-1. \tag{2.27}$$

According to classical results on the Beltrami equation (cf. [16, p. 87]) we can find a solution to $\mathscr{B}g=0$ with $g\in\mathscr{C}^{[s/m]-1+\varepsilon}(\mathbb{C})$ for every $0<\varepsilon<1$ and $g_w\neq0$. Hence

$$Z(x,t)=g(x+ix^mt) \tag{2.28}$$

is a non-trivial first integral of the vector field (2.2′) which belongs to $\mathscr{C}^{[s/m]-1+\varepsilon}$ for every $0<\varepsilon<1$.

Consequently, thanks to (2.28) and the preceding discussion, we can apply Lemma 2.2 to obtain a $\mathscr{C}^{[s/m]-m-1+\varepsilon}$ diffeomorphism ($0<\varepsilon<1$ arbitrary) transforming (2.2′) into a multiple of (2.7′). This fact together with Lemma 2.1 completes the proof of Theorem 1.1. □

Remark. When $m=1$ Theorem 1.1 states that the vector field (1.1) with $s\geqslant3$ can be normalized by a diffeomorphism that belongs to $\mathscr{C}^{[s]-2+\varepsilon}$ for all $\varepsilon<1$. In some particular cases a stronger statement is true.

For instance, when $a=a(t)$ is a function of t alone then our regularity assumption automatically gives $a\in\mathscr{C}^s$, and

$$(x,t)\mapsto\left(x\exp\left\{-\int_0^t\Im a(s)\,ds\right\},\int_0^t\Re a(s)\,ds\right)$$

is a $\mathscr{C}^{s+1}$ diffeomorphism which normalizes (1.1).

On the other hand, suppose that $a=a(x)$ is a function of x alone. Assume that a is real, $a(0)=1$, $a\in\mathscr{C}^{s-1}$ and that $x\mapsto xa(x)$ is $\mathscr{C}^s$. Let

$$B(x)=\int_0^x\frac{1}{a(x')}\frac{1-a(x')}{x'}\,dx'.$$

Then $B\in\mathscr{C}^{s-1}$ and

$$(x,t)\mapsto(x\exp\{B(x)\},t) \tag{2}$$

is a $\mathscr{C}^{s-1}$ diffeomorphism that normalizes the vector field (1.1).

We point out that we have no example which shows that the regularity of the diffeomorphism obtained in Theorem 1.1 is optimal. □

We now turn to the proof of Theorem 1.2. In fact, we will be able to prove a more general result. We start with the real analytic case:

Proposition 2.3. *Let $\mathscr{L}$ be a real-analytic vector field given by*

$$\mathscr{L} = \frac{\partial}{\partial t} - it^m a(x,t)\frac{\partial}{\partial x}, \quad \Re a(0,0) \neq 0, \tag{2.29}$$

where m is a positive integer. Then there is a real-analytic diffeomorphism at the origin $(x,t) \mapsto (X,T)$ which transforms $\mathscr{L}$ into a non-vanishing multiple of the generalized Mizohata operator $\frac{\partial}{\partial T} - iT^m \frac{\partial}{\partial X}$.

Proof. The proof is a direct consequence of the Cauchy–Kowalevsky theorem. Let Z be a local real-analytic solution to the Cauchy problem

$$\mathscr{L}Z = 0, \quad Z(x,0) = x. \tag{2.30}$$

One readily sees that

$$Z(x,t) = x + t^{m+1}\alpha(x,t) + i\frac{t^{m+1}}{m+1}\beta(x,t),$$

where α, β are real and $\beta(0,0) \neq 0$. In the coordinates $X = x + t^{m+1}\alpha(x,t)$, $T = t\sqrt[m+1]{\beta(x,t)}$ we have $Z = X + iT^{m+1}/(m+1)$ and thus $\mathscr{L}$ is a multiple of $\frac{\partial}{\partial T} - iT^m \frac{\partial}{\partial X}$. $\square$

Notice that when $\mathscr{L}$ is only $\mathscr{C}^\infty$ the same reasoning applies if $\mathscr{L}$ admits a *non-trivial* first integral Z (that is, if Z is a solution to the problem $\mathscr{L}Z = 0$, $Z(0,0) = 0$, $Z_x(0,0) \neq 0$) with the property that $x \mapsto Z(x,0)$ is real analytic. Indeed its complexification $\zeta \mapsto Z(\zeta, 0)$ defines a biholomorphism at the origin, which we denote by h. If we set $Z_1 \doteq h^{-1} \circ Z$ then $\mathscr{L}Z_1 = 0$, $Z_1(x,0) = x$ and the argument in Proposition 2.3 shows that $\mathscr{L}$ is smoothly equivalent to the operator $\frac{\partial}{\partial T} - iT^m \frac{\partial}{\partial X}$. We shall exploit this argument in the classification of the vector fields (2.29) described below when m is an even integer.

A complete set of invariants for such vector fields under $\mathscr{C}^\infty$ diffeomorphisms was described by Sjöstrand [12] in the case $m = 1$. Also, when $m \geqslant 3$ is odd, Ninomiya [8] found a necessary and sufficient condition for (2.29) to be locally integrable.

Suppose now that m is even. In this case $\mathscr{L}$ is hypo-elliptic [13]; consequently, it is locally integrable and furthermore defines a hypocomplex structure in a neighborhood of the origin (cf. [15, Theorem III.6.3, p. 158; see also Theorem III.6.2, p. 154]). More explicitly, the following holds: $\mathscr{L}$ admits a non-trivial first-integral Z_0, which is an open map, and if u is a solution to $\mathscr{L}u = 0$ then $u = H \circ Z_0$ for some holomorphic function $H(z)$ in the complex plane. In particular any two non-trivial first integrals of $\mathscr{L}$ are biholomorphically related.

Denote by $\mathscr{F}_0$ the set of all germs of smooth non-singular curves (i.e. one-dimensional embedded submanifolds) at the origin in $\mathbb{C}$, where we identify two curves that are biholomorphically equivalent. From the preceding discussion we can naturally define for each vector field of the form (2.29), with m even, an element

$\mathfrak{c}(\mathscr{L})\in\mathscr{F}_0$: if Z is a non-trivial first integral of $\mathscr{L}$ we set $\mathfrak{c}(\mathscr{L})$ as being the holomorphic equivalence class of the curve $x\mapsto Z(x,0)$ in $\mathscr{F}_0$. Conversely, given a germ of a smooth (non-parametrized) curve γ, there is a smooth vector field $\mathscr{L}$ of the form (2.29) such that $\mathfrak{c}(\mathscr{L})$ represents the holomorphic equivalence class of γ. To see this, one finds holomorphic coordinates such that γ is parametrized by $x\to x+ip(x)$ with $p(0)=p'(0)=0$. Put $Z=x+i(\frac{t^{m+1}}{m+1}+p(x))$ and

$$\mathscr{L}=\frac{\partial}{\partial t}-i\frac{t^m}{1+ip'(x)}\frac{\partial}{\partial x}.$$

We have $\mathscr{L}Z=0$, while γ is parametrized by $x\to Z(x,0)$. Thus $\mathfrak{c}(L)$ is the holomorphic equivalence class of the given γ. Also notice that when $\mathscr{L}$ is the Mizohata operator then $\mathfrak{c}(\mathscr{L})$ is the class of $\mathbb{R}\times\{0\}$.

Theorem 1.2 follows from the following:

Theorem 2.4. *Consider the vector fields*

$$\mathscr{L}_1=\frac{\partial}{\partial t}-it^m a_1(x,t)\frac{\partial}{\partial x},\quad \Re a_1(0,0)\neq 0,$$

$$\mathscr{L}_2=\frac{\partial}{\partial T}-iT^m a_2(X,T)\frac{\partial}{\partial X},\quad \Re a_2(0,0)\neq 0,$$

where $m\geqslant 2$ is even and a_j are smooth. Then there is a smooth diffeomorphism $(x,t)\mapsto(X,T)$ at the origin transforming $\mathscr{L}_1$ into a non-vanishing multiple of $\mathscr{L}_2$, if and only if $\mathfrak{c}(\mathscr{L}_1)=\mathfrak{c}(\mathscr{L}_2)$.

Proof. We first assume the existence of a smooth diffeomorphism $\varphi:(x,t)\mapsto(X,T)$ at the origin such that $\varphi_*\mathscr{L}_1$ is a multiple of $\mathscr{L}_2$. A fortiori it must satisfy $T(x,0)=0$ and moreover $x\mapsto X(x,0)$ is a diffeomorphism at the origin in $\mathbb{R}$. Let Z_j $(j=1,2)$ be non-trivial first integrals of $\mathscr{L}_j$. Since $\mathscr{L}_1(Z_2\circ\varphi)=\varphi_*(\mathscr{L}_1)(Z_2)=0$ it follows that $\tilde{Z}\doteq Z_2\circ\varphi$ is also a non-trivial first integral of $\mathscr{L}_1$. Thus there is a biholomorphism H at the origin in the complex plane such that $\tilde{Z}=H\circ Z_1$. In particular, we obtain

$$H(Z_1(x,0))=Z_2(X(x,0),T(x,0))=Z_2(X(x,0),0).$$

Since $x\mapsto Z_2(X(x,0),0)$ is just a reparametrization of the curve $X\mapsto Z_2(X,0)$ we have verified that $\mathfrak{c}(\mathscr{L}_1)=\mathfrak{c}(\mathscr{L}_2)$.

Conversely assume that $\mathfrak{c}(\mathscr{L}_1)=\mathfrak{c}(\mathscr{L}_2)$. With the above notation we obtain a diffeomorphism $\psi\in\mathscr{C}^\infty(\mathbb{R})$ near the origin such that $H(Z_1(x,0))=Z_2(\psi(x),0)$ for some biholomorphism H defined near the origin. Replacing Z_2 by $Z_2/(Z_2)_X(0,0)$ and Z_1 by $(H\circ Z_1)/(Z_2)_X(0,0)$ we can assume that

$$Z_1(x,0)=Z_2(\psi(x),0),\,(Z_2)_X(0,0)=1.\tag{2.31}$$

From $\mathscr{L}_j Z_j = 0$ together with (2.31) we obtain

$$Z_1(x,t) = \Re Z_1(x,t) + i[t^{m+1}\theta(x,t) - \Im Z_1(x,0)],$$

$$Z_2(X,T) = \Re Z_2(X,T) + i[T^{m+1}\Theta(X,T) - \Im Z_2(X,0)],$$

where θ and Θ are smooth and real, and do not vanish at the origin. We must determine a diffeomorphism $(x,t)\mapsto(X(x,t),T(x,t))$ such that $Z_2(X(x,t),T(x,t)) = Z_1(x,t)$. We shall take it of the form

$$X(x,t) = \psi(x) + t^{m+1}A(x,t), \quad T(x,t) = tB(x,t)$$

with $B(0,0)\neq 0$ (this condition guarantees that $x\mapsto X(x,t), t\mapsto T(x,t)$ is indeed a smooth diffeomorphism at the origin). We can write

$$\Im Z_2(\psi(x) + t^{m+1}A, 0) = \Im Z_2(\psi(x), 0) + t^{m+1}G(x,t;A),$$

where G is smooth in all three arguments and satisfies $G(0,0;A) = 0$ for all A. We also have, thanks to our hypothesis, that

$$\Re Z_2(\psi(x) + t^{m+1}A, tB) - \Re Z_1(x,0) = t^{m+1}F(x,t;A,B),$$

$$\Re Z_1(x,t) - \Re Z_1(x,0) = t^{m+1}\tilde{F}(x,t),$$

where $\tilde{F}(x,t)$ is smooth in x and t, while F is of the form

$$F(x,t;A,B) = Af(x,t;A,B) + Bg(x,t;A,B),$$

with f and g smooth in all arguments and satisfying

$$f(0,0;A,B) = c_0\neq 0, g(0,0;A,B) = c_1, \forall A,B.$$

Hence A and B must solve the system:

$$\begin{cases} F(x,t;A,B) = \tilde{F}(x,t), \\ B^{m+1}\Theta(\psi(x) + t^{m+1}A, tB) - G(x,t;A) = \theta(x,t). \end{cases} \tag{2.32}$$

This system has the particular solution $(0,0;A_0,B_0)$ with

$$B_0 = \left\{\frac{\theta(0,0)}{\Theta(0,0)}\right\}^{\frac{1}{m+1}}, A_0 = \frac{\tilde{F}(0,0) - c_1B_0}{c_0}.$$

It suffices to apply the implicit function theorem to obtain the sought pair of smooth functions $(A(x,t),B(x,t))$ satisfying $A(0,0) = A_0$ and $B(0,0) = B_0$. $\square$

3. Normalizations near a circle: the real-analytic case

We consider an analytic vector field near the circle $r = |z| - 1 = 0 (z = (1 + r)e^{i\theta})$ given by

$$\mathscr{L} = \frac{\partial}{\partial\theta} - ira(r,\theta)\frac{\partial}{\partial r}, \quad a(r, \theta + 2\pi) = a(r,\theta), \tag{3.1}$$

in which $a(r,\theta)$ is a complex-valued real-analytic function satisfying $\Re a(0,\theta) \neq 0$. Replacing $a(r,\theta)$ with $a(r,-\theta)$ if necessary, one may assume that

$$\Re a(0,\theta) > 0. \tag{3.2}$$

Let $\gamma \subset \mathbb{C}$ be the unit circle with the clockwise orientation. For $\mathscr{L}$ given by (3.1), let $\varphi^t(r,\theta)$ be the flow of the vector field $-\mathscr{L}^{\mathrm{c}}$, where $\mathscr{L}^{\mathrm{c}}$ is the complexification of $\mathscr{L}$ defined on some $U_\varepsilon = \{(r,\theta) \in \mathbb{C}^2 : |r| + |\Im\theta| < \varepsilon\}$ for some small $\varepsilon > 0$. Then $\varphi^{2\pi}(r,0) = (h(r), -2\pi)$. Obviously, $h(r)$ is holomorphic with $h(0) = 0$. We shall call h the *holonomy* of (3.1) with respect to γ. It was proved by Meziani [7] that $\lambda = \frac{1}{2\pi}\int_0^{2\pi} a(0,\theta)\, d\theta$ is an invariant of the differential operator, and that $h'(0) = e^{2\pi i\lambda}$. Meziani [7] also showed that if $\mathscr{L} \in \mathscr{C}^\infty$ and $Im\,\lambda \neq 0$ then, for each finite k, there exists a $\mathscr{C}^k$ diffeomorphism defined near the unit circle which sends $\mathscr{L}$ into a multiple of $\mathscr{L}_\lambda = \frac{\partial}{\partial\theta} - i\lambda r\frac{\partial}{\partial r}$.

As mentioned in the introduction, Meziani [7] constructed a holomorphic holonomy transformation by truncating the $\mathscr{C}^\infty$ vector fields, and used the holonomy transformation to normalize the vector field (3.1). In fact, Meziani's argument proved essentially that, in the real analytic case, the real-analytic vector field (3.1) is equivalent to a multiple of $\mathscr{L}_\lambda$, if and only if the corresponding holonomy transformation is linearizable; in particular, all real analytic vector fields are real-analytically equivalent to $\mathscr{L}_\lambda$ if $\lambda \notin \mathbb{R}$.

It turns out that the holonomy transformations are much more relevant in the real analytic case, which we will now describe. We shall only consider the case $\lambda \in \mathbb{R}$. Let Σ_λ be the set of vector fields (3.1)–(3.2) with

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} a(0,\theta)\, d\theta > 0.$$

We have the following.

Lemma 3.1. *Fix $\lambda > 0$. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two real-analytic vector fields in Σ_λ, and let h_1, h_2 be the corresponding holonomies of $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Then $\mathscr{L}_1$ and $\mathscr{L}_2$ are equivalent up to a multiple by an analytic diffeomorphism defined near the unit circle γ, if and only if h_1, h_2 are conjugate by a holomorphic transformation fixing the origin of $\mathbb{C}$. Moreover, up to a holomorphic conjugate, each germ of holomorphic function $h(z)$ with $h(0) = 0$ and $h'(0) = e^{2\pi i\lambda}$ is the holonomy of some real analytic vector field in Σ_λ.*

Proof. Assume that $\psi_*\mathscr{L}_1 = u\mathscr{L}_2$. Then, restricted to γ, ψ is a real-analytic diffeomorphism. Next, we want to show that ψ preserves the orientation of γ also. Write $\psi : r' = \psi_2(r,\theta) = r\tilde{\psi}_2(r,\theta), \theta' = \psi_1(r,\theta)$, $\mathscr{L}_j = \frac{\partial}{\partial\theta} - ira_j(r,\theta)\frac{\partial}{\partial r}$. Then

$$-ir\tilde{\psi}_2 a_2\circ\psi = \frac{\frac{\partial}{\partial\theta}\psi_2 - ira_1\frac{\partial}{\partial r}\psi_2}{\frac{\partial}{\partial\theta}\psi_1 - ira_1\frac{\partial}{\partial r}\psi_1}.$$

Dividing both sides by r and setting $r = 0$ yields

$$-i\frac{\partial}{\partial\theta}\psi_1(0,\theta)a_2(0,\psi_1(0,\theta)) = \frac{\partial}{\partial\theta}\log|\tilde{\psi}_2(0,\theta)| - ia_1(0,\theta).$$

Note that $\tilde{\psi}_2$ is a periodic real function. Hence, $\frac{\partial}{\partial\theta}\log|\tilde{\psi}_2(0,\theta)|$ vanishes for some θ_0. Since $\Re a_j(0,\theta)$ are positive, then $\frac{\partial}{\partial\theta}\psi_1(0,\theta)$ is positive either. Thus, ψ preserves the orientation of γ. Complexify $\mathscr{L}_1, \mathscr{L}_2$ and ψ. Each $\mathscr{L}_j$ defines a holomorphic foliation on U_ε for some $\varepsilon > 0$; also the complexification of ψ sends leaves of foliation defined by $\mathscr{L}_1$ to leaves of the foliation of $\mathscr{L}_2$. Since the circle γ is contained in a leaf and ψ sends γ into itself and preserves the orientation, then ψ yields a conjugate between two holonomies of $\mathscr{L}_1$ and $\mathscr{L}_2$. More specifically, let $\Gamma\subset\{(r,\theta)\in\mathbb{C}^2\}$ be given by $\theta = 0$, and let ψ send $\mathscr{L}_1$ into a multiple of $\mathscr{L}_2$. Let ψ^c be the complexification of the real analytic map $\psi : r' = \psi_2(r,\theta), \theta' = \psi_1(r,\theta)$. Let φ_j^t be the flow of $-\mathscr{L}_j$. Denote by $f_1{:}\Gamma\to\Gamma' = \psi^c(\Gamma)$ the restriction of ψ^c to Γ. Let φ_2^t be the flow of the vector field $-\mathscr{L}_2$. Let $t_0\geqslant 0$ be the smallest number so that $\varphi_2^{t_0}(\psi(0,0)) = (0,2k\pi)$ for some integer k. For small $|r|$, there is a unique $t(r)$ close to t_0 so that $\varphi_2^t(\psi(r,0)) = (f_2(r),k)$. By the uniqueness theorem of ODEs, one sees that $h_1 = f_1^{-1}f_2^{-1}h_2f_2f_1$.

Conversely, let

$$\mathscr{L}_j = \frac{\partial}{\partial\theta} - ira_j(r,\theta)\frac{\partial}{\partial r}, \quad j = 1,2$$

with $\frac{1}{2\pi}\int_0^{2\pi} a_j(0,\theta)\,d\theta = \lambda$ and $a_j(r,\theta+2\pi) = a_j(r,\theta)$. We assume that the corresponding holonomies h_1, h_2 of $\mathscr{L}_1$ and $\mathscr{L}_2$ are conjugate, i.e., that $h_2 = f\circ h_1\circ f^{-1}$ for some holomorphic transformation f. We want to show that, up to a multiple, $\mathscr{L}_1$ and $\mathscr{L}_2$ are equivalent under a real analytic diffeomorphism. Denote by $\varphi_j^t(r,\theta)$ the flow of $-\mathscr{L}_j$. We have

$$\varphi_j^t(r,\theta) = (r_j^t(r,\theta),\theta - t), \quad r_j^t(r,\theta+2\pi) = r_j^t(r,\theta), \quad r_j^0(r,\theta) = r.$$

It is convenient to make a change of coordinates for each $\mathscr{L}_j$ such that

$$a_j(r,\theta) = \lambda + O(r).$$

(See Lemma 4.2 below.) Then

$$r_j^t(r,\theta) = e^{i\lambda t}r + E_j(r,\theta)$$

with $E_j(r,\theta)=O(|r|^2)$ analytic. Put $f(r)=\mu r+O(|r|^2)$. For small $r\in\mathbb{R}$ and for $\theta\in\mathbb{R}$, we shall find a unique

$$t(r,\theta)\in\left(\theta+\frac{1}{\lambda}\arg\mu-\pi/2,\theta+\frac{1}{\lambda}\arg\mu+\pi/2\right)$$

such that

$$r_2^{-t}(f(r_1^\theta(r,\theta)),0)\in\mathbb{R},\quad \text{for } t=t(r,\theta);\tag{3.3}$$

consequently, we have a well-defined real map

$$\Phi:(r,\theta)\to\varphi_2^{-t(r,\theta)}(f(r_1^\theta(r,\theta)),0).$$

We have

$$r_2^{-t}(f(r_1^\theta(r,\theta)),0)=\mu re^{i\lambda(\theta-t)}+E_3(r,\theta,t)\tag{3.4}$$

with $E_3(r,\theta,t)=O(|r|^2)$ being analytic. Now one sees that there exists a unique $t(r,\theta)=\theta+\frac{1}{\lambda}\arg\mu+O(|r|)$ satisfying (3.3). Also, $t(r,\theta)$ is analytic in r,θ.

Next, we need to show that $\Phi(r,\theta+2\pi)=\Phi(r,\theta)+(0,2\pi)$. By the definition of holonomy, we have

$$\varphi_j^{2\pi}(r,0)=(h_j(r),-2\pi)$$

for $r\in\mathbf{C}$ and $|r|$ small. Hence, $r_j^{\theta+2\pi}(r,\theta+2\pi)=r_j^{\theta+2\pi}(r,\theta)=h_j(r_j^\theta(r,\theta))$ for (r,θ) in some U_ε. Now

$$\begin{aligned}
r_2^{-t(r,\theta+2\pi)}(f(r_1^{\theta+2\pi}(r,\theta+2\pi)),0)&=r_2^{-t(r,\theta+2\pi)}(f\circ h_1(r_1^\theta(r,\theta)),0)\\
&=r_2^{-t(r,\theta+2\pi)}(h_2\circ f(r_1^\theta(r,\theta)),0)\\
&=r_2^{-t(r,\theta+2\pi)}(h_2\circ f(r_1^\theta(r,\theta)),-2\pi)\\
&=r_2^{-t(r,\theta+2\pi)}(\varphi_2^{2\pi}(f(r_1^\theta(r,\theta)),0))\\
&=r_2^{-t(r,\theta+2\pi)+2\pi}(\varphi_2^{0}(f(r_1^\theta(r,\theta)),0))\\
&=r_2^{-t(r,\theta+2\pi)+2\pi}(f(r_1^\theta(r,\theta)),0).
\end{aligned}$$

In particular, the last quantity is real when r,θ are real [cf. (3.3)]. Since $t(r,\theta+2\pi)-2\pi$ remains in $(\theta+\frac{1}{\lambda}\arg\mu-\pi/2,\theta+\frac{1}{\lambda}\arg\mu+\pi/2)$, the uniqueness of $t(r,\theta)$ yields $t(r,\theta+2\pi)-2\pi=t(r,\theta)$. Therefore, $\Phi(r,\theta+2\pi)=\Phi(r,\theta)+(0,2\pi)$. Next, we need to verify that Φ sends $\mathscr{L}_1$ into a multiple of $\mathscr{L}_2$. It suffices to show that the complexification of Φ sends leaves of the complexified $\mathscr{L}_1$ into ones of the complexified $\mathscr{L}_2$, i.e., that $\Phi\circ\varphi_1^t(r,\theta)$, as $t\in\mathbb{C}$ varies and $(r,\theta)\in\mathbb{C}^2$ is fixed, remains in a leaf of $\mathscr{L}_2$. To this end, write $(r_1, \theta_1)(t) = \varphi_1^t(r, \theta)$. Then $\varphi_1^{\theta_1}(r_1, \theta_1) = \varphi_1^{\theta_1+t}(r, \theta) = \varphi_1^{\theta}(r, \theta)$. Thus

$$\Phi\varphi_1^t(r, \theta) = \varphi_2^{-t\circ\varphi_1^t(r,\theta)}(f(r_1^\theta(r, \theta)), 0)$$

is in the leaf passing through $f(r_1^\theta(r, \theta), 0)$ for $t \in \mathbb{C}$ with small $|Im\, t|$.

It remains to show that given a germ of holomorphic function $h(r) = e^{2\pi i\lambda}r + O(|r|^2)$, there is a real analytic operator (3.1)–(3.2) of which the holonomy with respect to the oriented circle is holomorphically conjugate to h. To this end, consider a germ of holomorphic vector field at $0 \in \mathbb{C}^2$ of the form

$$v = -iz_1\frac{\partial}{\partial z_1} + iz_2 b(z)\frac{\partial}{\partial z_2}, \quad b(0) = \lambda \neq 0. \tag{3.5}$$

Let $\psi_t(z)$ be the flow of v. For small ε, $\psi_{2\pi}(\varepsilon, z_2) = (\varepsilon, h_\varepsilon(z_2))$, and h_ε is a germ of holomorphic function defined near the origin with $h'_\varepsilon(0) = e^{2\pi i\lambda}$ for $\lambda = b(0)$. Note that the holomorphic conjugate class of h_ε is independent of ε. By a theorem of Pérez–Marco and Yoccoz [10], there exists a holomorphic vector field (3.5) to which the corresponding h_ε for some small $\varepsilon \neq 0$ is conjugate to the given holomorphic function $h(z_2)$. The solution curve of (3.5) for the initial value (ε, z_2) can be written as $(\varepsilon e^{-it}, z_2(t))$ with

$$\dot{z_2} = iz_2 b(\varepsilon e^{-it}, z_2).$$

Note that $(e^{-it}, z_2(t))$ is exactly the solution curve of

$$\dot{\theta} = -1, \quad \dot{z_2} = iz_2 b(\varepsilon e^{i\theta}, z_2)$$

with the initial value $(0, z_2)$. With respect to the oriented circle γ, $h_\varepsilon(r)$ is precisely the holonomy transformation of

$$\mathscr{L} = \frac{\partial}{\partial\theta} - irb(\varepsilon e^{-i\theta}, r)\frac{\partial}{\partial r}.$$

The proof of the lemma is complete. □

For each positive integer n, the above lemma establishes a one-to-one correspondence between holomorphic conjugate classes of germs of holomorphic functions $h(z)$ with $h(0) = 0$ and $h'(0) = e^{i2\pi\lambda}$ and real analytic equivalence classes of real-analytic differential operators (3.1)–(3.2) with $n - 1 < \lambda \leqslant n$. The proof of Theorem 1.4 is now immediate.

Proof of Theorem 1.4. By results of Siegel [11] and Bruno [3], all germs of holomorphic function $f(z) = e^{2\pi i\lambda}z + O(|z|^2)$ are linearizable, if λ is irrational and satisfies the Bruno condition. By a result of Yoccoz [18], the germ of holomorphic function $f(z) = e^{2\pi i\lambda}z + z^2$ is not linearizable, if λ is irrational and violates the Bruno condition. Now Theorem 1.4 follows from Lemma 3.1. □

For a real-analytic differential operator $\mathscr{L}$ of the form (3.1)–(3.2), one says that $\mathscr{L}$ is $\mathscr{C}^\omega$ *solvable* if given a real-analytic function $f(r,\theta)$ defined near γ then there is a solution u to $\mathscr{L}u = f$ with u real analytic near γ. Of course we are assuming that f satisfies the natural compatibility conditions: $\mu(f) = 0$ for all distributions μ supported on $\{0\} \times S^1$ and satisfying ${}^t\mathscr{L}\mu = 0$. It turns out that when λ is irrational then all such distributions are multiples of $\delta(r) \otimes 1_\theta$ and consequently the only compatibility condition is

$$\int_0^{2\pi} f(0,\theta)\, d\theta = 0.$$

Proposition 3.2. *Let $\mathscr{L}$ be a real-analytic vector field* (3.1)–(3.2) *with irrational λ. Then $\mathscr{L}$ is $\mathscr{C}^\omega$-solvable if and only if $\mathscr{L}$ is real-analytically equivalent to $\mathscr{L}_\lambda$ and* $(\log|e^{2\pi n i\lambda} - 1|)/n$ *is bounded.*

Proof. According to a result of Bergamasco–Meziani [2], we know that if λ is irrational then $\mathscr{L}_\lambda$ is $\mathscr{C}^\omega$ solvable if and only if $(\log|e^{2\pi n i\lambda} - 1|)/n$ is bounded. Hence what has to be proved is that when $\mathscr{L}$ is $\mathscr{C}^\omega$ solvable then it can be real analytically reduced to $\mathscr{L}_\lambda$.

The function $Z(r,\theta) = r^{1/\lambda}e^{i\theta}$ is a first integral of $\mathscr{L}_\lambda$ defined for $r \neq 0$. We then must find a smooth diffeomorphism

$$r \mapsto rR(r,\theta),\ \theta \mapsto \theta + \Theta(r,\theta) \tag{3.6}$$

such that

$$W(r,\theta) \doteq (rR)^{1/\lambda}e^{i(\theta+\Theta)}$$

is a solution to $\mathscr{L}W = 0$.

Observe that

$$\mathscr{L}\{\lambda^{-1}\log r + i\theta\} = i(1 - \lambda^{-1}a).$$

On the other hand, the function $f = i(1 - \lambda^{-1}a)$ satisfies the compatibility condition and consequently there is a real-analytic function u solving

$$\mathscr{L}u = -i(1 - \lambda^{-1}a).$$

Then

$$W(r,\theta) = \exp\{\lambda^{-1}\log r + i\theta + u(r,\theta)\} = r^{1/\lambda}e^{i\theta + u(r,\theta)}$$

satisfies $\mathscr{L}W = 0$. We set

$$R(r,\theta) = \exp\{\lambda\Re u(r,\theta)\},\ \Theta(r,\theta) = \Im u(r,\theta).$$

We must show that (3.6) is a real-analytic diffeomorphism near $r = 0$. Along $r = 0$, the determinant of its Jacobian matrix is given by

$$J(0, \theta) = e^{\lambda \Re u(0,\theta)}(1 + \Im u_\theta(0, \theta)).$$

Setting $r = 0$ in the expression $\mathrm{L}u = -i(1 - \lambda^{-1}a)$ gives $u_\theta(0, \theta) = -i(1 - \lambda^{-1}a(0, \theta))$ and thus $\Im u_\theta(0, \theta) = -(1 - \lambda^{-1}\Re a(0, \theta))$. Consequently

$$J(0, \theta) = \lambda^{-1}e^{\lambda \Re u(0,\theta)}\Re a(0, \theta) \neq 0 \quad \forall \theta.$$

Therefore (3.6) is a real-analytic diffeomorphism defined near γ and the proof of the theorem is complete. □

Proposition 3.3. *Let $\mathscr{L}$ be a real-analytic vector field of the form* (3.1)–(3.2) *with $\lambda \in \mathbb{R}$ and assume that there is a non-constant real-analytic function u defined in $(-\delta, \delta) \times S^1$ and satisfying $\mathscr{L}u = 0$. Then λ is rational and $\mathscr{L}$ is real-analytically equivalent to $\mathscr{L}_\lambda$.*

Proof. Write

$$u(r, \theta) = \sum_j u_j(\theta)r^j,$$

where $u_j(\theta)$ are 2π-periodic and real analytic. Let k be the smallest positive integer such that u_k does not vanish identically. Then from the relation $\mathscr{L}u = 0$ we obtain

$$u_k'(\theta) - ika(0, \theta)u_k(\theta) = 0. \tag{3.7}$$

Considering (3.7) as an ordinary differential equation in $\mathbb{R}$ its solution is given by

$$u_k(\theta) = u_k(0)\exp\left\{ik\int_0^\theta a(0, \theta')d\theta'\right\}.$$

Since u is a non-trivial 2π-periodic function, then $n \doteq k\lambda \in \mathbb{Z}$. Replacing $u(r, \theta)$ by $u(r, \theta)/u_k(0)$, we may assume that

$$u(r, \theta) = r^k\exp\left\{ik\int_0^\theta a(0, \theta')d\theta'\right\} + O(r^{k+1}).$$

Thus we can write $u(r, \theta) = R^k(r, \theta)e^{in\Theta(r,\theta)}$ where $R(r, \theta) = r\sqrt[k]{|\frac{u(r,\theta)}{r^k}|}$ and $(r, \theta) \to (R(r, \theta), \Theta(r, \theta))$ is a real-analytic diffeomorphism. Since $r^k\exp\{in\theta\}$ is a non-trivial first integral of $\mathscr{L}_\lambda$ it is clear that this diffeomorphism sends $\mathscr{L}$ into a non-vanishing multiple of $\mathscr{L}_\lambda$. □

Since germs of periodic holomorphic mappings are always linearizable near a fixed point, Lemma 3.1 also gives:

Corollary 3.4. *Let $\mathscr{L}$ be a real-analytic vector field given by* (3.1)–(3.2). *Assume that λ is a positive rational. Then $\mathscr{L}$ is equivalent to $\mathscr{L}_\lambda$ if and only if the holonomy of $\mathscr{L}$ is periodic.*

Notice that for each positive integer λ the moduli space of all real analytic vector fields (3.1)–(3.2) can be identified to the Ecalle–Voronin moduli space for germs of holomorphic functions which are tangent to the identity (cf. [17]). See also [1] for results on classifying entire functions tangent to the identity under germs of smooth transformations.

4. Normalizations near a circle: the finite smooth case

We are given a $\mathscr{C}^s$ smooth vector field $\mathscr{L}$ which is tangent to a closed $\mathscr{C}^1$ real curve γ in the real plane to constant order one, while $\mathscr{L}$ and $\overline{\mathscr{L}}$ are $\mathbb{C}$-linearly independent in a deleted neighborhood of γ. As we have shown in section two, γ is actually $\mathscr{C}^s$. Moreover if we apply a $\mathscr{C}^s$ diffeomorphism sending γ into the unit circle defined by $r = |z| - 1 = 0$, then near the unit circle $\mathscr{L}$ takes the form

$$\frac{\partial}{\partial\theta} - ira(r,\theta)\frac{\partial}{\partial r}, \tag{4.1}$$

in which $ra(r,\theta)$ is a complex-valued $\mathscr{C}^s$ function with

$$\Re a(0,\theta) > 0.$$

For a $\mathscr{C}^s$ function $f(r,\theta)$ we shall write its Taylor expansion as

$$f(r,\theta) = \sum_{n=0}^{[s]} f_n(\theta) r^n + o(|r|^{[s]})$$

with $f_j \in \mathscr{C}^{s-j}$. We shall need the following version of Whitney's extension theorem.

Lemma 4.1. *Let $f_j(\theta) \in \mathscr{C}^{s-j}$ be 2π-periodic, $j = 0, 1, \ldots, [s]$. Then there exists $f(r,\theta) \in \mathscr{C}^s$ which is 2π-periodic in the θ variable and satisfies*

$$f(r,\theta) = \sum_{j=0}^{[s]} \frac{f_j(\theta)}{j!} r^j + o(|r|^{[s]}).$$

Proof. We follow the argument in [4, Section 1.3, Theorem 1.3.3]. Let $\phi(t) \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(t)\, dt = 1$. For a 2π-periodic function $h(\theta) \in \mathscr{C}^s$ and for $k \in \mathbb{Z}_+$ we define

$$g(r,\theta) = r^k \int_{\mathbb{R}} h(\theta - rt)\phi(t)\, dt.$$

Then $g(r,\theta)$ is 2π-periodic in θ, is of class $\mathscr{C}^{k+s}$ and satisfies

$$\frac{\partial^j}{\partial r^j} g(0,\theta) = 0, \text{ for } j<k; \quad \frac{\partial^k}{\partial r^k} g(0,\theta) = k!h(\theta).$$

The proof of Lemma 4.1 is now standard. □

Lemma 4.2. *Let L be a* $\mathscr{C}^s$ $(s\geqslant 1)$ *vector field given by* (4.1) *and let* λ *be defined as in* (1.4) *with* $\Re\lambda > 0$. *Let* q_λ *be equal to* 1 *if* $\lambda \in \mathbb{Z}$, $q_\lambda = +\infty$ *if* $\lambda \in \mathbb{C}\setminus\mathbb{Q}$ *and otherwise we set* q_λ *to be the largest positive integer such that* $j\lambda \notin \mathbb{Z}$ *for* $1<j\leqslant q_\lambda$. *There exists a* $\mathscr{C}^s$ *smooth change of coordinates such that L becomes a multiple of*

$$L = \frac{\partial}{\partial\theta} - ir(\lambda + b(r,\theta))\frac{\partial}{\partial r}$$

with $b(r,\theta) = o(|r|^{\min\{q_\lambda,[s]-1\}})$ *and* $b \in \mathscr{C}^{s-1}$.

Proof. Put $N = \min\{q_\lambda, [s]-1\}$. We shall consider a diffeomorphism F of the form

$$r^* = rR(r,\theta), \quad \theta^* = \theta + \Theta(r,\theta) \tag{4.2}$$

with

$$R(r,\theta+2\pi) = R(r,\theta) \in \mathscr{C}^{s+1}, \quad \Theta(r,\theta+2\pi) = \Theta(r,\theta) \in \mathscr{C}^s.$$

Up to a multiple, F_*L is of the form

$$\frac{\partial}{\partial\theta^*} - ir^*(\lambda + b(r^*,\theta^*))\frac{\partial}{\partial r^*}$$

with $b \in \mathscr{C}^{s-1}$. Explicitly, we have

$$(R + rR_r)a(r,\theta) + iR_\theta = R(\lambda + b(rR, \theta+\Theta))(1 + \Theta_\theta - ira(r,\theta)\Theta_r).$$

In order to achieve that $b(r^*,\theta^*) - \lambda = o(|r^*|^N)$ it suffices to prove that it is possible to achieve:

$$a(R + rR_r) + iR_\theta = \lambda R(1 + \Theta_\theta - ira\Theta_r) + o(|r|^N).$$

Dividing both sides by $R(r,\theta)$, setting $\tilde{R} = \log R$, and comparing the coefficients of r^k for $k\geqslant 0$ of the identity obtained yields

$$i\tilde{R}_0'(\theta) - \lambda\Theta_0'(\theta) = \lambda - a_0, \tag{4.3}$$

$$\begin{aligned} &i\tilde{R}_k'(\theta) - \lambda\Theta_k'(\theta) + ka_0(\theta)\tilde{R}_k(\theta) \\ &\quad + ik\lambda a_0(\theta)\Theta_k(\theta) = -a_k(\theta) + E_k(\theta), \quad k>0 \end{aligned} \tag{4.4}$$

in which $E_k(\theta)$ is a polynomial in $a_j(\theta) \in \mathscr{C}^{s-1-j}$, $\tilde{R}_j(\theta)$, and $\Theta_j(\theta)$ for $j < k$. Since $(2\pi)^{-1} \int_0^{2\pi} a(0,\theta)\, d\theta = \lambda$, Eq. (4.3) has a unique 2π-periodic solution $\tilde{R}_0(\theta)$, $\Theta_0(\theta) \in \mathscr{C}^s$ satisfying

$$\int_0^{2\pi} \tilde{R}_0(\theta)\, d\theta = \int_0^{2\pi} \Theta_0(\theta)\, d\theta = 0. \tag{4.5}$$

Set $u_k = \tilde{R}_k + i\lambda\Theta_k$, and assume for the sake of induction that we have determined $\tilde{R}_j, \Theta_j \in \mathscr{C}^{s-j}$ for $j < k$. Now (4.4) becomes

$$u_k' = ika_0 u_k + p_k, \tag{4.6}$$

in which $p_k \in \mathscr{C}^{s-k-1}$ is 2π-periodic. Set $A_k(\theta) = -ik \int_0^\theta a_0(\theta)\, d\theta$. Then

$$u_k(\theta) e^{A_k(\theta)} = u_k(0) + \int_0^\theta p_k(\theta) e^{A_k(\theta)}\, d\theta \in \mathscr{C}^{s-k}.$$

We shall determine $u_k(0)$ such that $u_k(2\pi) = u_k(0)$; consequently, $u_k(\theta + 2\pi) = u_k(\theta)$ for all real θ. Note that $A_k(2\pi) = -2\pi ik\lambda \notin 2\pi i\mathbb{Z}$. Thus, we can take

$$u_k(0) = \frac{1}{e^{-2\pi i\lambda k} - 1} \int_0^{2\pi} p_k(\theta) e^{A_k(\theta)}\, d\theta.$$

Summing up we have determined $\tilde{R}_k, \Theta_k \in \mathscr{C}^{s-k}$. Of course, there exist $R_k \in \mathscr{C}^{s-k}$ such that $\log\left(\sum R_k(\theta) r^k\right) = \sum \tilde{R}_k(\theta) r^k$. By Lemma 4.1, there exist $R(r,\theta) \in \mathscr{C}^{s+1}, \Theta(r,\theta) \in \mathscr{C}^s$ defined near the origin, of which the formal power series expansion of $rR(r,\theta), \Theta(r,\theta)$ in r have coefficients R_k, Θ_k, respectively. Now $(r,\theta) \to (rR(r,\theta), \Theta(r,\theta))$ transforms L into the desired form. $\square$

Remark 4.3. In general, one cannot solve (4.6) for $k = q_\lambda + 1$. However, one can solve

$$u_k' = ika_0 u_k + p_k - c_k e^{-A_k(\theta)}, \quad k = q_\lambda + 1,$$

for some constant c_k. Hence one can find a $\mathscr{C}^{q_\lambda+1}$ diffeomorphism sending $\mathscr{L}$ into a multiple of

$$\frac{\partial}{\partial\theta} - ir(\lambda - ic_{q_\lambda+1} r^{q_\lambda+1} + o(|r|^{q_\lambda+1}))\frac{\partial}{\partial r}$$

when $q_\lambda < [s] - 1$. If $c_{q_\lambda+1} \neq 0$ one may apply a further transformation $(r,\theta) \to (c'r, \theta + c'')$ for some real constants $c' \neq 0, c''$ in order to achieve $c_{q_\lambda+1} = i$.

Theorem 4.4. *Let* $\mathscr{L}$ *be a* $\mathscr{C}^s(1\leqslant s<\infty)$ *vector field given by*

$$\frac{\partial}{\partial\theta}-ira(r,\theta)\frac{\partial}{\partial r},\quad \frac{1}{2\pi}\int_0^{2\pi}a(0,\theta)\,d\theta=\lambda=1/(\alpha+i\beta)$$

with $\alpha>0$ *and let* q_λ *have the meaning as in Lemma* 4.2. *Set*

$$s_*=\min\left\{\left[\frac{\min\{q_\lambda,[s]-1\}}{\alpha}\right],[s]-1\right\}$$

and

$$s'=\begin{cases} s_*-1, & \alpha\geqslant 1,\\ \min\{s_*-1,[\alpha s_*]\}, & 0<\alpha<1,\quad \alpha s_*\notin\mathbb{Z}_+,\\ \alpha s_*-1, & 0<\alpha<1,\quad \alpha s_*\in\mathbb{Z}_+. \end{cases}\tag{4.7}$$

Assume that

$$s'\geqslant 1.\tag{4.8}$$

Then there exists a $\mathscr{C}^{s'}$ *diffeomorphism sending* $\mathscr{L}$ *into a multiple of* $\mathscr{L}_\lambda$.

Proof. By Lemma 4.2, there exists a $\mathscr{C}^s$ diffeomorphism sending $\mathscr{L}$ into a multiple of

$$\widetilde{\mathscr{L}}=\frac{\partial}{\partial\theta}-ir(\lambda+b(r,\theta))\frac{\partial}{\partial r}$$

with $b(r,\theta)=o(|r|^{s_1})$ and $b\in\mathscr{C}^{s-1}$ for $s_1=\min\{q_\lambda,[s]-1\}$. For $r>0$ we set $w=W(r,\theta)=r^{1/\lambda}e^{i\theta}$. Notice that W defines a diffeomorphism from $\{(r,\theta)\mid r>0\}$ onto $\mathbb{C}\backslash\{0\}$. Moreover $W_*\widetilde{\mathscr{L}}$ is a multiple of the Beltrami vector field:

$$\mathscr{B}=\frac{\partial}{\partial\bar{w}}+\mu(w)\frac{\partial}{\partial w},$$

where μ is defined by

$$\mu(w)=\frac{\bar{\lambda}wb(W^{-1}(w))}{\lambda\bar{w}(\lambda+\bar{\lambda}+b(W^{-1}(w)))}.$$

Explicitly we have

$$b(W^{-1}(w))=b\left(|w|^{1/\alpha},\frac{1}{2i}\log\frac{w}{\bar{w}}-\frac{\beta}{\alpha}\log|w|\right).$$

Since we also have

$$\frac{\partial^{j+k}b}{\partial r^j\partial\theta^k}(r,\theta)=o(|r|^{s_1-j})$$

we obtain

$$\frac{\partial^{j+k}}{\partial w^j\partial\bar{w}^k}\{b(W^{-1}(w))\} = O(|w|^{\frac{s_1}{\alpha}-(j+k)})$$

and consequently

$$\frac{\partial^{j+k}\mu}{\partial w^j\partial\bar{w}^k}(w) = O(|w|^{\frac{s_1}{\alpha}-(j+k)}) \tag{4.8$'$}$$

for all $j+k\leqslant[s]-1$.

Since we could have assumed from the beginning that b vanishes identically outside an interval $|r|<\delta$ we can assume that μ vanishes identically outside a small disc centered at the origin and that $|\mu(w)|\leqslant\mu_0<1$. From (4.8′) we derive that $\mu\in W^{s_*,\infty}(\mathbb{C})$. Since (4.7) gives $s_*\geqslant 1$ we can apply the same reasoning as in the proof of Theorem 1.1 to conclude the existence of a solution $F\in\bigcap_{p<\infty}W^{s_*+1,p}(\mathbb{C})$ solving $\mathscr{B}F=0$ and satisfying $F_w\neq 0$. Notice that in particular $F\in\mathscr{C}^{s_*+\varepsilon}$ for every $\varepsilon<1$.

We can assume that $F(w)=w+o(|w|)$. Replacing F by

$$F(w)-\sum_{j=2}^{s_*}c_jF(w)^j$$

for suitable chosen constants c_j, we may further assume that $F(w)=w+O(|w|^{s_*+\varepsilon})$. Write $F(w)=w(1+\hat{F}(w))$. We have $\hat{F}(w)\in\mathscr{C}^{s_*+\varepsilon-1}$ for $w\neq 0$ and

$$\frac{\partial^{j+k}\hat{F}}{\partial w^j\partial\bar{w}^k}(w) = O(|w|^{s_*+\varepsilon-(j+k+1)}), j+k\leqslant s_*-1.$$

On the other hand, we have

$$\left|\frac{\partial^{j+k}W}{\partial r^j\partial\theta^k}\right| + \left|\frac{\partial^{j+k}\bar{W}}{\partial r^j\partial\theta^k}\right| = O(r^{\alpha-j}),$$

from which an easy induction argument gives

$$\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}\{\hat{F}(W(r,\theta))\} = O(r^{s_*+\varepsilon-j-k-1}),\ \alpha\geqslant 1,$$

$$\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}\{\hat{F}(W(r,\theta))\} = O(r^{\alpha(s_*+\varepsilon-1)-j-k}),\ \ 0<\alpha<1,$$

for $j+k\leqslant s_*-1, r>0$. At this point we make the key observation that [cf. (4.7)] $s_*-1-s'+\varepsilon>0$ for $\alpha\geqslant 1$, and that

$$\alpha(s_*+\varepsilon-1)-s'>0 \text{ for } \varepsilon \text{ close to } 1.$$

Hence $(r,\theta)\to\hat{F}(W(r,\theta))\in\mathscr{C}^{s'}$ for $r\geqslant 0$ and also $\hat{F}(W(r,\theta))=o(r^{s'})$. Write

$$1+\hat{F}(r^{\frac{1}{\lambda}}e^{i\theta})=\tilde{R}^{\frac{1}{\lambda}}e^{i\tilde{\Theta}}$$

where $\tilde{R},\tilde{\Theta}\in\mathscr{C}^{s'}$ are real valued and 2π-periodic in the θ variable. We have $\tilde{R}(r,\theta)=1+o(|r|^{s'})$ and $\tilde{\Theta}(r,\theta)=o(|r|^{s'})$. Now

$$F(W(r,\theta))=(r\tilde{R})^{\frac{1}{\lambda}}e^{i(\theta+\Theta(r,\theta))}$$

and

$$\varphi^{+}:(r,\theta)\to(rR,\theta+\Theta)$$

is a $\mathscr{C}^{s'}$ diffeomorphism sending $\{(r,\theta):0\leqslant r<\delta_0\}$ onto itself. Notice also that $\varphi^{+}(r,\theta)-(r,\theta)=o(|r|^{s'})$ and that $\varphi^{+}_{*}\mathscr{L}$ is a multiple of $\mathscr{L}_\lambda$.

By a similar argument, we can construct φ^{-} which now sends the open set $\{(r,\theta):-\delta_0<r\leqslant 0\}$ onto itself and transforms $\mathscr{L}$ into a multiple of $\mathscr{L}_\lambda$. Set, finally, $\varphi(r,\theta)\doteq\varphi^{\pm}(r,\theta)$ for $\pm r\geqslant 0$. Then $\varphi\in\mathscr{C}^{s'}$ sends $\mathscr{L}$ into a multiple of $\mathscr{L}_\lambda$.

The proof of Theorem 4.4 is now complete. $\square$

Theorem 4.5. *Let λ be a complex number with $\Re\lambda>0$. There exists a $\mathscr{C}^\infty$ vector field* (4.1) *with $\Re a(r,\theta)>0$ and $\frac{1}{2\pi}\int_0^{2\pi}a(0,\theta)\,d\theta=\lambda$ which is not equivalent to a multiple of $\mathscr{L}_\lambda$ by any $\mathscr{C}^\infty$ diffeomorphism defined near $r=0$.*

Proof. Notice first that $\lambda=p/q$ with p,q positive integers then Remark 4.3 shows how to exhibit a real-analytic vector field $\mathscr{L}$ of the form (4.1) that is not equivalent to a multiple of $\mathscr{L}_\lambda$ under any $\mathscr{C}^{q+1}$ smooth transformation.

We now consider the case when $\lambda\in\mathbb{C}\backslash\mathbb{Q}$. We start by considering a $\mathscr{C}^\infty$ diffeomorphism $w=f(z)$ at the origin in $\mathbb{C}$ whose formal Taylor expansion at 0 is of the form

$$z+\sum_{j\geqslant 2}c_jz^j. \tag{4.9}$$

We assume that the radius of convergence of (4.9) is equal to 0. Up to a multiple, $f_*\frac{\partial}{\partial\bar{z}}$ becomes

$$X=\frac{\partial}{\partial\bar{w}}+\frac{f_{\bar{z}}(z)}{\overline{f_z(z)}}\frac{\partial}{\partial w}.$$

As in the proof of Theorem 4.4 we shall write $w=W(r,\theta)=r^{1/\lambda}e^{i\theta}$, $r>0$. Consider also the map $(r,\theta)\to z=Z(r,\theta)\doteq f^{-1}(W(r,\theta))$. For a vector field L given by (3.1) we have

$$LW=iW\left(1-\frac{1}{\lambda}a\right),L\bar{W}=-i\bar{W}\left(1+\frac{1}{\bar{\lambda}}a\right).$$

A simple computation shows that, up to a multiple, $Z_*^{-1}\frac{\partial}{\partial \bar{z}}$ becomes

$$\mathscr{L} = \frac{\partial}{\partial \theta} - ira(r,\theta)\frac{\partial}{\partial r},$$

where

$$a \doteq \lambda \frac{\overline{f_z(z)} + f_{\bar{z}}(z)\frac{\overline{f(z)}}{f(z)}}{\overline{f_z(z)} - f_{\bar{z}}(z)\frac{\overline{\lambda f(z)}}{\lambda f(z)}}.$$

We know that $\mathscr{L}$ is $\mathscr{C}^\infty$ for $r>0$, while $\mathscr{L}(r,\theta) - \mathscr{L}_\lambda(r,\theta) = O(|r|^\infty)$ due to our choice of f. Thus $\mathscr{L}$ extends to a $\mathscr{C}^\infty$ vector field defined near $r=0$ (by setting $\mathscr{L} = \mathscr{L}_\lambda$ for $r<0$). Notice also that this gives $(2\pi)^{-1}\int_0^{2\pi} a(0,\theta) = \lambda$.

Assume that there is a $\mathscr{C}^\infty$ diffeomorphism g sending $\mathscr{L}_\lambda$ into a multiple of $\mathscr{L}$. Then $g(r,\theta) = (rR(r,\theta), \theta + \Theta(r,\theta))$. Replacing $g(r,\theta)$ by $g(-r,\theta)$ if necessary, we may assume that $R(r,\theta)>0$.

Away from $w=0, f^{-1}\circ W\circ g\circ W^{-1}$ sends $\frac{\partial}{\partial \bar{w}}$ into a multiple of $\frac{\partial}{\partial \bar{z}}$; the composition is also bounded. Thus $f^{-1}\circ W\circ g\circ W^{-1}$ extends to a holomorphic function $z = h(w)$ defined near $w=0$. Thus we have

$$f^{-1}\circ W\circ g = h\circ W,$$

that is,

$$f^{-1}(r^{1/\lambda}R^{1/\lambda}(r,\theta)e^{i(\theta+\Theta(r,\theta))}) = h(r^{1/\lambda}e^{i\theta}) \tag{4.10}$$

If we write $h(w) = \sum b_n w^n$ and $F(r) \doteq h(r^{1/\lambda})$ then

$$F(r) = \sum_{n\geqslant 0} b_n r^{n/\lambda}. \tag{4.11}$$

Now notice that $\lambda \notin \mathbb{Q}$ gives

$$m/\lambda + l = m'/\lambda, m, m', l \in \mathbb{Z}_+ \Leftrightarrow m = m', l = 0.$$

Write the formal Taylor expansion of $f^{-1}(w)$ at $w=0$ as $\sum a_n w^n$. Since $\Re\lambda>0$ the first term in (4.10) tells us that $F(r)$ has a formal expansion

$$F(r) \sim \sum_{n,j,k\geqslant 0} \mathscr{C}_{jk}^n r^{\frac{n}{\lambda}+j+k}, \tag{4.12}$$

where $\mathscr{C}_{0,0}^n = a_n R(0,0)^{n/\lambda} e^{in\Theta(0,0)}$. Of course, the above formal expansion is understood as follows: if γ_n is a sequence of complex number satisfying $\Re\gamma_n \to +\infty$ as $n\to+\infty$, and if $F(r)$ is complex-valued function in $r\in\mathbb{R}_+$, denote $F(r)\sim\sum c_n r^{\gamma_n}$ if

$$F(r) - \sum_{\Re\gamma_n \leqslant \Re\gamma_N} c_n e^{\gamma_n \log r} = o(|r|^{\Re\gamma_N}).$$

One readily sees that the coefficients c_n are uniquely determined, whenever the formal expansion exists.

According to (4.11) and (4.12) we conclude that $\mathscr{C}^n_{0,0} = b_n$, that is,

$$a_n R^{n/\lambda}(0,0)e^{in\Theta(0,0)} = b_n.$$

This contradicts the divergence of the formal power expansion of $f(z)$ at $z = 0$ and consequently completes the proof of the theorem. □

In conclusion we mention that also in the case of global normalization we have no example which shows that our regularity results are optimal.

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