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Conformal maps and non-reversibility of elliptic area-preserving maps[⋆]

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1 Introduction

It has been long observed that area-preserving maps and reversible maps share similar results. This was certainly known to G.D. Birkhoff [5] who showed that these two types of maps have periodic orbits near a general elliptic fixed point. The KAM theory, developed by Kolmogorov-Arnold-Moser for Hamiltonian systems [9], [1] and area preserving maps [15], has also been extended a great deal to reversible systems and maps (see [16], [2], [21]). A natural question is if area-preserving maps and Hamiltonian systems are reversible.

In this paper we shall prove

Theorem 1.1. *There exist non-reversible elliptic real analytic area-preserving maps with eigenvalues not roots of unity.*

We want to mention that the Birkhoff fixed-point theorem and Moser invariant curves are applicable to some non-reversible area-preserving maps constructed in the proof of Theorem 1.1. In [8], the author proved that there exist reversible elliptic real analytic maps which admit first-integrals but cannot be transformed into the Moser-Webster normal form [17] by

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any convergent transformation; consequently, such reversible maps are not equivalent to area-preserving maps under any real analytic transformation.

Recall that a germ of real analytic map φ of $\mathbf{R}^2$ with $\varphi(0) = 0$ is *reversible* if $\varphi^{-1} = \tau\varphi\tau^{-1}$ for some real analytic involution $\tau(\tau^2 = \mathrm{Id})$ with $\tau(0) = 0$. Note that the proof of Theorem 1.1 will show the existence of area-preserving maps that are not *weakly* reversible in the sense of Arnol'd and Sevryuk [3]. We should mention that it is easy to see the non-reversibility with respect to a given involutive symmetry. Roughly speaking, the periodic points of a general elliptic area-preserving map which are symmetric with respect to a given involution would not survive under a small perturbation of area-preserving maps. To find non-reversible area-preserving maps, we have to look for the dynamics of area-preserving maps beyond the real space; namely, we shall seek the obstruction to the reversibility in the complexified space.

We start our construction with holomorphic symplectic maps of $\mathbf{C}^2$ of the form

$$\begin{aligned} \varphi\colon \xi' = {}&\lambda\xi f(\xi,\eta), \quad \eta' = \overline{\lambda}\eta g(\xi,\eta), \\ &|\lambda| = f(0) = g(0) = 1, \end{aligned} \tag{1.1}$$

where λ is not a root of unity, and f, g are holomorphic functions. Here a holomorphic symplectic map is meant to preserve $d\xi\wedge d\eta$. The holomorphic map φ (1.1) preserves the totally real space $\mathbf{R}^2\colon \eta = \overline{\xi}$ if and only if

$$g(\xi,\eta) = \overline{f(\overline{\eta},\overline{\xi})} \equiv \overline{f}(\eta,\xi). \tag{1.2}$$

It turns out that if φ is reversible, there exist convergent solutions T to the following functional equation

$$\overline{F}^{-1}(\xi) = T\circ F\circ T^{-1}(\xi), \quad T(0) = 0, \tag{1.3}$$

where the conformal map F is the first component of $\xi \to \varphi(\xi, 0)$.

We shall prove that Theorem 1.1 can be reduced to the following

Theorem 1.2. *There exist holomorphic functions* $F(\xi) = \lambda\xi + O(2)$ *with* λ *not a root of unity such that the conformal maps* $F, \overline{F}^{-1}$ *are not conjugate near the origin, i.e.,* (1.3) *has no convergent solution* T.

The proof of Theorem 1.2 is based on the observation that, when (1.3) has a convergent solution T, the repelling periodic orbits and attracting periodic orbits of the same period with comparable distance to the origin must occur in pairs, if they exist; see Proposition 5.1 below for details. The same observation will also allow us to show the following.

Theorem 1.3. *There exists a conformal map $F(z) = \lambda z + O(2)$ with λ not a root of unity such that F and $\overline{F}^{-1}$ are conjugate near the origin, while for each conformal map G commuting with F, periodic points of F, that are contained in the real analytic curve $\Gamma_G \subset \mathbf{C}\colon G(z) - \overline{z} = 0$ and have eigenvalues of absolute value one, accumulate at the origin.*

The functional equation (1.3) resembles the classical Schröder functional equation

$$R_\lambda = U \circ F \circ U^{-1}, \quad R_\lambda(\xi) = \lambda\xi, \tag{1.4}$$

arising from linearizing the conformal map F. Indeed, one can see that if the Schröder functional equation (1.4) has a convergent solution, all formal solutions T to (1.3) are convergent, assuming λ is not a root of unity. In particular, a theorem of Bruno [6], which generalizes a theorem of Siegel [22], implies that (1.3) has convergent solutions T when λ satisfies the Bruno condition.

We would like to mention that results of Moser [14] and Moser and Webster [17] imply that real analytic area-preserving maps are reversible near a hyperbolic fixed point, while hyperbolic reversible maps are locally equivalent to area-preserving maps. See also Devaney [7] for symmetric periodic orbits, homoclinic orbits, and other results about reversible maps and systems on smooth manifolds. The non-reversibilities of Hamiltonian systems and area-preserving maps have been studied by other people. Arnol'd and Sevryuk [3] constructed non-reversible Hamiltonian systems with degenerate eigenvalues. Formal obstructions to the reversibility of area-preserving maps were investigated by Quispel and Capel [18] for maps with a parabolic fixed point and by Roberts and Capel [19] for maps tangent to the identity. In some global aspect, Lamb [11] studied the persistence of non-reversibility of area-preserving flows and diffeomorphisms of $\mathbf{R}^2$, where one can also find Mather's example of non-reversible area-preserving flow of $\mathbf{R}^2$. The reader is referred to a recent survey of Lamb and Roberts [12] on the reversibility of dynamical systems.

The paper is organized as follows. Section 2 contains a simple proof of a complex version of Theorem 1.1. Using a theorem of Yoccoz [23] we then give an example of formally linearizable holomorphic symplectic map that is not weakly reversible. Section 2 also establishes the relationship between the reversibility of area-preserving maps and that of conformal maps. In Sect. 3 we discuss the reality condition (1.2) for area-preserving maps and complete the reduction from Theorem 1.1 to Theorem 1.2. Section 4 gives some estimates for the periodic orbits of a certain conformal map and their eigenvalues. Section 5 is devoted to the proofs of Theorem 1.2 and Theorem 1.3 by considering the hyperbolic and elliptic periodic orbits of conformal maps.

2 Normalizations

In this section we shall give a simple proof for a complex version of Theorem 1.1. This section also contains the main ingredients for the proof of Theorem 1.1. Throughout the paper, $[f]_k$ stands for the sum of all homogeneous terms of degree k of a formal power $f(\xi,\eta)$. We denote $f(\xi,\eta) \prec g(\xi,\eta)$ if f is majorized by g, i.e, if coefficients of f, g satisfy $|f_{\alpha\beta}| \leq g_{\alpha\beta}$ for all $\alpha, \beta \geq 0$.

Consider a holomorphic symplectic map of $\mathbf{C}^2$ defined by (1.1). Assume that λ is not a root of unity, i.e., $\lambda^n \neq 1$ for all positive integers n. Birkhoff [4] showed that there is a formal symplectic transformation $\Phi(\xi,\eta) = (\xi,\eta) + O(2)$, which is unique under a certain normalizing condition, such that

$$\hat{\varphi} = \Phi\varphi\Phi^{-1}\colon \xi = \xi\Lambda(\xi\eta), \quad \eta = \eta/\Lambda(\xi\eta), \tag{2.1}$$

where Λ is a formal power series in $\xi\eta$ with $\Lambda(0) = \lambda$. In particular, φ is formally reversible.

A smooth formal curve in $\mathbf{C}^2$ passing through the origin is defined as $\Gamma_f\colon f(\xi,\eta) = 0$, where f is a formal power series with $f(0) = 0, df(0) \neq 0$. Two such curves Γ_f, Γ_g are considered to be the same if $g = uf$ for some formal power series u ($u(0) \neq 0$). One says that Γ_f is invariant under a formal map φ if $\Gamma_{f\circ\varphi} = \Gamma_f$.

The following lemma shows that a holomorphic symplectic map has only two invariant smooth formal curves.

Lemma 2.1 (Birkhoff [4]). *A smooth formal curve, invariant under the formal map* (2.1) *of which λ is not a root of unity, is either ξ or η axis.*

Proof. Let Γ be defined by $f = 0$ with $f(0) = 0, df(0) \neq 0$. Then

$$f \circ \hat{\varphi}(\xi,\eta) = u(\xi,\eta)f(\xi,\eta). \tag{2.2}$$

The linear terms in (2.2) give

$$[f]_1(\lambda\xi, \overline{\lambda}\eta) = u(0)[f]_1(\xi,\eta).$$

Since λ is not a root of unity, it is clear that either $u(0) = \lambda$ and $[f]_1(\xi,\eta) = a\xi$ with $a \neq 0$, or $u(0) = \overline{\lambda}$ and $[f]_1(\xi,\eta) = a\eta$. Without loss of generality, we may assume that $[f]_1(\xi,\eta) = \xi$. Setting $\xi = 0$ in (2.2) yields

$$f(0, \overline{\lambda}\eta) = u(0,\eta)f(0,\eta).$$

If $f(0,\eta) = c_k\eta^k + O(k+1)$, then $\overline{\lambda}^k c_k = \lambda c_k$. Hence, $c_k = 0$. This shows that $f(\xi,\eta) = \xi v(\xi,\eta)$ with $v(0) = 1$; consequently, Γ is the ξ-axis. □

According to Arnol'd and Sevryuk [3], one says that a germ of real analytic transformation φ at the origin is *weakly reversible* if $\varphi^{-1} = \tau\varphi\tau^{-1}$ for some real analytic transformation τ which is not necessary to be an involution.

Lemma 2.2. *Let φ be a holomorphic map defined by (1.1). Assume that λ is not a root of unity and $\varphi^{-1} = \tau\varphi\tau^{-1}$ for some formal map τ with $\tau(0) = 0$. Then τ interchanges ξ and η axes. Moreover,*

$$G^{-1} = T \circ F \circ T^{-1} \tag{2.3}$$

with F the first component of $\varphi(\xi, 0)$, G the second component of $\varphi(0, \eta)$, and T the second component of $\tau(\xi, 0)$.

Proof. From Lemma 2.1 it follows that either τ preserves both coordinate axes, or τ interchanges the two axes. The former cannot occur since the linear parts of φ^{-1} and φ, when restricted to the ξ-axis, are not conjugate. It is straightforward that $G^{-1} = T \circ F \circ T^{-1}$. ☐

In what follows, when f is a power series in two variables, f_ξ, f_η are meant to be the partial derivatives with respect to the first and second variables, respectively. More explicitly,

$$f_\xi(\xi, \eta) = \sum \alpha f_{\alpha\beta}\xi^{\alpha-1}\eta^\beta,$$
$$f_\xi(\eta, \xi) = (f_\xi)(\eta, \xi) = \sum \alpha f_{\alpha\beta}\eta^{\alpha-1}\xi^\beta, \text{ etc.}$$

Lemma 2.3. *Let $f(\xi, \eta)$ be a holomorphic function with $f(0) = 1$. Then there exists a unique holomorphic function $g(\xi, \eta)$ with $g(0) = 1$ such that (1.1) is a holomorphic symplectic map. Moreover, if $f|_{\xi=0} = 1$, then $g|_{\xi=0} = 1$.*

Proof. Put $\tilde{f}(\xi, \eta) = \xi f(\xi, \eta)$ and $\tilde{g}(\xi, \eta) = \eta g(\xi, \eta)$. Obviously, φ, given by (1.1), preserves $d\xi \wedge d\eta$ if and only if $\tilde{g}$ is a solution to

$$\tilde{f}_\xi(\xi, \eta)\tilde{g}_\eta(\xi, \eta) - \tilde{g}_\xi(\xi, \eta)\tilde{f}_\eta(\xi, \eta) = 1, \quad \tilde{g}|_{\eta=0} = 0. \tag{2.4}$$

By the Cauchy-Kowalewski theorem, the above equations have a unique holomorphic solution $\tilde{g}$. Assume that $f|_{\xi=0} = 1$. Then $\tilde{f}_\xi|_{\xi=0} = 1$ and $\tilde{f}_\eta|_{\xi=0} = 0$. Thus, (2.4) implies that $\tilde{g}_\eta|_{\xi=0} = 1$. This shows that $g|_{\xi=0} = 1$. ☐

We are ready to prove the following

Theorem 2.4. *Let φ be the holomorphic symplectic map (1.1) with λ not a root of unity. Assume that $f|_{\xi=0} = 1$. If φ is weakly reversible, the conformal map $F\colon \xi \to \lambda\xi f(\xi, 0)$ is linearizable by convergent transformations.*

Proof. Assume that φ is weakly reversible with respect to a holomorphic transformation τ. Under the assumptions, Lemma 2.3 says that $g(0,\eta)=1$, i.e., the second component of $\varphi(0,\eta)$ is already a linear transformation. Therefore, Theorem 2.4 follows from Lemma 2.2. □

Example. Consider the symplectic rational map

$$\varphi_\lambda\colon \xi\to\lambda\xi(1+\xi),\quad \eta\to\overline{\lambda}\eta/(1+2\xi)$$

with λ not a root of unity. By a theorem of Yoccoz [23], the conformal map $\xi\to\lambda\xi(1+\xi)$ is not linearizable when λ does not satisfy the Bruno condition. Theorem 2.4 implies that for such λ, φ_λ is not weakly reversible near the origin.

Remarks. (a) By Lemma 2.3 one can see that φ_λ is linearizable by the formal map

$$\xi\to\xi U_\lambda(\xi),\quad \eta\to\eta/U_\lambda'(\xi),$$

where $U_\lambda(\xi)=\xi+O(2)$ is the formal transformation linearizing $\xi\to\lambda\xi(1+\xi)$. If λ is not a root of unity and violates the Bruno condition then φ_λ is not linearizable by any convergent transformation.

For the proof, assume that a holomorphic map $\varPhi$, which is not necessarily symplectic, linearizes φ_λ. Substituting $(D\varPhi)^{-1}(0)\varPhi$ for $\varPhi$, one may assume that $\varPhi(\xi,\eta)=(\xi,\eta)+O(2)$. By Lemma 2.1, $\varPhi$ preserves $\xi=0$ and $\eta=0$. Now one readily sees that $\xi\to\lambda\xi(1+\xi)$ is linearizable by the first component of $\xi\to\varPhi(\xi,0)$, which is a contradiction.

(b) To author's knowledge, it remains open if there exists a formally linearizable real analytic area-preserving map of $\mathbf{R}^2$ that is not linearizable by any convergent (area-preserving) map. However, see a result of Rüssmann [20] about the convergence of linearization under a Diophantine condition.

We now turn to holomorphic symplectic map φ (1.1) satisfying the reality condition (1.2). Following notations in Lemma 2.2, the functional equation (2.3) becomes (1.3).

We need some standard facts about the linearization of conformal maps. Let $F\colon \xi\to\lambda\xi+O(2)$ be a conformal map with λ not a root of unity. Then there exists a unique formal transformation U such that

$$U\circ F\circ U^{-1}=R_\lambda,\quad U(\xi)=\xi+O(2). \tag{2.5}$$

The only formal power series g with $g\circ R_\lambda=R_\lambda\circ g$ are μz with $\mu\in\mathbf{C}$. Denote by $\mathcal{C}_F$ the set of centralizers of F, i.e., the set of conformal maps G with $G\circ F\circ G^{-1}=F$ and $G'(0)\neq 0$. Then $\mathcal{C}_F$ is a group containing F^n for $n=0,\pm1,\pm2,\dots$. If G is a centralizer, then $U\circ G\circ U^{-1}$, commuting with R_λ, is a linear transformation. When F is not linearizable, $\mathcal{C}_F$ consists of convergent transformations $U^{-1}\circ R_\mu \circ U$ with $|\mu| = 1$.

We now introduce some notations. Given a conformal map $F\colon \xi \to \lambda\xi + O(2)$ with λ not a root of unity, let U be the formal map satisfying (2.5). Define $\mathcal{C}'_F$ to be the set of convergent maps

$$\overline{U}^{-1}\circ R_\mu \circ U(\xi) \tag{2.6}$$

for $\mu \in \mathbf{C}^*$, and $\mathcal{I}_F \subset \mathcal{C}'_F$ to be the set of such convergent maps with $|\mu| = 1$. Notice that when $\mathcal{C}'_F$ is nonempty, $\mathcal{C}'_F$ and $\mathcal{C}_F$ admit a non-canonical identification

$$T = \overline{U}^{-1}\circ R_\mu \circ U \in \mathcal{C}'_F \mapsto \overline{T}_0 \circ T = U^{-1}\circ R_{\overline{\mu}_0\mu}\circ U \in C_F, \tag{2.7}$$

where $T_0 = \overline{U}^{-1}\circ R_{\mu_0}\circ U \in \mathcal{C}'_F$.

Proposition 2.5. *Let $F\colon \xi \to \lambda\xi + O(2)$ be a conformal map with λ not a root of unity. Then $\mathcal{I}_F$ consists of conformal maps T satisfying*

$$\overline{F}^{-1} = T\circ F\circ T^{-1}, \quad |T'(0)| = 1. \tag{2.8}$$

Proof. Take $T \in \mathcal{I}_F$. From $F = U^{-1}\circ R_\lambda\circ U$, it follows that $T\circ F\circ T^{-1} = \overline{F}^{-1}$. Conversely, if T satisfies (2.8), the first identity in (2.8) implies that

$$\overline{U}^{-1}\circ R_\lambda\circ \overline{U} = \overline{F}^{-1} = T\circ U^{-1}\circ R_\lambda \circ U\circ T^{-1}. \tag{2.9}$$

Hence, $T = \overline{U}^{-1}\circ R_\mu\circ U$ for some $\mu \in \mathbf{C}^*$. Now $|T'(0)| = 1$ implies that $|\mu| = 1$. The lemma is proved. □

Notice that Proposition 2.5 implies that $(T\circ F)\circ(\overline{T}\circ\overline{F}) = \mathrm{Id} = (\overline{F}\circ T)\circ(F\circ\overline{T})$. Thus (2.8) is equivalent to the decomposition

$$F = T_1\circ T_2, \quad T_j\circ \overline{T}_j = \mathrm{Id}, \quad j = 1, 2, \tag{2.10}$$

represented by

$$F = \overline{T}\circ(T\circ F) = (F\circ\overline{T})\circ T.$$

Also, (2.10) yields

$$\overline{F}^{-1} = T_1^{-1}\circ F\circ T_1 = T_2\circ F\circ T_2^{-1}. \tag{2.11}$$

In light of (2.8) and (2.10) we say that a conformal map F is *reversible* if (2.8) or (2.10) holds.

Lemma 2.6. *Let $F\colon \xi \to \lambda\xi + O(2)$ be a non-linearizable conformal map. If λ is not a root of unity, then F is not conjugate to $\overline{F}^{-1}$ by any conformal map $T(z) = \mu z + O(2)$ with $|\mu| \neq 1$.*

Proof. Assume for the sake of contradiction that $\overline{F}^{-1} = T \circ F \circ T^{-1}$ with T given in the lemma. Let U be the formal map satisfying (2.5). Then (2.9) implies that $\overline{U} \circ T \circ U^{-1} = R_\mu$. Substituting $\overline{F}^{-1}, T^{-1}$ for F, T if necessary, one obtains

$$\overline{U}^{-1}(\mu\xi) = T(U^{-1}(\xi)), \quad |\mu| > 1.$$

It is clear that $\overline{U}^{-1}$ is majorized by $U^*(\xi) = \xi + O(2)$, where U^* is the formal solution to

$$U^*(|\mu|\xi) = \hat{T}(U^*(\xi))$$

with

$$\hat{T}(\xi) = |\mu|\xi + \sum_{j>1} |T_j|\xi^j.$$

Note that $U^* \circ T \circ U^{*-1}(\xi) = |\mu|\xi$. A theorem of Koenigs [10] says that U^*, and hence U, converges. The contradiction yields the proof of the lemma. $\square$

Proposition 2.7. *Let φ be a real analytic area-preserving map defined by* (1.1)-(1.2) *in which λ is not a root of unity. Let $F(\xi)$ be the first component of $\varphi(\xi, 0)$. Consider the statements: (i) φ is weakly reversible. (ii) $\overline{F}^{-1}$ and F are conjugate by a convergent transformation. (iii) $\mathcal{I}_F$ is nonempty. (iv) C'_F is nonempty. Then (i) implies (ii), while (ii)-(iv) are equivalent.*

Proof. Let U be the formal map satisfying (2.5).

(i) $\Rightarrow$ (ii) This follows from (2.3) and $G = \overline{F}$.

(ii) $\Rightarrow$ (iii) Assume that $\overline{F}^{-1} = T \circ F \circ T^{-1}$ for a convergent transformation T. Then (2.9) implies that $\overline{U} \circ T \circ U^{-1} = R_\mu$. If $|\mu| = 1$, then T is contained in $\mathcal{I}_F$. If $|\mu| \neq 1$, Lemma 2.6 says that U converges. Then $\mathcal{I}_F$ contains all $\overline{U}^{-1} \circ R_\mu \circ U$ for $|\mu| = 1$.

(iii) $\Rightarrow$ (iv) It is trivial.

(iv) $\Rightarrow$ (ii) Assume that $\overline{U}^{-1} \circ R_\mu \circ U$ is convergent. From $F = U^{-1} \circ R_\lambda \circ U$ it follows that

$$\overline{F}^{-1} = \overline{U}^{-1} \circ R_\lambda \circ \overline{U} = (\overline{U}^{-1} \circ R_\mu \circ U) \circ F \circ (\overline{U}^{-1} \circ R_\mu \circ U)^{-1}.$$

Thus $\overline{F}^{-1}$ and F are conjugate. $\square$

Now, Theorem 1.1 follows from Theorem 1.2, Proposition 2.7, and Lemma 3.2 of the next section.

3 Reality conditions

In this section we shall complete the reduction from Theorem 1.1 to Theorem 1.2.

Consider holomorphic maps (1.1)-(1.2), which preserve the totally real space $\mathbf{R}^2\colon \eta = \overline{\xi}$. Given a holomorphic function $f(\xi,\eta)$ with $f(0)=1$, decompose

$$f(\xi,\eta) = f^+(\xi,\eta) + f^-(\xi,\eta), \quad f^+(0)=1, \quad f^-(0)=0 \tag{3.1}$$

with

$$f^+ = (f(\xi,\eta) + \overline{f}(\eta,\xi))/2, \quad f^- = (f(\xi,\eta) - \overline{f}(\eta,\xi))/2.$$

Notice that f^+, f^- are uniquely determined by the conditions

$$\overline{f}^+(\xi,\eta) = f^+(\eta,\xi), \quad f^+(0)=1; \tag{3.2}$$

$$\overline{f}^-(\xi,\eta) = -f^-(\eta,\xi), \quad \Im\{f^-(0)\}=0. \tag{3.3}$$

Now (1.2) becomes

$$g(\xi,\eta) = f^+(\xi,\eta) - f^-(\xi,\eta). \tag{3.4}$$

Lemma 3.1. *Let f^- be a holomorphic function satisfying* (3.3)*. Then there exists a unique holomorphic function f^+ satisfying* (3.2) *such that* (1.1), (3.1) *and* (3.4) *define a holomorphic symplectic map preserving* $\mathbf{R}^2\colon \eta = \overline{\xi}$.

Proof. Consider a holomorphic map φ given by (1.1)-(1.2). Put

$$f(\xi,\eta) = 1 + \hat{f}(\xi,\eta), \quad g(\xi,\eta) = 1 + \hat{g}(\xi,\eta).$$

For brevity, denote $\hat{f}, \hat{g}$ by f, g, respectively. The identity $\det \frac{\partial\varphi}{\partial(\xi,\eta)} = 1$ is equivalent to

$$f + \xi f_\xi + g + \eta g_\eta = \xi\eta f_\eta g_\xi - (f + \xi f_\xi)(g + \eta g_\eta).$$

By (3.1) and (3.4) one gets the functional equation

$$2f^+ + \xi f^+_\xi + \eta f^+_\eta = \eta f^-_\eta - \xi f^-_\xi - (f + \xi f_\xi)(g + \eta g_\eta) + \xi\eta f_\eta g_\xi. \tag{3.5}$$

One needs to prove that (3.5) has a unique convergent solution f^+ satisfying (3.2) with $f^+(0)=0$.

In homogeneous terms, (3.5) can be rewritten as

$$\begin{aligned}(2+k)[f^+]_k &= \eta[f^-_\eta]_{k-1} - \xi[f^-_\xi]_{k-1} \\ &\quad - \sum_{i+j=k;\, i,j>0} ([f + \xi f_\xi]_i \cdot [g + \eta g_\eta]_j - [\eta f_\eta]_i \cdot [\xi g_\xi]_j).\end{aligned} \tag{3.6}$$

This shows that given f^- there exists a unique formal solution f^+ to (3.5) with $f^+(0)=0$.

Next, we want to show that the formal power series f^+ satisfies $\overline{f}^+(\xi,\eta) = f^+(\eta,\xi)$. We have $[f^-]_1(\xi,\eta) = a\xi - \overline{a}\eta$. From (3.6) it follows that $[f^+]_1(\xi,\eta) = -\frac{1}{3}(\overline{a}\eta + a\xi)$. Hence, $[\overline{f}^+]_1(\xi,\eta) = [f^+]_1(\eta,\xi)$. Assume for induction that $[\overline{f}^+]_j(\xi,\eta) = [f^+]_j(\eta,\xi)$ holds for $j<k$. Then

$$[g]_j(\xi,\eta) = [\overline{f}]_j(\eta,\xi) = [f^+]_j(\xi,\eta) - [f^-]_j(\xi,\eta).$$

Note that $\overline{f}(\xi,\eta) = g(\eta,\xi)$ implies that

$$(\overline{f}_\xi)(\xi,\eta) = \sum \beta g_{\alpha\beta}\eta^\alpha\xi^{\beta-1} = g_\eta(\eta,\xi), \quad (\eta g_\eta)(\eta,\xi) = \xi(g_\eta(\eta,\xi)).$$

Conjugating (3.6) yields

$$\begin{aligned}
&(2+k)[\overline{f}^+]_k(\xi,\eta)\\
&\quad = \eta[\overline{f}^-_\eta]_{k-1}(\xi,\eta) - \xi[\overline{f}^-_\xi]_{k-1}(\xi,\eta)\\
&\qquad - \sum_{i+j=k;i,j>0} ([\overline{f}+\xi\overline{f}_\xi]_i\cdot[\overline{g}+\eta\overline{g}_\eta]_j - [\eta\overline{f}_\eta]_i\cdot[\xi\overline{g}_\xi]_j)(\xi,\eta)\\
&\quad = -\eta[f^-_\xi]_{k-1}(\eta,\xi) + \xi[f^-_\eta]_{k-1}(\eta,\xi)\\
&\qquad - \sum_{i+j=k;i,j>0} ([g+\eta g_\eta]_i\cdot[f+\xi f_\xi]_j - [\xi g_\xi]_i\cdot[\eta f_\eta]_j)(\eta,\xi)\\
&\quad = (2+k)[f^+]_k(\eta,\xi).
\end{aligned}$$

Finally, we need to show the convergence of f^+. Put

$$h^\pm(\xi,\eta) = \sum_{k>0}(k+2)\sum_{\alpha+\beta=k}|f^\pm_{\alpha\beta}|\xi^\alpha\eta^\beta.$$

Obviously, we have

$$\eta f^\pm_\eta - \xi f^\pm_\xi \prec h^\pm; \quad f+\xi f_\xi, g+\eta g_\eta, \eta f_\eta, \xi g_\xi \prec h^+ + h^-.$$

Now (3.6) yields

$$h^+ \prec h^- + 2(h^+ + h^-)^2.$$

Thus h^+ is majorized by the convergent solution $w = w(\xi,\eta)$ to

$$w = h^-(\xi,\eta) + 2(w + h^-(\xi,\eta))^2, \quad w(0)=0.$$

Therefore, h^+, and hence f^+, is convergent. This prove the lemma. □

We now show the following realization lemma, by which the reduction from Theorem 1.2 to Theorem 1.1 is complete.

Lemma 3.2. *Let $p(\xi)$ be a holomorphic function with $p(0) = 1$. There exists a unique holomorphic function $q(\xi)$ with $q(0) = 0$ such that $\xi \to \lambda\xi p(\xi)$ is the first component of $\varphi(\xi, 0)$, where φ is the area-preserving map corresponding to $f^-(\xi,\eta) = q(\xi) - \overline{q}(\eta)$. Furthermore, the first component of $\tilde{\varphi}(\xi, 0)$ is the same as that of $\varphi(\xi, 0)$, if $\tilde{\varphi}$ is the real analytic area-preserving map corresponding to $q(\xi) - \overline{q}(\eta) + \xi\eta h(\xi, \eta)$ with h a convergent power series satisfying $\overline{h}(\xi, \eta) = -h(\eta, \xi)$.*

Proof. Following the notations in the proof of Lemma 3.1, put $f = 1 + \hat{f}, g = 1 + \hat{g}$. For brevity, denote $\hat{f}, \hat{g}$ by f, g. Setting $\eta = 0$ in (3.5) yields the functional equation

$$\begin{aligned}(3.7)\quad 2f^+(\xi,0) + \xi f^+_\xi(\xi,0) = -\xi f^-_\xi(\xi,0) - \{f^+(\xi,0) + f^-(\xi,0)\\ + \xi(f^+_\xi(\xi,0) + f^-_\xi(\xi,0))\} \cdot \{f^+(\xi,0) - f^-(\xi,0)\}.\end{aligned}$$

One sees that the coefficients of $f^+(\xi, 0)$ are determined uniquely by $f^-(\xi, 0)$, i.e., $\tilde{\varphi}(\xi, 0) = \varphi(\xi, 0)$.

Let $p(\xi) = 1 + \tilde{p}(\xi)$ be a holomorphic function with $\tilde{p}(0) = 0$. One readily sees that there exists a unique holomorphic solution q to

$$(3.8)\qquad -2q(\xi) + 2\tilde{p}(\xi) + \xi\tilde{p}'(\xi) = -(\tilde{p}(\xi) + \xi\tilde{p}'(\xi))(\tilde{p}(\xi) - 2q(\xi)).$$

Put $f^-(\xi,\eta) = q(\xi) - \overline{q}(\eta)$ and $k(\xi) = \tilde{p}(\xi) - q(\xi)$. Applying the substitutions

$$q(\xi) = f^-(\xi, 0), \quad \tilde{p}(\xi) = k(\xi) + f^-(\xi, 0)$$

to (3.8), one gets

$$\begin{aligned}2k(\xi) + \xi k_\xi(\xi) = -\xi f^-_\xi(\xi,0) - \{k(\xi) + f^-(\xi,0)\\ + \xi(k_\xi(\xi) + f^-_\xi(\xi,0))\} \cdot \{k(\xi) - f^-(\xi,0)\}.\end{aligned}$$

Comparing with (3.7), one sees that $k(\xi)$ is the unique solution $f^+(\xi, 0)$ to (3.5). This shows that $f^+(\xi, 0) = k(\xi) = \tilde{p}(\xi) - q(\xi)$. Therefore, $f(\xi, 0) = p(\xi)$ as desired. □

Note that the claim in the introduction, that the Birkhoff fixed-point theorem and Moser invariant curves are still applicable to some non-reversible area-preserving maps, follows from Proposition 2.7, Lemma 3.2, and Theorem 1.2 also.

Anticipating the proof of Theorem 1.2 in Sect. 5 let us record here an argument. Take a conformal map $F\colon \xi \to \lambda\xi p(\xi)$ which is not conjugate to $\overline{F}^{-1}$. Lemma 3.2 says that there exists a holomorphic function $q(\xi)$ with $q(0) = 1$ such that F is the first component of $\varphi_c(\xi, 0)$, where φ_c is the real analytic area-preserving map corresponding to

$$f^-(\xi,\eta) = q(\xi) - \overline{q}(\eta) + ic\xi\eta, \quad c \in \mathbf{R}.$$

Proposition 2.7 implies that φ_c is not weakly reversible. Write

$$f^-(\xi,\eta)=a\xi+b\xi^2+ic\xi\eta-\overline{a}\eta-\overline{b}\eta^2+O(3).$$

Returning to (3.6) one readily sees that $[f^+]_1,[f^+]_2$ are independent of c. Hence, $f(\xi,\eta)=1+ic\xi\eta+\dots$, where the omitted terms are either independent of c or of order >2. Now a straightforward computation shows that $\Lambda(\xi\eta)$ in the Birkhoff normal form (2.1) has the expansion $\Lambda=\lambda+(ic+\tilde{c})\xi\eta+O(|\xi\eta|^2)$, where $\tilde{c}$ depends only on $a,b,\overline{a},\overline{b}$. For $ic+\tilde{c}\neq 0$, φ_c is not formally linearizable, i.e., it is an area-preserving map with a general elliptic fixed point at the origin. Therefore, the Birkhoff fixed-point theorem and Moser invariant curves are applicable to φ_c for $ic+\tilde{c}\neq 0$.

4 Estimates of periodic orbits and eigenvalues

In this section we shall establish the existence of periodic orbits and estimate the eigenvalues of conformal maps under suitable conditions. The method in this section was used previously by Moser [13] who showed the existence of hyperbolic and elliptic periodic orbits of general elliptic area-preserving maps. See also a paper of Zehnder [24] for the homoclinic points of such maps.

Set

$$\mathbf{a}_n=(a_2,\dots,a_n),\quad \mathbf{a}_n^+=(a_2,\dots,\widehat{a_{n+1}},\dots,a_{2n}),$$
$$\mathbf{a}=(a_2,a_3,\dots),\quad \mathbf{d}_n=\mathbf{d}_n(\lambda)=\left(\frac{1}{\lambda-\lambda^2},\dots,\frac{1}{\lambda-\lambda^n}\right),$$
$$\mathbf{d}_n^+=\mathbf{d}_n^+(\lambda)=\left(\frac{1}{\lambda-\lambda^2},\dots\widehat{\frac{1}{\lambda-\lambda^{n+1}}},\dots\frac{1}{\lambda-\lambda^{2n}}\right),$$
$$I_n=\{i;2\le i\le 2n,i\neq n+1\},\quad \delta_n=\delta_n(\lambda)=\min_{k\in I_n}\{|\lambda-\lambda^k|\}.$$

Let $\mathcal{F}$ be the set of conformal mappings

$$F(z)=\lambda z+\sum_{k>1}a_kz^k,\quad \lambda^k\neq 1,\quad |a_k|<1,\quad k=2,3,\dots. \tag{4.1}$$

For each $F\in\mathcal{F}$ there exists a unique polynomial

$$U_n=z+\sum_{k=2}^{2n}u_kz^k,\quad u_n=0 \tag{4.2}$$

such that $F_n(z)=U_n\circ F\circ U_n^{-1}(z)=\lambda z+A_{n,1}z^{n+1}+O(2n+1)$. Note that

$$\begin{aligned} u_k &= \frac{1}{\lambda-\lambda^k}\{a_k + u_k^*(\mathbf{a}_{k-1}, u_2, \dots, u_{k-1}, \lambda)\} \\ &= \frac{1}{\lambda-\lambda^k}\{a_k + \tilde{u}_k(\mathbf{a}_{k-1}, \mathbf{d}_n^+, \lambda)\}, \end{aligned} \tag{4.3}$$

where $u_k^*, \tilde{u}_k$ are polynomials of integer coefficients. Thus

$$F_n \colon z_1 = \lambda z + A_{n,1} z^{n+1} + B_{n,1} z^{2n+1} + E_{n,1}(z), \tag{4.4}$$

$$A_{n,1} = a_{n+1} + \tilde{a}_n(\mathbf{a}_n), \quad B_{n,1} = a_{2n+1} + \tilde{a}_{2n+1}(\mathbf{a}_n^+), \tag{4.5}$$

$$E_{n,1}(z) = z^{2n+2} E'_{n,1}(z, \mathbf{a}). \tag{4.6}$$

Note that for (4.4) to hold it suffices that $\delta_n(\lambda) \neq 0$. This will be used later on. Note also that $\tilde{a}_n(\mathbf{a}), \tilde{a}_{2n+1}(\mathbf{a}_n^+)$, and $E'_{n,1}(z, \mathbf{a})$ depend on λ and $\mathbf{d}_n^+$ also. For simplicity, such dependence will not be expressed within formulae unless an emphasis is needed.

Lemma 4.1. *Let $D(r) = \{z \in \mathbf{C} | |z| < r\}$. There exists $d_n > 1$ such that if $n > 16$ and $F \in \mathcal{F}$, then*

$$F, U_n, U_n^{-1} \colon D(r) \to D((3/2)^{\frac{1}{n}} r), \quad r \leq 8\delta_n^{d_n}. \tag{4.7}$$

Proof. Note that

$$\delta_n \leq 2\sin(\pi/n) < 2\pi/n < 1/2, \quad n > 16.$$

Since $F \prec z/(1-z)$, it is clear that (4.7) holds for F if d_n is sufficiently large. From (4.3) it follows that

$$U_n(z) \prec z + \delta_n^k(z^2 + \cdots + z^{2n}) \prec z + \delta_n^k \frac{z^2}{1-z}$$

for a large k. This shows that (4.7) holds for U_n for sufficiently large d_n. One has

$$U_n^{-1}(z) = z - U_n(U_n^{-1}(z)) + U_n^{-1}(z).$$

Hence, $U_n^{-1} \prec U_n^*$ for

$$U_n^*(z) = z + \delta_n^k \frac{U_n^{*2}(z)}{1 - U_n^*(z)}.$$

Clearly, (4.7) holds for U_n^* for some large d_n. This shows that (4.7) is valid for some large d_n. □

Put $D^\infty = D(1) \times D(1) \dots$. For a function $f(z, \mathbf{a})$, define

$$\|f\|_{D(r)\times D^\infty} = \sup\{|f(z, \mathbf{a})|; z \in D(r), \mathbf{a} \in D^\infty\}.$$

Note that $E'_{n,1}(z, a)$ given in (4.6), for instance, depends on λ. Thus, the norm $\|E'_{n,1}\|_{D(r)\times D^\infty}$ depends on λ also.

Lemma 4.2. *Let* F *be given by* (4.1) *with* $\delta_n(\lambda) \neq 0$*, and* F_n *be given by* (4.4)-(4.6). *The iterate* $z_k = F_n^k(z)$ *can be expressed as*

$$z_k = \lambda^k z + A_{n,k} z^{n+1} + B_{n,k} z^{2n+1} + E_{n,k}(z), \quad 1 \leq k \leq n \tag{4.8}$$

with

$$A_{n,k} \equiv A_{n,k}(\mathbf{a}) = \lambda^{k-1} \frac{1-\lambda^{nk}}{1-\lambda^n} A_{n,1}(\mathbf{a}) \tag{4.9}$$

$$= k\lambda^{k-1} A_{n,1}(\mathbf{a}) + (1-\lambda^n)\tilde{A}_{n,k}(\mathbf{a}), \tag{4.10}$$

$$B_{n,k} \equiv B_{n,k}(\mathbf{a}) = k\lambda^{k-1} B_{n,1}(\mathbf{a}) + (n+1)\frac{k(k-1)}{2}\lambda^{k-2} A_{n,1}^2(\mathbf{a}), \tag{4.11}$$

$$E_{n,k} = z^{2n+1}(zE'_{n,k}(z,\mathbf{a}) + (1-\lambda^n)E''_{n,k}(z,\mathbf{a})) \tag{4.12}$$

and

$$\begin{aligned} \|(A_{n,k}, \tilde{A}_{n,k}, B_{n,k})\|_{D^\infty} &\leq \delta_n^{-d'_n}, \\ \|(E'_{n,k}, E''_{n,k})\|_{D(\delta_n^{d_n}) \times D^\infty} &\leq \delta_n^{-d'_n} \end{aligned} \tag{4.13}$$

for some $d'_n > d_n$.

Proof. By (4.7) one knows that F_n^k maps $D(2\delta_n^{d_n})$ into $D(4\delta_n^{d_n})$ for $1 \leq k \leq n$. The Cauchy inequality implies that

$$\|A_{n,1}\|_{D^\infty} \leq 4\delta_n^{d_n}/(2\delta_n^{d_n})^{n+1} < \delta_n^{-d'_n}$$

for some large d'_n. One can get similar estimate for $B_{n,1}$ for a possibly larger d'_n. Note that

$$1 - \lambda^{jn} = (1-\lambda^n)\sum_{i=0}^{j-1} \lambda^{in},$$

$$\frac{1-\lambda^{nk}}{1-\lambda^n} - k = -(1-\lambda^n)\sum_{j=1}^{k-1}\sum_{i=0}^{j-1} \lambda^{in}.$$

From (4.9) we see that

$$\tilde{A}_{n,k} = -\lambda^{k-1} A_{n,1} \sum_{j=1}^{k-1}\sum_{i=0}^{j-1} \lambda^{in}.$$

Thus

$$\|\tilde{A}_{n,k}\|_{D^\infty} \leq (k-1)^2 \|A_{n,1}\|_{D^\infty}.$$

Take $E''_{n,1}=0$. Now (4.4) and (4.7) yield

$$
\begin{aligned}
(2\delta_n^{d_n})^{(2n+2)}\|E'_{n,1}\|_{D(2\delta_n^{d_n})\times D^\infty} &= \|E_{n,1}\|_{D(2\delta_n^{d_n})\times D^\infty} \\
&\le 1+\|A_{n,1}\|_{D^\infty}+\|B_{n,1}\|_{D^\infty}\le \delta_n^{-d'_n}
\end{aligned}
\tag{4.14}
$$

for some large d'_n. Thus we obtain (4.9)-(4.13) for $k=1$.

For induction, we assume that (4.9)-(4.13) hold and then prove that they hold for a possibly larger d'_n when k is in place of $k+1(\le n)$. Start with

$$
\begin{aligned}
z_{k+1} = \lambda(\lambda^k z &+ A_{n,k}z^{n+1}+B_{n,k}z^{2n+1}) \\
&+ A_{n,1}(\lambda^k z+A_{n,k}z^{n+1})^{n+1}+B_{n,1}(\lambda^k z)^{2n+1} \\
&+ z^{2n+1}(z\tilde{E}'_{n,k+1}(z,\mathbf{a})+(1-\lambda^n)\tilde{E}''_{n,k+1}(z,\mathbf{a})),
\end{aligned}
\tag{4.15}
$$

where $E'_{n,k+1}$ and $\tilde{E}''_{n,k+1}$ are polynomials in

$$
\lambda, z, \tilde{A}_{n,1}, B_{n,1}, \tilde{A}_{n,k}, B_{n,k}, E'_{n,k}(z,\mathbf{a}), E''_{n,k}(z,\mathbf{a}), E'_{n,1}(z_k,\mathbf{a})
\tag{4.16}
$$

with integer coefficients. Note that F_n^k maps $D(\delta_n^{d_n})$ into $D(2\delta_n^{d_n})$. Hence, (4.14) implies that

$$
\|E'_{n,1}(z_k(\cdot,\cdot\cdot),\cdot\cdot)\|_{D(\delta_n^{d_n})\times D^\infty}\le \delta_n^{-d'_n}
$$

for some large d'_n. Thus, $\tilde{E}'_{n,k+1}$ and $\tilde{E}''_{n,k+1}$ satisfy (4.13) for some large d'_n. By (4.9), the coefficient of z^{n+1} on the right-hand side of (4.15) equals

$$
\lambda A_{n,k}+A_{n,1}\lambda^{k(n+1)}=\lambda^k\frac{1-\lambda^{nk}}{1-\lambda^n}A_{n,1}+\lambda^{(n+1)k}A_{n,1},
$$

which is $A_{n,k+1}$, defined by (4.9) with $k+1$ in place of k. The coefficient of z^{2n+1} of the combined first 3 terms on the right-hand side of (4.15) is

$$
\begin{aligned}
\lambda B_{n,k}&+\lambda^k B_{n,1}+(n+1)A_{n,k}A_{n,1}+(1-\lambda^n)\tilde{B}_{n,k+1} \\
&=(k+1)\lambda^k B_{n,1}+(n+1)\frac{k(k+1)}{2}\lambda^{k-1}A_{n,1}^2 \\
&\quad+(n+1)(1-\lambda^n)\tilde{A}_{n,k}A_{n,1}+(1-\lambda^n)\tilde{B}_{n,k+1},
\end{aligned}
\tag{4.17}
$$

where $\tilde{B}_{n,k+1}$ is a polynomial in λ, z, $\tilde{A}_{n,k}$, $B_{n,k}$, and $B_{n,1}$. Note that (4.17) is obtained by using (4.10) and (4.11). Obviously, the two terms in (4.17) become $B_{n,k+1}$, as defined by (4.11), while the remaining two terms are absorbed into $\tilde{E}''_{n,k+1}$. One readily sees (4.12) with $E'_{n,k+1}$, $E''_{n,k+1}$ being polynomials in quantities (4.16) with integer coefficients. Also $A_{n,k+1}$, $\tilde{A}_{n,k+1}$, and $B_{n,k+1}$ are polynomials in (4.16). This proves (4.13) with $k+1$ in place of k. The lemma is proved by induction. □

Note that

$$A_n \equiv A_{n,n} = \lambda^{n-1}\frac{1-\lambda^{n^2}}{1-\lambda^n}A_{n,1} = n\overline{\lambda}A_{n,1} + (1-\lambda^n)\tilde{A}_n,$$
$$B_{n,n} = B_n + (1-\lambda^n)\tilde{B}_n$$

for

$$B_n = n\overline{\lambda}B_{n,1} + \frac{n(n^2-1)}{2}\overline{\lambda}^2 A_{n,1}^2.$$

Rewrite $U_n \circ F^n \circ U_n^{-1}$ as

$$F_n^n\colon z_n = \lambda^n z + A_n z^{n+1} + B_n z^{2n+1} + E_n(z) \tag{4.18}$$

with

$$E_n(z) = z^{2n+1}(zE_n'(z,\mathbf{a}) + (1-\lambda^n)E_n''(z,\mathbf{a})). \tag{4.19}$$

The estimates (4.13) then read

$$\|(A_n, B_n)\|_{D^\infty} \leq \delta_n^{-d_n'}, \quad \|(E_n', E_n'')\|_{D(\delta_n^{d_n})\times D^\infty} \leq \delta_n^{-d_n'} \tag{4.20}$$

for some large d_n', which is now fixed.

Corollary 4.3. *There exists $d_n'' > d_n'$ such that F has no periodic orbit in the deleted disk $D^*(\delta_n^{d_n''})$ with period smaller than n.*

Proof. From (4.18)-(4.20) it follows that for $1 \leq k < n$

$$|F_n^k(z) - z| \geq |z|(|\lambda^k - 1| - 5\delta_n^{-d_n'}|z|^n) \geq |z|(\delta_n - 5\delta_n^{-d_n'}|z|^n).$$

Hence, $F_n^k, k = 1, \ldots, n-1$ have no fixed point in $D^*(\delta_n^{d_n''})$ for some $d_n'' > d_n'$. □

Proposition 4.4. *Let $\{\epsilon_k\}_{k=1}^\infty$ be a decreasing sequence with $0 < \epsilon_k < 1$. Let F, U_n, F_n, F_n^n be given by (4.1), (4.2), (4.4), and (4.18), respectively. Assume that for F_n*

$$|A_n| = |A_n(\lambda, \mathbf{a})| > \epsilon_{n+1}. \tag{4.21}$$

There exists $d_n''' > 2d_n''$ such that for $r_n = \epsilon_{n+1}^{\frac{1}{n}}\delta_n^{d_n'''}$, all periodic points of F in the deleted disk $D^*(2r_n)$ with period at most n form a single orbit in $D^*(r_n)$ with period n, provided

$$0 < |1-\lambda^n| < \frac{\epsilon_{n+1}^2}{2^{n+1}}\delta_n^{nd_n'''}. \tag{4.22}$$

Proof. For $d_n''' > 2d_n''$, Corollary 4.3 says that F has no periodic point in $D^*(4r_n)$ of period smaller than n. To find periodic orbits with period n, we use (4.18) to rewrite $z_n - z = 0$ as

$$A_n z^n + \frac{\lambda^n - 1}{1 + s(z)} = 0 \tag{4.23}$$

with

$$s(z) = \frac{B_n z^n + z^{n+1} E_n' + (1 - \lambda^n) z^n E_n''}{A_n}.$$

By (4.20) and (4.21), there exists $d_n''' > 2d_n''$, independent of ϵ_{n+1}, such that

$$|s(z)| < 1/2, \quad |z| = 4r_n.$$

Now (4.21) and (4.22) imply that

$$|A_n z^n|_{|z| = \frac{1}{2} r_n} > \frac{\epsilon_{n+1}^2}{2^n} \delta_n^{n d_n'''} > 2|\lambda^n - 1| > \left| \frac{\lambda^n - 1}{1 + s(z)} \right|_{|z| = 4r_n}.$$

By Rouché's theorem, (4.23) has no solution in $D(4r_n) \setminus D(\frac{1}{2} r_n)$ and has n solutions in $D(\frac{1}{2} r_n)$, counted with multiplicities. Obviously, all these solutions are nonzero. Let $\tilde{p}_n$ be one of such solutions. Corollary 4.3 implies that $F_n^k(\tilde{p}_n), k = 0, 1, \ldots, n-2$ are distinct. Therefore, the fixed points of F_n^n in $D^*(4r_n)$ form one single orbit in $D^*(\frac{1}{2} r_n)$. From (4.7) it follows that the periodic points of F in $D^*(2r_n)$ of period at most n form one single orbit in $D^*(r_n)$ with period n. □

Proposition 4.5. *Let notations and assumptions in Proposition* 4.4 *be kept. Assume further that*

$$\left| n^2 + n - 2\Re\left(\frac{n B_n}{A_n^2}\right) \right| > \epsilon_{2n+1} \delta_n^{2d_n'}. \tag{4.24}$$

There exists $d_n'''' > d_n'''$ such that if

$$0 < |1 - \lambda^n| < \left(\epsilon_{n+1}^4 \epsilon_{2n+1} \delta_n^{d_n''''}\right)^n \tag{4.25}$$

the eigenvalue γ_n of the periodic orbit in Proposition 4.4 *satisfies*

$$||\gamma_n|^2 - 1| > \frac{1}{32\pi^2} \epsilon_{2n+1} \delta_n^{2d_n'} |1 - \lambda^n|^2. \tag{4.26}$$

Proof. Let $p_k = U_n(\tilde{p}_k), k = 1, \ldots, n$ be the single orbit of F in $D^*(r_n)$. Set $z = \tilde{p}_n$. From $z_n = z$ and (4.18) it follows that

$$z^n = \frac{1}{A_n}(1 - \lambda^n - B_n z^{2n} - z^{-1} E_n(z)). \tag{4.27}$$

The eigenvalue of the periodic orbit, obtained by differentiating (4.18), equals

$$\begin{aligned}\gamma_n &= \lambda^n + (n+1)A_n z^n + (2n+1)B_n z^{2n} + \partial_z E_n(z)\\ &= 1 + n(1-\lambda^n) + nB_n z^{2n} - (n+1)z^{-1}E_n(z) + \partial_z E_n(z)\\ &= 1 + n(1-\lambda^n) + \frac{nB_n}{A_n^2}(1-\lambda^n)^2 + \tilde{\gamma}_n, \qquad (4.28)\end{aligned}$$

in which the last two identities are derived by using (4.27) twice, while

$$\begin{aligned}\tilde{\gamma}_n = \frac{nB_n}{A_n^2}&(2(\lambda^n - 1)(B_n z^{2n} + z^{-1}E_n(z)) + (B_n z^{2n} + z^{-1}E_n(z))^2)\\ &- (n+1)z^{-1}E_n(z) + \partial_z E_n(z).\end{aligned}$$

The Cauchy inequality yields

$$\begin{aligned}|\partial_z E_n(z)| \leq \|E_n\|_{D(2|z|)\times D^\infty}/|z| &\leq 2^{2n+1}|z|^{2n}(2|z|\cdot\|E_n'\|_{D(2|z|)\times D^\infty}\\ &+ |1-\lambda^n|\cdot\|E_n''\|_{D(2|z|)\times D^\infty}) \leq \delta_n^{-d_n''''}|1-\lambda^n|^{2+\frac{1}{n}}\end{aligned}$$

for some large d_n''''. The norms of the remaining terms in $\tilde{\gamma}_n$ can be bounded from above in a simpler way, except that $|A_n|$ is bounded from below by using (4.21). Thus, for some positive d_n'''' one has

$$|\tilde{\gamma}_n| \leq \epsilon_{n+1}^{-2}\delta_n^{-d_n''''}|1-\lambda^n|^{2+\frac{1}{n}}. \qquad (4.29)$$

We need to compute $|\gamma_n|$ more explicitly. For a real number x, denote by $\{x\}$ the closest integers to x (and the smaller one if there are two such integers). Put

$$\lambda = \mathrm{e}^{\mathrm{i}2\pi\alpha}, \quad n\alpha = \{n\alpha\} + \alpha_n.$$

By (4.22) we know that $0 < |\alpha_n| < \frac{1}{4}$. Consequently, one has

$$\sqrt{2}\pi|\alpha_n| < |1-\lambda^n| < 4\pi|\alpha_n|. \qquad (4.30)$$

Also

$$1 - \lambda^n = 1 - \mathrm{e}^{\mathrm{i}\alpha_n} = -i\alpha_n + \frac{1}{2}\alpha_n^2 + R_3(\alpha_n), \quad |R_3(\alpha_n)| \leq 4\alpha_n^3.$$

Thus (4.28) yields

$$\gamma_n = 1 - in\alpha_n + \left(\frac{n}{2} - \frac{nB_n}{A_n^2}\right)\alpha_n^2 + \hat{\gamma}_n$$

with

$$\hat{\gamma}_n = \tilde{\gamma}_n + nR_3(\alpha_n) + \frac{nB_n}{A_n^2}\left(\left(\frac{1}{2}\alpha_n^2 + R_3(\alpha_n)\right)^2 - i\alpha_n(\alpha_n^2 + 2R_3(\alpha_n))\right).$$

Finally, we arrive at

$$
\begin{aligned}
|\gamma_n|^2 - 1 &= \left(n - 2\Re\left(\frac{nB_n}{A_n^2}\right)\right)\alpha_n^2 + 2\Re\hat{\gamma}_n \\
&\quad + \left|-in\alpha_n + \left(\frac{n}{2} - \frac{nB_n}{A_n^2}\right)\alpha_n^2 + \hat{\gamma}_n\right|^2 \\
&= \left(n^2 + n - 2\Re\left(\frac{nB_n}{A_n^2}\right)\right)\alpha_n^2 + h''_n \equiv h'_n + h''_n. \qquad (4.31)
\end{aligned}
$$

To estimate h''_n, note that by (4.30) one has $|\alpha_n| \leq |1-\lambda^n|$. Using (4.20), (4.21), and(4.29), one gets

$$
|h''_n| \leq \frac{1}{\epsilon_{n+1}^4}\delta_n^{-\tilde{d}''''_n}|1-\lambda^n|^{2+\frac{1}{n}}
$$

for some large $\tilde{d}''''_n$. Combining (4.24) and (4.30)-(4.31) yields

$$
\begin{aligned}
||\gamma_n|^2 - 1| \geq |h'_n| - |h''_n| &\geq \frac{1}{16\pi^2}\epsilon_{2n+1}\delta^{2d'_n}|1-\lambda^n|^2 \\
&\quad - \delta_n^{-\tilde{d}''''_n}|1-\lambda^n|^{2+\frac{1}{n}} > \frac{1}{32\pi^2}\epsilon_{2n+1}\delta_n^{2d'_n}|1-\lambda^n|^2,
\end{aligned}
$$

provided (4.25) holds for some large d''''_n, which is now fixed. □

5 Periodic orbits of conformal maps

In this section, we shall find hyperbolic or elliptic periodic orbits of real analytic area-preserving maps in the complexified space. The hyperbolic orbits will serve as the obstruction to the reversibility of area-preserving maps, while the elliptic orbits will show certain symmetry in respect to the centralizers of the associated conformal maps of certain reversible area-preserving maps. The symmetric periodic orbits of a general elliptic reversible map in the real space were known to Birkhoff [5].

Proposition 5.1. *Let $F\colon z \to \lambda z + O(2)$ be a conformal map with λ not a root of unity. Suppose that the functional equation (1.3) has a convergent solution. Assume that there exist $\kappa > 1$ and a sequence of positive numbers $r_j \to 0$ such that the deleted disk $D^*(r_j)$ contains all periodic orbits of F in $D^*(\kappa r_j)$ with period n_j. Then there exists j_0 such that for $j > j_0$, F has the same number of repelling periodic orbits of period n_j as that of attracting periodic orbits in $D^*(r_j)$ of period n_j.*

Proof. We may assume that F is not linearizable near the origin. Otherwise, the proposition is trivial since F has no periodic points near the origin.

Returning to the functional equation (1.3), we assume that it has a convergent solution $T(z) = \mu z + O(2)$. Since F is not linearizable, Lemma 2.6 says that $|\mu| = 1$. In particular, there exists j_0 such that T sends $D(r_j)$ into $D(\kappa r_j)$ for $j > j_0$. Fix $j > j_0$. Assume for the sake of contradiction that in $D(r_j)$, F has s repelling periodic orbits $O_1, \dots, O_s$ of period n_j and t attracting periodic orbits of period n_j with $s > t$. Take p_k in O_k for $k = 1, \dots, s$. From (1.3) it follows that $T(p_k)$ is a periodic point of $\overline{F}^{-1}$ with period n_j, of which the full orbit is contained in $D(\kappa r_j)$. This implies that $w = \overline{T(p_k)}$ is a periodic point of F in $T(D(r_j)) \subset D(\kappa r_j)$ with period n_j. Now (1.3) yields $(\overline{F}^{-n_j})'(\overline{w}) = (F^{n_j})'(p_k)$, i.e., $(F^{-n_j})'(w) = \overline{(F^{n_j})'(p_k)}$. Therefore, $\overline{T(p_1)}, \dots, \overline{T(p_s)}$ are attracting periodic points of F with period n_j, of which the full orbits are contained in $D(\kappa r_j)$. Since $s > t$, two of the attracting periodic points must be in the same orbit. Thus, $F^m(\overline{T}(\overline{p}_k)) = \overline{T}(\overline{p}_l)$ for some m and $k \neq l$. We have $\overline{F}^m(T(p_k)) = T(p_l)$. Now (1.3) yields $T(p_k) = (T \circ F \circ T^{-1})^m(T(p_l))$, i.e., $p_k = F^m(p_l)$. This contradicts that p_k, p_l are in different orbits. The proposition is proved. □

By Proposition 5.1, Theorem 1.2 follows from the following.

Theorem 5.2. *Let* $\epsilon_1, \epsilon_2, \dots$ *be a sequence of positive numbers. Given a conformal map* $z \to \lambda z + \sum_{n>1} a_n z^n$ *with* $|a_n| < 1$, *there exists* $G(z) = \hat{\lambda} z + \sum_{n>1} \hat{a}_n z^n$ *with* $\hat{\lambda}$ *not a root of unity,* $|\hat{\lambda} - \lambda| < \epsilon_1$, *and* $|\hat{a}_n - a_n| \leq \epsilon_n$ *for* $n > 1$ *such that all periodic points in the deleted disk* $D^*(2r_{n_j})$ *of period* n_j *form one single hyperbolic periodic orbit in* $D^*(r_{n_j})$.

Proof. Fix $\lambda, a_2, a_3, \dots$ with $|a_n| < 1$. Replacing ϵ_n by smaller positive numbers if necessary, one may assume that $\{\epsilon_n\}_{n=1}^{\infty}$ decreases and $0 < \epsilon_n < 1$, and that $|\hat{a}_n| < 1$ if $|\hat{a}_n - a_n| \leq \epsilon_n$.

Consider positive integers m_0, n_j and l_j with

$$n_j = 3^{l_j}, \quad l_{j+1} > 4 l_j. \tag{5.1}$$

Put

$$\begin{gathered}\alpha = \alpha'_{n_j} + \alpha''_{n_j}, \quad \alpha'_{n_j} = \frac{m_0}{n_0} + \sum_{i \leq j} \frac{1}{n_i}, \quad \alpha''_{n_j} = \sum_{i > j} \frac{1}{n_i}, \\ \lambda_n = \mathrm{e}^{2\pi i \alpha'_n}, \quad \hat{\lambda} = \mathrm{e}^{2\pi i \alpha}.\end{gathered} \tag{5.2}$$

Clearly, (5.1) implies that

$$0 < \alpha_{n_j} \equiv \alpha - \{n_j \alpha\} = n_j \alpha''_{n_j} < \frac{2 n_j}{n_{j+1}}. \tag{5.3}$$

Put

$$k\alpha'_{n_j} = \{k\alpha'_{n_j}\} + \alpha_{n_j,k}.$$

It is clear that

$$|\alpha_{n_j,k}| \geq 1/n_j, \quad 0 < k < n_j.$$

Hence, (5.3) and $n_{j+1} > 4n_j$ imply that

$$|\alpha_{n_j,k}| = |\alpha_{n_j,k-n_j} + \alpha_{n_j}| > \frac{1}{2n_j}, \quad n_j < k < 2n_j.$$

Now (4.30) yields

$$\delta_{n_j} = \delta_{n_j}(\hat{\lambda}) = \min_{0<k(\neq n_j)<2n_j} \{|1 - \hat{\lambda}^k|\} > \frac{1}{2n_j}. \tag{5.4}$$

In particular, $\hat{\lambda}$ is not a root of unity.

While n_j still need to determined, we put

$$\hat{a}_n = a_n, \quad n \neq n_j + 1,\ 2n_j + 1.$$

We shall determine the triples $\{a_{n_j+1}, a_{2n_j+1}, l_{j+1}\}$ recursively for $0 \leq j < \infty$.

Fix $m_0, l_0 > 2$ such that for $\hat{\lambda}$ defined by (5.2), the inequality $|\hat{\lambda} - \lambda| < \epsilon_1$ always holds. Now $\hat{a}_k = a_k, k \in I_{n_0}$ have been determined. Put

$$G_n(z) = \lambda_n z + \sum_{n \in I_n} \hat{a}_n z^n + \sum_{1<n \notin I_n} a_n z^n$$

for $n = n_0$. Take the unique polynomial $U_{n_0}(z)$ of the form (4.2) such that

$$U_n \circ G_n \circ U_n^{-1}(z) = \lambda_n z + A_{n,1} z^{n+1} + B_{n,1} z^{2n+1} + O(|z|^{2n+2})$$

holds for $n = n_0$. By (4.5) one has

$$A_{n,1} = a_{n+1} + \tilde{b}_{n+1}(\hat{\mathbf{a}}_n, \mathbf{d}_n, \lambda_n), \quad B_{n,1} = a_{2n+1} + \tilde{b}_{2n+1}(\hat{\mathbf{a}}_{2n}, \mathbf{d}_n^+, \lambda_n)$$

for $n = n_0$, where $\tilde{b}_n, \tilde{b}_{2n+1}$ are polynomials of integer coefficients. Note that $\lambda_n^n = 1$. Returning to (4.18), one sees that $U_n \circ G_n^n \circ U_n^{-1}$ is given by

$$z_n = z + A_n z^{n+1} + B_n z^{2n+1} + E_n(z)$$

with

$$A_n = n\overline{\lambda}_n a_{n+1} + n\overline{\lambda}_n \tilde{b}_{n+1}(\hat{\mathbf{a}}_n, \mathbf{d}_n, \lambda_n), \tag{5.5}$$

$$B_n = n\overline{\lambda}_n a_{2n+1} + \frac{n(n^2-1)}{2}\overline{\lambda}_n^2 A_n^{*2} + n\overline{\lambda}_n \tilde{b}_{2n+1}(\hat{\mathbf{a}}_{2n}, \mathbf{d}_n^+, \lambda_n) \tag{5.6}$$

for $n = n_0$, where A_n, B_n are polynomials with integer coefficients satisfying estimates (4.20).

First, we choose $\hat{a}_{n_0+1}$ as follows. Put $\hat{a}_{n_0+1} = a_{n_0+1}$ if

$$|A_{n_0}(\hat{\mathbf{a}}_{n_0}, a_{n_0+1}, \mathbf{d}_n, \lambda_{n_0}, \overline{\lambda}_{n_0})| > \epsilon_{n_0+1},$$

and put $\hat{a}_{n_0+1} = a_{n_0+1} + \epsilon_{n_0+1}$ otherwise. In the latter case, (5.5) implies that

$$|A_n(\hat{\mathbf{a}}_{n+1}, \mathbf{d}_n, \lambda_n, \overline{\lambda}_n)| > \epsilon_{n+1} \tag{5.7}$$

for $n = n_0$. This shows that in either case, we have (5.7).

Next, we put $\hat{a}_{2n_0+1} = a_{2n_0+1}$ if

$$\left| n_0^2 + n_0 - 2\Re\left(\frac{n_0 B_{n_0}}{A_{n_0}^2}\right) \right| > \epsilon_{2n_0+1}\delta_{n_0}^{2d'_{n_0}},$$

and put

$$\hat{a}_{2n_0+1} = a_{2n_0+1} + \lambda_{n_0}\epsilon_{2n_0+1}\frac{A_{n_0}^2}{|A_{n_0}|^2} \tag{5.8}$$

otherwise. Then in the latter case, (5.6) and (5.8) imply that

$$B_{n_0}(\hat{\mathbf{a}}_{2n_0+1}, \cdot) = B_{n_0}((\hat{\mathbf{a}}_{2n_0}, a_{2n_0+1}), \cdot) + n_0\epsilon_{2n_0+1}\frac{A_{n_0}^2}{|A_{n_0}|^2}.$$

Hence

$$\left| n_0^2 + n_0 - 2\Re\left(\frac{n_0 B_{n_0}}{A_{n_0}^2}\right) \right| \geq \epsilon_{2n_0+1}\frac{2n_0^2}{|A_{n_0}|^2} - \epsilon_{2n_0+1}\delta_{n_0}^{2d'_{n_0}}.$$

Using (4.20) one obtains

$$\left| n^2 + n - 2\Re\left\{\frac{nB_n(\hat{\mathbf{a}}_{2n+1}, \mathbf{d}_n^+, \lambda_n, \overline{\lambda}_n)}{A_n^2(\hat{\mathbf{a}}_{n+1}, \mathbf{d}_n, \lambda_n, \overline{\lambda}_n)}\right\} \right| > \epsilon_{2n+1}\delta_n^{2d'_n} \tag{5.9}$$

for $n = n_0$. This shows that (5.9) holds for $n = n_0$ in either case.

Note that with $n = n_0$ inequalities (5.7) and (5.9) involve λ_{n_0}. Choose large l_{j_0+1} such that (5.7) and (5.9) remain true when $\lambda_n (n = n_0)$ is replaced by $\hat{\lambda}$ of the form (5.2). Recursively, one finds all $l_j, \hat{a}_{n_j}$ such that (5.7) and (5.9), for which λ_n is replaced by $\hat{\lambda}$, hold for all $n = n_j$. In fact, we should choose a possibly larger l_{j_0+1} such that

$$|1 - \hat{\lambda}^n| < \left(\epsilon_n^4 \epsilon_{2n+1}\delta_n^{d''''_n}(\hat{\lambda})\right)^n \tag{5.10}$$

holds for $n = n_0$ also. To achieve (5.10), note that (4.30) and (5.3) yield

$$|1 - \hat{\lambda}^{n_j}| < 4\pi\alpha_{n_j} < \frac{8\pi n_j}{n_{j+1}}.$$

On the other hand, (5.4) gives us

$$\left[\epsilon_{n_j}^4 \epsilon_{2n_j+1} \delta_{n_j}^{d''''_{n_j}}(\hat{\lambda})\right]^{n_j} > \left[\epsilon_{n_j}^4 \epsilon_{2n_j+1} (\frac{1}{2n_j})^{d''''_{n_j}}\right]^{n_j}.$$

Obviously, when n_{j+1} is large in relative to n_j, (5.10) holds for $n = n_j$.

With (5.7), (5.9), and (5.10), we apply Proposition 4.5 to $G = \lim_{j\to\infty} G_{n_j}$. From (4.26) it follows that all the periodic orbits of G in $D^*(r_{n_j})$ with period n_j are hyperbolic. Thus, we have determined $\hat{\lambda}$ and G. The proof of Theorem 5.2 is complete. $\square$

To prove Theorem 1.3, we need the following

Lemma 5.3. *There exists a constant $L > 1$ such that for*

$$H(z) = z + \sum_{n>1} c_n z^n, \quad |c_n| < 1, \quad n > 1, \tag{5.11}$$

one has

$$H \circ \overline{H}^{-1}(z) = z + \sum_{n>1} a_n z^n, \tag{5.12}$$

$$a_n = c_n - \bar{c}_n + \tilde{a}_n(\mathbf{c}_{n-1}, \bar{\mathbf{c}}_{n-1}), \quad |a_n| < L^n, \tag{5.13}$$

where $\mathbf{c}_{n-1} = (c_2, \ldots, c_{n-1})$, and $\tilde{a}_n$ are polynomials of integer coefficients.

Proof. The existence of polynomial $\tilde{a}_n$ in (5.13) is clear. We now estimate a_n. One has

$$\overline{H}(z) - z \prec \frac{z^2}{1-z}.$$

Since $\overline{H}^{-1}(z) = z - \overline{H} \circ \overline{H}^{-1}(z) + \overline{H}^{-1}(z)$, then $\overline{H}^{-1}(z) - z \prec K(z) - z$ for

$$K = z + \frac{K^2}{1-K}, \quad K(0) = 0.$$

Thus

$$H \circ \overline{H}^{-1}(z) \prec \frac{K(z)}{1 - K(z)},$$

in which the right-hand side is a convergent power series independent of H. Therefore, there exists $L > 1$ such that $|a_n| < L^n$ for $n > 1$. $\square$

Proof of Theorem 1.3. We should consider set of $F \in \mathcal{F}$ of the form

$$F = T_1 \circ T_2, \quad T_1 = R_\lambda, \quad T_2 = R_{L^2} \circ H \circ \overline{H}^{-1} \circ R_{L^{-2}}, \tag{5.14}$$

where H, of the form (5.11), is to be determined. It is clear that $T_2 \circ \overline{T}_2 = \mathrm{Id}$. Hence,

$$R_{\overline{\lambda}} \circ F \circ R_\lambda = \overline{F}^{-1}. \tag{5.15}$$

Note that Lemma 5.3 implies that the coefficients of $F = F_H$, given by (5.14), have the form

$$a_n = \lambda \frac{c_n - \overline{c}_n}{L^{2n-2}} + a_n^*(\mathbf{c}_{n-1}, \overline{\mathbf{c}}_{n-1}), \quad |a_n| < 1, \tag{5.16}$$

where a_n^* are polynomials in $\mathbf{c}_{n-1}, \overline{\mathbf{c}}_{n-1}$.

The argument for the existence of periodic points in the proof of Theorem 5.2 can be modified as follows. Fix $c_2, c_3, \dots$ with $|c_k| < 1$ and define H by (5.11). Take a decreasing sequence of positive numbers $\tilde{\epsilon}_n$ with $0 < \tilde{\epsilon}_n < 1$ such that $|\hat{c}_n - c_n| \leq \tilde{\epsilon}_n$ implies that $|\hat{c}_n| < 1$. Put

$$\epsilon_n = \frac{\tilde{\epsilon}_n}{L^{2n-2}}.$$

Then $\{\epsilon_n\}$ is a decreasing sequence with $0 < \epsilon_n < 1$, which is applicable to Proposition 4.4. Recursively, by choosing $\hat{c}_n = c_n$ or $\hat{c}_n = c_n + i\tilde{\epsilon}_n$ for the sequence $n = n_j$ in the proof of Theorem 5.2, one obtains inequality (5.7) in the form

$$|A_n(\hat{\mathbf{a}}_{n+1}, \mathbf{d}_n, \lambda_n, \overline{\lambda}_n)| > \frac{\tilde{\epsilon}_{n+1}}{L^{2n-2}} = \epsilon_{n+1},$$

in which $\hat{a}_{n+1}$ is determined by $\hat{c}_2, \dots, \hat{c}_{n+1}$ via (5.16). This gives us (4.21) for the above specific sequence $\{\epsilon_n\}$. Let $\hat{F}$ be the conformal map (5.14) corresponding to $\hat{H}(z) = \hat{\lambda} z + \sum_{k>1} \hat{c}_k z^k$. By Proposition 4.4, one knows the existence of a single periodic orbit of $\hat{F}$ in $D^*(r_{n_j})$ with period n_j, while no other periodic orbit of the same period exists in $D^*(2r_{n_j})$. By Proposition 5.1 we conclude that all such periodic orbits are elliptic for large j. We should point out that (4.24), excluded eventually by the conclusion that the periodic orbits are elliptic, cannot be achieved within such a special class of F as in (5.14), since the leading term in (5.16) prevents us to add an extra term with an *arbitrary* argument to a_n via c_n as we did in (5.8) for the general case.

To complete the proof of the theorem, we want to show the symmetry of the above periodic orbits of $F \equiv \hat{F}$ with respect to the centralizers of F.

Fix $G \in \mathcal{C}_F$. Since F has periodic orbits accumulating at the origin, then F is not linearizable. Hence, $|G'(0)| = 1$. By (2.10) and (2.11), Proposition 2.5 implies that $T_0 = \overline{T}_1 = R_{\overline{\lambda}}$ is in $\mathcal{C}'_F$. From the non-canonical identification (2.7) one sees that $T = R_{\overline{\lambda}} \circ G$ is in $\mathcal{C}'_F$, and hence in $\mathcal{I}_F$ since $|G'(0)| = 1$. Using Proposition 2.5 again, one gets

$$\overline{F}^{-1} = T \circ F \circ T^{-1}, \quad T \circ \overline{T} = \mathrm{Id}.$$

Choose j_0, dependent of G, such that $T(D(r_{n_j}))$ is contained in $D(2r_{n_j})$ for $j > j_0$. Let $p_1, \ldots, p_{n_j}$ be the unique periodic orbit in $D^*(r_{n_j})$. Then $T(p_1), \ldots, T(p_{n_j})$ form a periodic orbit of $\overline{F}$ contained in $D(2r_{n_j})$, i.e., $\overline{T(p_1)}, \ldots, \overline{T(p_{n_j})}$ form a periodic orbit of F. Thus, the uniqueness of the periodic orbit implies that $\{\overline{T}(\overline{p}_1), \ldots, \overline{T}(\overline{p}_{n_j})\} = \{p_1, \ldots, p_{n_j}\}$. In other words, $z \to \overline{T}(\overline{z})$ becomes an involution sending the periodic orbit onto itself. Since $n_j = 3^{l_j}$ is odd, the involution fixes at least one point in the orbit. Without loss of generality, one may assume that $\overline{T}(\overline{p}_{n_j}) = p_{n_j}$. Therefore, $G(p_{n_j}) = \lambda \overline{p}_{n_j}$, i.e.,

$$\tilde{\Gamma}_G\colon G(z) - \lambda\overline{z} = 0 \tag{5.17}$$

contains a periodic point p_{n_j} of F in $D(r_{n_j})$ for all $j > j_0$. To reach the conclusion of the theorem, note that (5.17) is not invariant under a change of coordinates. Take μ with $\mu^2 = \overline{\lambda}$ and $\Re\mu > 0$. Then (5.17) becomes

$$\Gamma_{\hat{G}}\colon \hat{G}(z) - \overline{z} = 0$$

for $\hat{G} = R_\mu \circ G \circ R_\mu^{-1}$. However, $\hat{G}$ is a centralizer of $\hat{F} = R_\mu \circ F \circ R_\mu^{-1}$, and conversely each centralizer of $\hat{F}$ is realized in this way. Therefore, $\hat{F}$ satisfies all the properties stated in Theorem 1.3. This completes the proof of the theorem. ☐

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