Math. Ann. 323, 227–257 (2002) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080200278

Quantitative estimates for periodic points of reversible and symplectic holomorphic mappings

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Received: 5 December 2000 / Published online: 4 April 2002 – © Springer-Verlag 2002

Abstract. We show that reversible holomorphic mappings of $\mathbf{C}^2$ have periodic points accumulating at an elliptic fixed point of general type. On the contrary, we also show the existence of holomorphic symplectic mappings that have no periodic points of certain periods in a sequence of deleted balls about an elliptic fixed point of general type. The radii of the balls are carefully chosen in terms of the periods, which allows us to show the existence of holomorphic mappings of $\mathbf{C}^2$ that are not reversible with respect to any C^1 involution with a holomorphic linear part, and that admit no invariant totally real and C^1 real surfaces.

Mathematics Subject Classification (2000): 32R05, 37J10

1 Introduction

Let φ be a biholomorphic mapping defined near $0 \in \mathbf{C}^2$ with $\varphi(0) = 0$. The origin is an elliptic fixed point of φ of *general type*, if in some local holomorphic coordinates φ is of the form

$$\begin{aligned} \xi_1 &= \lambda\xi e^{a_s(\xi\eta)^s} + o(2s+1), \\ \eta_1 &= \overline{\lambda}\eta e^{-a_s(\xi\eta)^s} + o(2s+1), \end{aligned} \qquad |\lambda| = 1, \quad 0 \neq a_s \in \mathbf{C}, \tag{1.1}$$

in which (ξ, η) are the holomorphic coordinates of $\mathbf{C}^2$. One says that φ is *symplectic* if $\varphi^* d\xi \wedge d\eta = d\xi \wedge d\eta$, and that it is *reversible* near the origin with respect to an involution τ ($\tau^2 = \mathrm{Id}$, $\tau(0) = 0$) if $\varphi^{-1} = \tau\varphi\tau^{-1}$. The main purpose of this paper is to study the existence of periodic points near fixed points of the holomorphic mappings that are either symplectic or reversible.

The existence of periodic points of area-preserving or reversible mappings of the real plane was established by G. D. Birkhoff, who showed that the mappings have periodic points accumulating at each elliptic fixed point of general type

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Supported in part by NSF grant DMS 0196036

The author is indebted to Sidney M. Webster for bringing the problem on the existence of periodic points of holomorphic symplectic mappings to his attention.

([1], [2]). The periodic points of reversible real mappings and systems in higher dimensional spaces were further studied by R. L. Devaney [3]. To motivate our results, let us recall Birkhoff's treatment for real planar mappings. Start with a twist mapping of the z-plane

$$T\colon z_1 = \lambda z e^{i(z\bar{z})^s}, \quad i = \sqrt{-1}, \quad |\lambda| = 1.$$

Away from the origin the fixed points of the n-th iterate T^n form circles surrounding the origin. Such circles of periodic points can, of course, be destroyed by perturbing T. However, these circles do survive in a weaker sense: If S is a higher order (smooth or real analytic) perturbation of T, each circle of periodic points of T is deformed slightly into a closed curve C, surrounding the origin, such that S^n sends each point on C to a point in the radial direction. We shall call such a curve C the *Birkhoff curve* of S of order n. Birkhoff then went on to find periodic points as follows: If S is additionally area-preserving, the intersection of the Birkhoff curve C with $S^n(C)$ is evidently non-empty and consists of fixed points of S^n (see [2]); if S is additionally reversible with respect to an involution τ and if the set of fixed points of τ is real line, then C intersects the curve of fixed point of τ and the intersection consists of periodic points of S with period dividing $2n$ (see [1]).

Returning to the holomorphic case, we first want to show that certain Birkhoff (holomorphic) curves still exist for a holomorphic mapping φ of the form (1.1), and that the Birkhoff curves, as in the real case, yield periodic points when φ is reversible. However, the Birkhoff curves behave quite differently in case of holomorphic symplectic mappings. The main results of this paper conclude that for a Birkhoff curve of C of order n, $\varphi^n(C)$ might not intersect with C in a certain neighborhood of the origin; see also Sect. 5 for a holomorphic symplectic mapping φ which has a Birkhoff curve C of order 1 with $\varphi(C) \cap C = \emptyset$.

To formulate our results, we need some notation. For a complex number λ with $|\lambda| = 1$, put

$$\kappa_n(\lambda) = \begin{cases} \left[\dfrac{|\lambda^n - 1|}{n}\right]^{\frac{1}{2s}} & \text{if } \lambda^n \neq 1, \\ \left(\dfrac{1}{n}\right)^{\frac{1}{2s}} & \text{otherwise.} \end{cases} \tag{1.2}$$

Let $B(r) \subset \mathbf{C}^2$ be the ball centered at the origin with radius r, and let $B^*(r) = B(r) \setminus \{0\}$. Recall that p is a period point of φ if $\varphi^n(p) = p$ for some positive integer n, and the smallest such positive integer n is called the *period* of the periodic point.

Theorem 1.1. *Let φ be a homeomorphism defined near the origin of $\mathbf{C}^2$ by* (1.1) *with $\lambda^2 \neq 1$. Suppose that φ is reversible with respect to a C^1 involution of which* *the linear part is* **C***-linear. There exists a constant* $c_0 > 0$ *such that for* n *sufficiently large,* φ *has a periodic orbit contained in the punctured ball* $B^*(c_0\kappa_n(\lambda))$ *with period dividing* $2n$*; in particular,* φ *has periodic points accumulating at the origin.*

Combining Theorem 1.1 and the normal form of Moser and Webster [8] for holomorphic reversible mappings, we obtain

Theorem 1.2. *Let* $\varphi\colon (\xi,\eta) \to (\lambda\xi, \overline{\lambda}\eta) + O(2)$ $(|\lambda| = 1)$ *be a holomorphic mapping defined near* $0 \in \mathbf{C}^2$*. Assume that* λ *is not a root of unity, and that* φ *is not equivalent to a linear mapping under any formal transformation of* $\mathbf{C}^2$*. If* φ *is reversible with respect to a holomorphic involution, then* φ *has periodic points accumulating at the origin.*

In contrast to Theorem 1.1 we shall prove

Theorem 1.3. *Let* $a(z)$ *be a holomorphic function with* $a(0) = 0$*, and let* m_n *be a sequence of positive integers. There exist* λ*,* $|\lambda| = 1$*, which is not a root of unity, a sequence of positive constants* $d_n \to \infty$ *and a holomorphic symplectic mapping* φ *of the form* (1.1) *such that* φ *is equivalent to*

$$\xi_1 = \lambda\xi e^{a(\xi\eta)}, \quad \eta_1 = \lambda^{-1}\eta e^{-a(\xi\eta)} \tag{1.3}$$

under a formal symplectic transformation, while φ *has no periodic points of period less than* m_n *in* $B^*(d_n\kappa_n(\lambda))$ *for a sequence of positive integers* $n = n_k \to \infty$*.*

Notice that the Birkhoff normalization says that a holomorphic symplectic mapping of $\mathbf{C}^2$, of which the eigenvalues are not roots of unity, is formally equivalent to a mapping of the form (1.3) with $a(\xi\eta)$ a formal power series. An immediate consequence of the Birkhoff normal form, Theorem 1.1, and Theorem 1.3 (for $m_n = 2n + 1$) is

Corollary 1.4. *Let* $a(z)$ *be a holomorphic function with* $a(z) \not\equiv 0 = a(0)$*. There exists a holomorphic symplectic mapping* φ *of the form* (1.1) *with convergent Birkhoff normal form* (1.3) *such that* φ *is not reversible with respect to any* C^1 *involution with a holomorphic linear part.*

Note that there exist formally linearizable holomorphic symplectic mappings that are not reversible with respect to any holomorphic transformation. The reader is also referred to [4] and [5] for the existence of real analytic area-preserving mappings and Hamiltonian systems that have non-resonant eigenvalues and are not reversible with respect to any real analytic involution.

Combining Theorem 1.3 with the Birkhoff fixed-point theorem we shall prove

Corollary 1.5. *There exists a holomorphic symplectic mapping* φ *of the form* (1.1) *with* λ *not a root of unity such that at the origin there are no germs of totally real and* C^1 *real surface that are invariant under* φ*.*

We shall see from Proposition 6.1 below that an elliptic symplectic holomorphic mapping of $\mathbf{C}^2$ have infinitely many totally real and formal surfaces that are invariant under the mapping, if the eigenvalues of the mapping are not root of unity.

Theorem 1.2 is analogous to the Birkhoff fixed-point theorem of reversible or area-preserving mappings of the real plane. The proof of Theorem 1.1 follows closely a proof of the Birkhoff-Lewis fixed-point theorem, given by J.K. Moser [7]. Theorem 1.3 demonstrates some subtle differences between the area-preserving real mappings and symplectic holomorphic ones. However, we should mention that it remains open if all holomorphic symplectic mappings have periodic points accumulates at each elliptic fixed point of general type. More specifically, we conclude the introduction with the following

Open problem. *Let* $\varphi\colon U \to U'$ *be a holomorphic symplectic mapping of* $\mathbf{C}^2$ *in the form* (1.1)*, where* U *is a neighborhood of the origin. Does there exist a sequence* $\{p_n\}_{n=1}^{\infty}$ *in* $U \setminus \{0\}$ *with* $p_n \to 0$ *such that, for each* n*,* $\{\varphi^j(p_n)\colon j = 1, 2, \dots\}$ *is a periodic orbit contained in* U*?*

2 Estimates for iterates

In this section we shall give some estimates for the iterates of mapping (1.1), in particular for the domain on which each iterate is defined. In fact, we shall mainly consider iterates of the mapping

$$\varphi\colon \begin{aligned} \xi_1 &= \lambda\xi e^{(\xi\eta)^s} + p(\xi,\eta),\\ \eta_1 &= \overline{\lambda}\eta e^{-(\xi\eta)^s} + q(\xi,\eta), \end{aligned} \qquad |\lambda| = 1, \tag{2.1}$$

where $p(\xi,\eta)$, $q(\xi,\eta)$ are continuous functions defined near the origin and satisfying

$$|p(\xi,\eta)| + |q(\xi,\eta)| = o(|\xi|^{2s+1} + |\eta|^{2s+1}). \tag{2.2}$$

With the above assumptions we shall establish estimates which are sufficient for the proof of the existence of periodic points of reversible mappings. To prove a version of the Birkhoff fixed-point theorem for area-preserving mappings, we shall also obtain some estimates for the Jacobian matrices of the iterates, under additional assumptions on the derivatives of p and q to be formulated later.

Note that the condition (2.2) means that for $\epsilon > 0$ there exists $\rho > 0$ such that

$$|(p(\xi,\eta), q(\xi,\eta))| \le \epsilon |(\xi,\eta)|^{2s+1}, \quad (\xi,\eta) \in B(\rho), \tag{2.3}$$

in which, and in what follows, we use the norm $|(\xi,\eta)| = \sqrt{|\xi|^2 + |\eta|^2}$. Throughout this section and next, we assume that $0 < \epsilon < 1$ and $0 < \rho < 1$, and we shall also denote by $c_j > 1$, $c'_j > 1$, etc., the constants dependent of ρ and s, but independent of n and λ.

Write the iterate φ^{j+1} as

$$\begin{aligned} \xi_{j+1} &= \lambda\xi_j e^{(\xi_j\eta_j)^s} + p(\xi_j, \eta_j) \\ &\equiv \lambda^{j+1} e^{(j+1)(\xi\eta)^s} + p_j(\xi, \eta), \\ \eta_{j+1} &= \overline{\lambda}\eta_j e^{-(\xi_j\eta_j)^s} + q(\xi_j, \eta_j) \\ &\equiv \overline{\lambda}^{j+1} e^{-(j+1)(\xi\eta)^s} + q_j(\xi, \eta). \end{aligned} \tag{2.4}$$

For short, we put

$$T \equiv T_c \colon \xi_1 = \lambda\xi e^{(\xi\eta)^s}, \quad \eta_1 = \overline{\lambda}\eta e^{-(\xi\eta)^s}.$$

Then

$$\varphi^{j+1} = T^{j+1} + (p_j, q_j).$$

To study the above iterates, we need to control both the distance of a point (ξ, η) to the origin and quantity $e^{(\xi\eta)^s}$ in expression (2.1). It is thus convenient to iterate φ from a domain of the form

$$B(r, \theta) \equiv \left\{(\xi, \eta) \colon |(\xi, \eta)| < r, |\operatorname{Re}\{(\xi\eta)^s\}| < \theta\right\},$$

in which r and θ are positive numbers. With $(\xi_j(\xi, \eta), \eta_j(\xi, \eta)) = \varphi^j(\xi, \eta)$, define

$$\begin{aligned} r_j(r, \theta) &= \sup\left\{|(\xi_j(\xi, \eta), \eta_j(\xi, \eta))| \colon (\xi, \eta) \in B(r, \theta)\right\}, \\ \theta_j(r, \theta) &= \sup\left\{|\operatorname{Re}\{(\xi_j\eta_j)^s(\xi, \eta)\}| \colon (\xi, \eta) \in B(r, \theta)\right\}, \\ \Delta\theta_j(r, \theta) &= \sup\left\{|\operatorname{Re}\{(\xi_j\eta_j)^s(\xi, \eta) - (\xi\eta)^s\}| \colon (\xi, \eta) \in B(r, \theta)\right\}. \end{aligned}$$

Proposition 2.1. *Let φ, ϵ and ρ be as in* (2.1) *and* (2.3), *and let r_j and $\Delta\theta_j$ be defined as above. There exists constant $c_1 > 1$ such that if r and θ satisfy*

$$0 < \theta < 1/n, \quad 0 < r < e^{-6}\rho, \quad \epsilon \max\{r^{2s}, 2nr^{4s}\} < c_1^{-1}\theta, \tag{2.5}$$

then

$$\Delta\theta_j(r, \theta) < c_1\epsilon j r^{4s}, \quad r_j(r, \theta) \le re^{3j\theta}, \quad j = 1, \ldots, 2n. \tag{2.6}$$

Proof. For $(\xi_1, \eta_1) = \varphi(\xi, \eta)$, one has

$$|(\xi_1, \eta_1)| \le |(\xi, \eta)|e^{|\operatorname{Re}(\xi\eta)|} + \epsilon|(\xi, \eta)|^{2s+1}, \quad (\xi, \eta) \in B(\rho). \tag{2.7}$$

One also has

$$\xi_1\eta_1 = \xi\eta + \lambda\xi e^{(\xi\eta)^s} q(\xi, \eta) + \overline{\lambda}\eta e^{-(\xi\eta)^s} p(\xi, \eta) + p(\xi, \eta)q(\xi, \eta),$$

from which one obtains

$$|(\xi_1\eta_1)^s - (\xi\eta)^s| \le c_1'\epsilon r^{4s}, \quad (\xi,\eta) \in B(\rho). \tag{2.8}$$

For brevity, we fix $(\xi,\eta) \in B(r,\theta)$ and denote $r_j(r,\theta)$ by r_j, $\xi_j(\xi,\eta)$ by ξ_j, etc. By (2.7) and (2.8), we first get

$$r_1 \le re^{\theta} + \epsilon r^{2s+1} \le re^{2\theta}, \quad \Delta\theta_1 \le c_1'\epsilon r^{4s},$$

in which the first inequality is obtained from $e^x + y < e^{x+y}$ for positive x and y. In particular, (2.6) holds for $j = 1$ if $c_1 \ge c_1'$. Assume that (2.6) holds for $j < 2n$. One knows that $\theta_j \le 2\theta$ and $r_j \le e^6 r$. Hence, φ^j sends $B(r,\theta)$ into $B(e^6 r) \subset B(\rho)$ for $r < e^{-6}\rho$. Now (2.7) and (2.8) imply that

$$\begin{aligned} r_{j+1} &\le re^{3j\theta}e^{\theta_j} + \epsilon r_j^{2s+1} \le re^{3j\theta}e^{2\theta} + e^{12s+6}\epsilon r^{2s+1} \le re^{3(j+1)\theta}, \\ \Delta\theta_{j+1} &\le \Delta\theta_j + c_1'\epsilon r_j^{4s} \le \Delta\theta_j + c_1' e^{24s}\epsilon r^{4s} \le c_1' e^{24s}\epsilon(j+1)r^{4s} \end{aligned}$$

for $c_1 = c_1' e^{24s}$. By induction the proof is complete. □

For an $m \times n$ matrix A, define $\|A\| = \max_{|v|=1}|Av|$, in which v stands for column-vectors and $|(v_1,\dots,v_n)^t| = \sqrt{|v_1|^2 + \cdots + |v_n|^2}$. Let F be a C^1 mapping from $\mathbf{C}^n$ to $\mathbf{C}^m$. Denote by F_z the $m \times n$ complex Jacobian matrix of F. The real Jacobian matrix of F is the $2m \times 2n$ matrix

$$JF = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}, \quad F = f + ig, \quad z = x + iy.$$

Define $\|JF\|(r) = \max_{|z|\le r}\|JF(z)\|$. Then one readily sees

$$\|JF\|(r) = \sup_{z'\ne z, |z|\le r, |z'|\le r} \frac{|F(z') - F(z)|}{|z' - z|} \le \|F_z\|(r) + \|F_{\overline{z}}\|(r).$$

Let $p_0 = p$, $q_0 = q$, p_j, and q_j be given by (2.4). Put

$$\begin{aligned} |(p_j,q_j)|(r,\theta) &\equiv \sup_{(\xi,\eta)\in B(r,\theta)} \{|(p_j(\xi,\eta), q_j(\xi,\eta))|\}, \\ \|J(p_j,q_j)\|(r,\theta) &\equiv \sup_{(\xi,\eta)\in B(r,\theta)} \|J(p_j,q_j)(\xi,\eta)\|. \end{aligned}$$

Also, put

$$|(p_j,q_j)|(r) = |(p_j,q_j)|(r,\infty), \quad \|J(p_j,q_j)\|(r) = \|J(p_j,q_j)\|(r,\infty).$$

Proposition 2.2. *Let φ, ϵ and ρ be given by* (2.1) *and* (2.3), *and let r and θ be as in* (2.5). *Then*

$$|(p_j, q_j)|(r, \theta) \le c_2 \epsilon n r^{2s+1}(nr^{2s} + 1) \tag{2.9}$$

for $1 \le j < 2n$. Moreover,

$$\|J(p_{n-1}, q_{n-1})\|(r, \theta) \le c_3 \epsilon n r^{2s} e^{c_3 n r^{2s}}, \tag{2.10}$$

provided φ is additionally C^1 and satisfies

$$\|J(p, q)\|(r) \le \epsilon r^{2s}.$$

Proof. From (2.4) we have

$$\begin{aligned} p_j &= \lambda p_{j-1} e^{(\xi_j \eta_j)^s} + \lambda^{j+1} \xi e^{(j+1)(\xi\eta)^s}\left[e^{(\xi_j\eta_j)^s - (\xi\eta)^s} - 1\right] + p(\xi_j, \eta_j),\\ q_j &= \overline{\lambda} q_{j-1} e^{-(\xi_j\eta_j)^s} + \overline{\lambda}^{j+1} \eta e^{-(j+1)(\xi\eta)^s}[e^{-(\xi_j\eta_j)^s + (\xi\eta)^s} - 1] + q(\xi_j, \eta_j). \end{aligned}$$

Put

$$w_j = \max\{|(p_j(\xi, \eta), q_j(\xi, \eta))| \colon (\xi, \eta) \in B(r, \theta)\}.$$

By Proposition 2.1 we know that r_j, $\Delta\theta_j$ satisfy (2.6); and hence, $r_j \le e^6 r$ and $\theta_j \le 2\theta$ for $j = 1, \ldots, 2n - 1$. Thus

$$\begin{aligned} w_j &\le e^{\theta_j} w_{j-1} + c_2' \epsilon (j r^{4s+1} + r^{2s+1}) \le e^{2\theta} w_{j-1} + c_2' \epsilon (2n r^{4s+1} + r^{2s+1})\\ &\le e^{2j\theta} w_0 + c_2' \epsilon (2n r^{4s+1} + r^{2s+1}) \sum_{l=0}^{j-1} e^{2l\theta} \le c_2'' \epsilon n (n r^{2s} + 1) r^{2s+1}, \end{aligned}$$

which yields (2.9) for $c_2 \ge c_2''$.

To estimate the norm of Jacobian matrices, write

$$\begin{aligned} J(\varphi^n - T^n) = (J\varphi) \circ \varphi^{n-1} \cdot J(\varphi^{n-1} - T^{n-1})&\\ + (J\varphi - JT) \circ \varphi^{n-1} \cdot J(T^{n-1})&\\ + ((JT) \circ \varphi^{n-1} - (JT) \circ T^{n-1}) \cdot J(T^{n-1})&. \end{aligned}$$

Since $r_j \le e^6 r$, then

$$\|(J\varphi) \circ \varphi^j\|(r, \theta) \le 1 + c_3' r^{2s}$$

for $j = 1, \ldots, n$. Since $\theta < 1/n$, then

$$\|J(T^j)(\xi, \eta)\|(r, \theta) \le c_3''(1 + n r^{2s})$$

for $j = 1, \ldots, n$. By $JT(\xi, \eta) = JT(0) + O(|\xi|^{2s} + |\eta|^{2s})$, one gets

$$\begin{aligned} \|(JT) \circ \varphi^{n-1} - (JT) \circ T^{n-1}\|(r, \theta) &\le c_3''' r^{2s-1} \|\varphi^{n-1} - T^{n-1}\|(r, \theta)\\ &\le c_3''' c_2 (1 + n r^{2s}) \epsilon n r^{4s} \le c_3''' c_2 \epsilon (1 + n r^{2s})^2 r^{2s}, \end{aligned}$$

in which the second inequality comes from (2.9). Thus

$$\begin{aligned}\|J(\varphi^n - T^n)\|(r,\theta) &\leq (1+c_3'r^{2s})\|J(\varphi^{n-1}-T^{n-1})\|(r,\theta)\\ &+ \tilde{c}_3(1+nr^{2s})^3\epsilon r^{2s} \leq (1+c_3'r^{2s})^{n-1}\|J(\varphi-T)\|(r,\theta)\\ &+ \tilde{c}_3(1+nr^{2s})^3\epsilon r^{2s}\sum_{k=0}^{n-1}(1+c_3'r^{2s})^k \leq c_3 n\epsilon r^{2s}e^{c_3nr^{2s}}.\end{aligned}$$

The proof of the proposition is complete. □

3 Birkhoff curves

In this section we shall first prove the existence of certain Birkhoff real surfaces of the mappings. Then we shall prove the existence of periodic points of reversible mappings. The section is concluded with a version of the Birkhoff fixed-point theorem to be applied to the study of invariant totally real surfaces of holomorphic symplectic mappings. Throughout the paper $\sqrt[k]{z}$ stands for the root with argument in $(-\pi/k, \pi/k]$ for $k>0$ and $z \neq 0$.

Let us first describe the Birkhoff curves in details. We start with the twist mapping T of the real plane, given by $z_1 = \lambda z e^{i|z|^{2s}}$ ($|\lambda| = 1$). Away from the origin the fixed points of T^n form circles of radius r satisfying $\lambda^n e^{inr^{2s}} = 1$. These circles of fixed points can also be described as follows. Consider real lines in the complex plane given by

$$l_w \subset \mathbf{C}\colon\ \operatorname{Im}\{wz\} = 0, \quad |w| = 1,$$

i.e., the lines given by $\overline{z} = w^2 z$. The set of fixed points of T^n that are contained in $l_w \setminus \{0\}$ consists of points $z \in l_w \setminus \{0\}$ such that T^n sends them in the radial direction, i.e., such that $T^n(z)/z > 0$. By choosing z depending on $w \in S^1$ continuously, one obtain a circle of fixed points of T^n.

Consider a complexification of T given by

$$T_c\colon\ \xi_1 = \lambda\xi e^{(\xi\eta)^s}, \quad \eta_1 = \overline{\lambda}\eta e^{-(\xi\eta)^s}.$$

By identifying $\mathbf{R}^2$ with the totally real plane defined by $\eta = \sqrt[2s]{-1}\,\overline{\xi}$, one obtains $T_c|_{\mathbf{R}^2} = T$. Away from the origin the fixed points of T_c^n form the holomorphic curve

$$\mathcal{B} = \{0 \neq (\xi,\eta)\colon \lambda^n e^{n(\xi\eta)^s} = 1\}.$$

As in the real case, each complex line

$$\tilde{l}_w \subset \mathbf{C}^2\colon\ \eta = w^2\xi$$

contains $x \neq 0$ such that $T_c^n(x)$ returns to the complex line $\tilde{l}_w$ with $\operatorname{Re}(\pi_1(T_c^n(x))/\pi_1(x)) > 0$. As the complex line $\tilde{l}_w$ varies with w in the annulus

$$A = \{w \in \mathbf{C}\colon 1/2 < |w| < 2\},$$

such points x sweep out the Birkhoff holomorphic curve $\mathcal{B}$.

In general, given a family $\{\Gamma_w\}$ of real surfaces in $\mathbf{C}^2$, we call $\mathcal{B} \subset \mathbf{C}^2$ a *Birkhoff set* of φ of order n with respect to $\{\Gamma_w\}$, if for each $x \in \mathcal{B}$, $\varphi^n(x)$ and x belong to the same real surface Γ_w for some $w \in A$.

Put

$$\mathcal{R}_n = \{\zeta \in \mathbf{C}^*\colon \lambda^n e^{n\zeta^{2s}} = 1\},$$

$$\mu(\zeta) = \frac{1}{s4^{4s+1}} \min\{1, \frac{1}{n|\zeta|^{2s}}\}. \tag{3.1}$$

Given a positive integer n, let $\xi_0(w)$ be a holomorphic function on the annulus A, which satisfies

$$\lambda^n e^{nw^{2s}\xi_0^{2s}(w)} = 1. \tag{3.2}$$

Note that the holomorphic curve $w \to (\xi_0(w), w^2\xi_0(w))$ consists of fixed points of T_c^n. Put $\zeta_0 = \xi_0(1)$. Then ζ_0 is in $\mathcal{R}_n$ and $\xi_0(w)$ is determined uniquely by ζ_0. It is also clear that

$$|\zeta_0|/2 < |\xi_0(w)| = |w^{-1}\zeta_0| < 2|\zeta_0|, \quad w \in A.$$

Put

$$A_{\zeta_0} = \bigcup_{1/2<|w|<2} \{(\xi,\eta)\colon |\xi - \xi_0(w)| < \mu(|\zeta_0|)|\zeta_0|, |\xi|/8 < |\eta| < 8|\xi|\}.$$

Then A_{ζ_0} is contained in $B(24|\zeta_0|) \setminus B(\frac{1}{24}|\zeta_0|)$. Denote by $D(r,\zeta) \subset \mathbf{C}$ the disk of radius r, centered at ζ; put $D(r) = D(r, 0)$.

Proposition 3.1. *Let $\xi_0(w)$, ζ_0, and A_{ζ_0} be defined as above. Let φ be a continuous mapping given by*

$$\xi_1 = \lambda\xi e^{(\xi\eta)^s} + p(\xi,\eta), \quad \eta_1 = \overline{\lambda}\eta e^{-(\xi\eta)^s} + q(\xi,\eta)$$

with $|\lambda| = 1$ and

$$|(p(\xi,\eta), q(\xi,\eta))| \le \epsilon|(\xi,\eta)|^{2s+1}, \quad (\xi,\eta) \in B(\rho).$$

Consider a family of real surfaces defined by

$$\Gamma_w \subset \mathbf{C}^2\colon \eta = w^2\xi + u(\xi, w), \quad u(0, w) = 0, \quad w \in A.$$

Suppose that $u(\xi, w)$, $u_\xi(\xi, w)$ and $u_{\overline{\xi}}(\xi, w)$ are continuous in ξ and w, and that

$$|u_\xi(\xi, w)| + |u_{\overline{\xi}}(\xi, w)| \le \epsilon, \qquad w \in A, \quad (\xi,\eta) \in B(\rho). \tag{3.3}$$

Let $\mathcal{B}$ be the set of points $(\xi, \eta) \in A_{\zeta_0}$ with $\varphi^j(\xi, \eta) \in B(16e^6|\zeta_0|)$ for $1 \le j \le 2n$ such that for some $w \in A$

$$(\xi, \eta) \in \Gamma_w, \quad (\xi_n, \eta_n) = \varphi^n(\xi, \eta) \in \Gamma_w, \quad \operatorname{Re}\{\xi_n/\xi\} > 0.$$

Then $\mathcal{B} \cap \Gamma_w$ is non-empty for each $w \in A$, provided

$$\epsilon < \mu^2(\zeta_0)/c_4. \tag{3.4}$$

If $u(\xi, w) \equiv 0$ and φ is additionally a C^1 mapping satisfying

$$\|J(p, q)(\xi, \eta)\| \le \epsilon|(\xi, \eta)|^{2s}, \quad (\xi, \eta) \in B(\rho)$$

for $\epsilon < \mu^2(\zeta_0)/(c_5 e^{c_5/\mu(\zeta_0)})$, then $\mathcal{B} \cap \Gamma_w$ is a single point for $w \in A$, and $\mathcal{B}$ is a continuous real surface parameterized by $w \to \mathcal{B}(w)$, $w \in A$.

Proof. Fix $w \in A$ and denote $\xi_0(w)$ by ξ_0 for short. Assume that both (ξ, η) and $\varphi^n(\xi, \eta)$ are on Γ_w. Then ξ and η satisfy $\eta = w^2\xi + u(\xi, w)$ and

$$\begin{aligned}\overline{\lambda}^n \eta e^{-n(\xi\eta)^s} + q_{n-1}(\xi, \eta) = w^2(\lambda^n \xi e^{n(\xi\eta)^s} + p_{n-1}(\xi, \eta))\\ + u(\lambda^n \xi e^{n(\xi\eta)^s} + p_{n-1}(\xi, \eta), w).\end{aligned}$$

Rewrite the above equation as

$$\lambda^{2n} e^{2n((w\xi)^2 + \xi u(\xi, w))^s} = 1 + E(\xi, w) \tag{3.5}$$

with $E = E^* + E^{**}$ and

$$\begin{aligned}E^*(\xi, w) &= \lambda^n e^{n(\xi\eta)^s} \xi^{-1}\left[w^{-2} q_{n-1}(\xi, \eta) - p_{n-1}(\xi, \eta)\right],\\ E^{**}(\xi, w) &= w^{-2}\xi^{-1}\left[u(\xi, w) - \lambda^n e^{n(\xi\eta)^s} u(\lambda^n \xi e^{n(\xi\eta)^s} + p_{n-1}(\xi, \eta), w)\right].\end{aligned}$$

For $0 < |\xi_0| < \rho/4$ and $|\xi - \xi_0| < \mu(\zeta_0)|\zeta_0|$, one has $|\zeta_0|/4 < |\xi| < 3|\zeta_0|$. Also, (3.3) and (3.4) imply that $|u(\xi, w)| \le \epsilon|\xi| < |\zeta_0|/4$. Thus

$$\begin{gathered}\tilde{r} \equiv |(\xi, \eta)| \le 4|\zeta_0|,\\ |w^2\xi^2 - w^2\xi_0^2| \le 4^3|\zeta_0||\xi - \xi_0|, \quad |\xi u(\xi, w)| \le \epsilon|3\zeta_0|^2,\\ |(1 - t)(w^2\xi^2 + \xi u(\xi, w)) + t w^2\xi_0^2| \le \max\{5|\xi|^2, |\zeta_0|^2\} \le 4^3|\zeta_0|^2\end{gathered} \tag{3.6}$$

for $0 \le t \le 1$. Hence,

(3.7)
$$\begin{aligned}|(w^2\xi^2 + \xi u(\xi, w))^s - (w\xi_0)^{2s}| &\le s4^{3(s-1)}|\zeta_0|^{2s-2}(4^3|\zeta_0||\xi - \xi_0| + \epsilon|3\zeta_0|^2)\\ &\le s4^{4s+1}\mu(\zeta_0)|\zeta_0|^{2s} \equiv \theta \le 1/n,\end{aligned}$$

in which (3.4) and (3.1) are used. Since $\lambda^n e^{n(w\xi_0(w))^{2s}} = 1$, then $(w\xi_0)^{2s}$ is pure imaginary. Hence, one gets

$$\tilde{\theta} = |\operatorname{Re}\{(\xi\eta)^s\}| = |\operatorname{Re}\left\{(\xi^2 + \xi u(\xi, w))^s - (w\xi_0)^{2s}\right\}| \le \theta. \tag{3.8}$$

Also, (3.1) and (3.6) imply that

$$
\begin{aligned}
\epsilon\tilde{r}^{2s} &\le 4^{2s}\epsilon|\zeta_0|^{2s} \le \frac{4^{2s}}{c_4}|\zeta_0|^{2s}\mu^2(\zeta_0) \le c_1^{-1}\theta,\\
2\epsilon n\tilde{r}^{4s} &\le 2\epsilon n 4^{4s}|\zeta_0|^{4s} \le \frac{4^{4s+1}}{c_4} n|\zeta_0|^{4s}\mu^2(\zeta_0)\\
&\le \frac{4^{4s+1}}{c_4}|\zeta_0|^{2s}\mu(\zeta_0) \le c_1^{-1}\theta,
\end{aligned}
$$

provided $c_4 \ge 4^{4s+1}c_1$; hence, (2.5) holds when

$$\xi \in D(\xi_0, \mu(|\zeta_0|)|\zeta_0|), \quad (\xi, \eta) \in \Gamma_w.$$

Therefore, estimates in Proposition 2.1 and Proposition 2.2 apply. By (2.6) we obtain

$$\varphi^j\,(D(\xi_0, \mu(\zeta_0)|\zeta_0|) \cap \Gamma_w) \subset B(4e^6|\zeta_0|), \quad j = 1, \dots, 2n.$$

By (2.9), we get

$$|E^*(\xi, w)| \le c_4'' n\epsilon|\zeta_0|^{2s}(1 + n|\zeta_0|^{2s}).$$

To estimate E^{**}, note that $\lambda^n = e^{-n(w\xi_0)^{2s}}$ and

$$
\begin{aligned}
|u(\xi) - {}&\lambda^n e^{n(\xi\eta)^s} u(\lambda^n \xi e^{n(\xi\eta)^s} + p_{n-1}(\xi, \eta))|\\
&\le |u(\xi)| \cdot |1 - e^{n(\xi\eta)^s - n(w\xi_0)^{2s}}|\\
&\quad + \epsilon e^6(|\xi| \cdot |1 - e^{n(\xi\eta)^s - n(w\xi_0)^{2s}}| + |p_{n-1}(\xi, \eta)|)\\
&\le c_4''' n\epsilon\mu(\zeta_0)|\zeta_0|^{2s+1}(1 + n|\zeta_0|^{2s}),
\end{aligned}
$$

in which (2.9), (3.3) and (3.8) are used. Thus, we have

$$|E(\xi, w)| \le \tilde{c}_4\epsilon(n|\zeta_0|^{2s} + 1)n|\zeta_0|^{2s} \le \tilde{c}_4\mu(\zeta_0)n|\zeta_0|^{2s}/c_4 < 1/2, \tag{3.9}$$

in which the last two inequalities come from (3.4) and (3.1), respectively. With (3.2), (3.7) and (3.9), we now put equation (3.5) into the form

$$\xi = K(\xi, w)$$

with

$$
\begin{aligned}
K(\xi, w) = \xi_0 + \xi\{1 - \sqrt{1 + w^{-2}\xi^{-1}u(\xi, w)}\}\\
+ \xi_0\left\{ \sqrt[2s]{1 + \frac{\ln(1 + E(\xi, w, w^{-1}))}{2n(w\xi_0)^{2s}}} - 1 \right\}.
\end{aligned}
\tag{3.10}
$$

Note that $|\sqrt[k]{1 + z} - 1| \le |z|$ for $k \ge 1$ and $|z| \le 1/2$. By (3.3), we get

$$|\sqrt{1 + w^{-2}\xi^{-1}u(\xi, w)} - 1| \le |w^{-2}\xi^{-1}u(\xi, w)| \le 4\epsilon \le \mu(\zeta_0)/8,$$

in which the last inequality comes from (3.4). Also note that $|\log(1+z)| \leq 2|z|$ for $|z| < 1/2$. From (3.9) and $|\zeta_0| = |w\xi_0(w)|$, it follows that

$$\left| \sqrt[2s]{1 + \frac{\ln\left(1 + E(\xi, w, w^{-1})\right)}{2n(w\xi_0)^{2s}}} - 1 \right| \leq \tilde{c}_4 \mu(\zeta_0)/c_4.$$

Take $c_4 > 8\tilde{c}_4$. Thus for $w \in A$, the continuous map $K(\xi, w)$ sends the closure of the disk $D(\xi_0, \mu(\zeta_0)|\zeta_0|)$ into the disk; the degree theory says that $\xi = K(\xi, w)$ has a solution $\xi = \xi(w)$ in the disk. Since $|\xi| > |\xi_0|/4$, then (2.9), (3.1) and (3.7) implies that

$$\mathrm{Re}\{\xi_n/\xi\} \geq \mathrm{Re}\{e^{n((\xi\eta)^s - (\xi_0(w)w)^{2s})}\} - |p_{n-1}(\xi, \eta)|/|\xi| > 0$$

for some choice of c_4.

Next, we want to show that $\mathcal{B}$ is a C^0 real surface, that is that $\xi = \xi(w)$ is continuous in $w \in A$. To this end we need to show that $K(\cdot, w)$ is a contraction map through estimating E_ξ and $E_{\overline{\xi}}$. We first have $u \equiv 0$ and $E = E^*$. With $\eta = w^2\xi$, we also have

$$\left| \frac{\partial}{\partial \xi} e^{n(\xi\eta)^s} \right| \leq se2^{2s} n|\xi|^{2s-1} \leq se8^{2s} n|\zeta_0|^{2s-1}.$$

Together with (2.10), we get

$$|E_\xi(\xi, w)| \leq c_5' \epsilon n |\zeta_0|^{2s-1} e^{c_5' n|\zeta_0|^{2s}}.$$

A similar estimate also holds for $E_{\overline{\xi}}$. Thus, we conclude that

$$|E_\xi(\xi, w)| + |E_{\overline{\xi}}(\xi, w)| \leq 2c_5' \epsilon n |\zeta_0|^{2s-1} e^{c_5' n|\zeta_0|^{2s}}.$$

Now, from (3.10) one sees that

$$|K_\xi(\xi, w)| + |K_{\overline{\xi}}(\xi, w)| \leq c_5'' \epsilon e^{c_5' n|\zeta_0|^{2s}} < 1/2, \quad \xi \in D(\xi_0, \mu(\zeta_0)|\zeta_0|)$$

for $c_5 = \max\{c_4, 2c_5'', c_5'\}$. Therefore, the mapping $\xi \to K(\xi, w)$ is a contraction from $D(\xi_0, \mu(\zeta_0)|\zeta_0|)$ into itself.

By the Picard iteration, one can show that $\xi(w)$ is unique and is continuous in w. The proof of the proposition is complete. □

Proof of Theorem 1.1. We assume that the homeomorphism φ is reversible with respect to a C^1 involution τ with a holomorphic linear part. It is easy to see that φ, given by (1.1), is differentiable at $(\xi, \eta) = 0$. Thus, the linear part of φ is also reversible with respect to that of τ. Since the linear part of τ is **C**-linear and that the eigenvalues of the linear part of φ are distinct, then τ is of the form $(\xi, \eta) \to (a\eta, b\xi) + o(|\xi| + |\eta|)$. Since τ is an involution, then $ab = 1$. By a linear transformation $(\xi, \eta) \to (\xi/\sqrt{a}, \sqrt{a}\eta)$, one may assume that $a = b = 1$.

Applying the dilation $(\xi,\eta)\to\sqrt[2s]{a_s}(\xi,\eta)$, one further achieves that $a_s=1$. Note that in the new (and from now on fixed) linear coordinates φ still satisfies the assumptions of the theorem.

Next, we want to show that the fixed points of τ form a C^1 real surfaces. Since the linear part of τ is $\tau_0\colon \xi\to\eta, \eta\to\xi$, then $\phi=(\mathrm{Id}+\tau_0^{-1}\circ\tau)/2$ is a C^1 change of coordinates. From $\tau^2=\mathrm{Id}=\tau_0^2$, it follows that $\phi\circ\tau=\tau_0\circ\phi$. Thus τ is equivalent to τ_0. Since the fixed points of τ_0 are given by $\eta=\xi$, the fixed points of τ form a C^1 real surface

$$\Gamma_1\colon \eta=\xi+u(\xi),\quad u(\xi)=o(|\xi|),$$

which fits into the family of C^1 real surfaces $\Gamma_w\colon \eta=w^2\xi+u(\xi), w\in A$.

We now find the periodic points for φ. Write $\lambda=e^{i2\pi\alpha}$ with $0\le\alpha<1$ and

$$n\alpha=\beta_n+l_n,\quad -1/2<\beta_n\le 1/2,\quad l_n\in\mathbf{Z}.$$

Let $K\ge 1$ be a fixed integer, and let $\zeta_0\neq 0$ be a solution to

$$n\zeta_0^{2s}=(-\beta_n+j)2\pi i,$$

where j is an integer satisfying $|j|\le K$. Since $n|\zeta_0|^{2s}\le 2(K+1)\pi$, then

$$\mu(\zeta_0)\ge\frac{1}{2(K+1)\pi}.$$

Fix ρ with $0<\rho<1$ such that

$$|u_\xi(\xi)|+|u_{\overline{\xi}}(\xi)|<\epsilon,\quad |(p(\xi,\eta),q(\xi,\eta))|\le\epsilon|(\xi,\eta)|^{2s+1}$$

for $(\xi,\eta)\in B(\rho)$ and

$$\epsilon=\frac{1}{4c_4(K+1)^2\pi^2}\le\frac{1}{c_4}\mu^2(\zeta_0).$$

Applying Proposition 3.1, there exists $(\xi^*,\eta^*)\in\Gamma_1$ such that $\varphi^n(\xi^*,\eta^*)$ is on Γ_1 and $\varphi^j(\xi^*,\eta^*)$ are contained in $B(16e^6|\zeta_0|)$ for $j=1,\dots,2n$; in particular (ξ^*,η^*) and $\varphi^n(\xi^*,\eta^*)$ are fixed by τ. Thus, $\varphi^{-n}(\xi^*,\eta^*)=\tau(\varphi^n(\xi^*,\eta^*))$ is well-defined, and

$$\varphi^{-n}(\xi^*,\eta^*)=\tau\varphi^n\tau^{-1}(\xi^*,\eta^*)=\varphi^n(\xi^*,\eta^*).$$

This shows that (ξ^*,η^*) is a fixed point of φ^{2n}.

We now identify the orbit located in $B^*(c_0\kappa_n(\lambda))$, as stated in Theorem 1.1. We choose ζ_0 as follows:

$$\zeta_0=\begin{cases}\left(-i\frac{2\pi\beta_n}{n}\right)^{\frac{1}{2s}} & \beta_n\neq 0,\\ \left(i\frac{2\pi}{n}\right)^{\frac{1}{2s}} & \text{otherwise.}\end{cases}$$

Then φ^{2n} has a periodic point (ξ^*, η^*) with $\xi^* \in D(\xi_0, \mu(\zeta_0)|\zeta_0|)$ and $\eta^* \in \Gamma_1$. Also, $\varphi^j(\xi^*, \eta^*)$ are contained in $B(16e^6|\zeta_0|)$. If $\beta_n \neq 0$, then $\lambda^n \neq 1$ and $2\pi|\beta_n| > |\lambda^n - 1|$. Hence,

$$|\zeta_0| < \left[\frac{2\pi|\beta_n|}{n}\right]^{\frac{1}{2s}} < \left[\frac{|\lambda^n - 1|}{n}\right]^{\frac{1}{2s}} = \kappa_n(\lambda).$$

If $\beta_n = 0$, then $\lambda^n = 1$ and

$$|\zeta_0| = \left[\frac{|2\pi|}{n}\right]^{\frac{1}{2s}} < 2\pi\kappa_n(\lambda).$$

Thus, (ξ^*, η^*) and its orbit are contained in $B^*(32\pi e^6\kappa_n(\lambda))$. The proof of the theorem is thus complete. □

Proof of Theorem 1.2. By the assumptions, the mapping $\varphi\colon (\xi, \eta) \to (\lambda\xi, \overline{\lambda}\eta) + O(2)$ is reversible with respect to a holomorphic involution. Since λ is not a root of unity and φ is not linearizable, then a theorem of Moser and Webster [8] says that there exists a change of holomorphic coordinates such that φ is of the form (1.1) with $a_s = 1$. Now Theorem 1.2 follows from Theorem 1.1. □

The following theorem is proved by Moser [7] for the case $s = 1$; see also Siegel and Moser [10] for the real analytic case.

Theorem 3.2. *Let φ be a C^1 transformation of the real plane, given by*

$$z_1 = \lambda z e^{a_s|z|^{2s}} + p(z), \quad p(0) = 0,$$

in which a_s, λ are complex numbers satisfying $a_s \neq 0$ and $|\lambda| = 1$, and $p(z)$ is a complex-valued C^1 function satisfying

$$|p_z(z)| + |p_{\overline{z}}(z)| = o(|z|^{2s}). \tag{3.11}$$

Let U be an open neighborhood of the origin, and let μ be a Borel measure on U that is positive on all non-empty open subsets of U. If φ preserves the measure μ, then $\operatorname{Re} a_s = 0$, *and there exists $c_6 > 0$ such that if n is sufficiently large, then φ has a periodic orbit in the punctured disk $D^*(c_6\kappa_n(\lambda, a_s))$ of period dividing n, in which*

$$\kappa_n(\lambda, a_s) = \begin{cases} \left[\dfrac{|\lambda^n - 1|}{n}\right]^{\frac{1}{2s}} & \text{if } \operatorname{Im} a_s \cdot \operatorname{Im}\lambda^n < 0, \\ \left(\dfrac{1}{n}\right)^{\frac{1}{2s}} & \text{otherwise.} \end{cases} \tag{3.12}$$

Proof. Since $p(0)=0$, then (3.11) implies that $p(z)=o(|z|^{2s+1})$. It is also clear that $\Re a_s = 0$; otherwise, either φ or its inverse sends a disk (of small radius) into a smaller disk, which contradicts that φ preserves the measure μ. Applying the dilation $z \to \sqrt[2s]{|a_s|}z$, one achieves that $a_s = \pm i$. Note that replacing φ by φ^{-1} results in $\lambda \to \overline{\lambda}$ and $a_s \to -a_s$, and that $\kappa_n(\lambda, a_s) = \kappa_n(\overline{\lambda}, -a_s)$. Thus, replacing φ with φ^{-1} if necessary one may further assume that $a_s = i$.

Identifying the z-plane with the totally real surface $M \subset \mathbf{C}^2\colon \eta = \sqrt[s]{i}\,\overline{\xi}$ by the embedding $z \to \sqrt[2s]{i}(z, \overline{z})$, one has $\widetilde{\varphi}|_M = \varphi$ for

$$\widetilde{\varphi}\colon \xi_1 = \lambda\xi e^{(\xi\eta)^s} + \widetilde{p}(\xi,\eta), \quad \eta_1 = \overline{\lambda}\eta e^{-(\xi\eta)^s} + \widetilde{q}(\xi,\eta),$$

in which $\tilde{p}(\xi,\eta) = \sqrt[2s]{i}\; p(\overline{\sqrt[2s]{i}}\,\xi)$ and $\tilde{q}(\xi,\eta) = \sqrt[2s]{i}\;\overline{p(\sqrt[2s]{i}\,\overline{\eta})}$. Write $\lambda = e^{2\pi i\alpha}$ and $n\alpha = l + \beta_n$ with $l \in \mathbf{Z}$ and $-1/2 < \beta_n \le 1/2$. Put

$$\zeta_0 = \begin{cases} \left(-i\,\frac{2\pi\beta_n}{n}\right)^{\frac{1}{2s}} & -1/2 < \beta_n < 0, \\ \left(i\,\frac{2\pi(1-\beta_n)}{n}\right)^{\frac{1}{2s}} & \text{otherwise.} \end{cases}$$

Then $\lambda^n e^{n\zeta_0^{2s}} = 1$; in particular, $\zeta_0 \in \mathcal{R}_n$. Also, $|\zeta_0|/c_7 < \kappa_n(\lambda, i) < c_7|\zeta_0|$ for some positive constant c_7. Note that ζ_0 (dependent of n) approaches to 0 as $n \to \infty$. On the other hand $\mu(\zeta_0)$ is bound from below by a positive constant. Thus for n sufficiently large, we have

$$\|J(\tilde{p},\tilde{q})\|(r) < \epsilon r^{2s}, \quad \text{for } \epsilon = \frac{\mu^2(\zeta_0)}{c_5 e^{c_5/\mu(\zeta_0)}}.$$

Applying Proposition 3.1, we obtain a real curve

$$C = \{(\xi(w), \xi(w)w^2)\colon |w| = 1\} \subset D(24|\zeta_0|) \setminus D(|\zeta_0|/24)$$

such that for $(\xi,\eta) \in C$, $(\xi_n,\eta_n) = \varphi^n(\xi,\eta)$ satisfies $\mathrm{Re}(\xi_n/\xi) > 0$. Next, we want to show that ξ_n/ξ is actually positive on C and that C is contained in M. To this end, note that M is the set of fixed points of the anti-holomorphic involution $\rho\colon (\xi,\eta) \to (\sqrt[s]{i}\,\overline{\eta}, \sqrt[s]{i}\,\overline{\xi})$ and that $\tilde{\varphi} = \rho\tilde{\varphi}\rho$. Assume now that $|w| = 1$. We have $\tilde{\varphi}^n(\xi(w), \xi(w)w^2) = c(w)(\xi(w), \xi(w)w^2)$ for some $c(w)$ with $\mathrm{Re}\, c(w) > 0$ and $\tilde{\varphi}^n(\sqrt[s]{i}\,\overline{\xi(w)}\overline{w}^2, \sqrt[s]{i}\,\overline{\xi(w)}) = \overline{c(w)}(\sqrt[s]{i}\,\overline{\xi(w)}\overline{w}^2, \sqrt[s]{i}\,\overline{\xi(w)})$. Put $\xi_0(w) = \zeta_0/w$. By the definition of ζ_0, we have $\sqrt[s]{i}\,\overline{\xi_0(w)}\overline{w}^2 = \xi_0(w)$, and hence $|\sqrt[s]{i}\,\overline{\xi(w)}\overline{w}^2 - \xi_0(w)| = |\xi - \xi_0(w)|$. Thus, $(\sqrt[s]{i}\,\overline{\xi(w)}\,\overline{w}^2, \sqrt[s]{i}\,\overline{\xi(w)})$ belongs to A_{ζ_0}, as does $(\xi(w), \xi(w)w^2)$. The uniqueness of solution (ξ,η) to $\xi_n\eta = \eta_n\xi$ as formulated in Proposition 3.1 implies that $\sqrt[s]{i}\,\overline{\xi(w)}\overline{w}^2 = \xi(w)$ and $c(w) = \overline{c(w)}$ for $|w| = 1$. The former means that C is contained in M. The latter and $\mathrm{Re}\, c(w) > 0$ imply that $c(w) > 0$. Now identify $C \cap M$ with a real curve $\tilde{C}$ in the z-plane. The original mapping φ sends a point on $\tilde{C}$ to a point in the radial direction, since z_n/z is positive on $\tilde{C}$. Since φ preserves the measure μ, then $\varphi^n(\tilde{C})$ must intersect $\tilde{C}$. Note that no two points on $\tilde{C}$ have the same argument. Thus φ^n actually fixes each point in $\tilde{C} \cap \varphi^n(\tilde{C})$. Finally, as we proved at the end of the proof of Theorem 1.1, one can take $c_6 = 32\pi c_7 e^6$ such that the whole periodic orbits is contained in $D(c_6\kappa_n(\lambda, a_s))$. The proof is complete. □

4 A subgroup of holomorphic symplectic mappings

In this section we shall introduce a subgroup $\mathcal{H}_0$ of holomorphic symplectic mappings of $\mathbf{C}^2$. The proof of Theorem 1.3 will make full use of the group structure of $\mathcal{H}_0$. We shall see that each transformation in $\mathcal{H}_0$ is already in the Birkhoff normal form, if it preserves the totally real plane $\mathbf{R}^2\colon \eta = \overline{\xi}$.

To describe $\mathcal{H}_0$, let $\mathcal{H}$ be the group of germs of biholomorphic transformations at the origin of $\mathbf{C}^2$, which preserve both coordinate axes. More specifically, $\mathcal{H}$ consists of all transformations of the form

$$\xi_1 = \xi f(\xi, \eta), \quad \eta_1 = \eta g(\xi, \eta), \tag{4.1}$$

where f, g are convergent power series satisfying $f(0)g(0) \neq 0$. Let $\mathcal{F}_0$ be the set of convergent power series

$$f(\xi, \eta) = \sum f_{ij}\xi^i\eta^j, \quad f_{ij} = 0, \quad i < j.$$

Let $\mathcal{F}_1 \subset F_0$ be the set of power series f satisfying the additional condition $f_{ii} = 0$ for $i > 0$. For $j = 0, 1$, denote by $\mathcal{H}_j$ the set of biholomorphic transformations (4.1) with $f, g \in \mathcal{F}_j$.

Given a power series $f(\xi, \eta)$, we set

$$\mathcal{N} f(\xi, \eta) = \sum_{i \geq 0} f_{ii}\xi^i\eta^i.$$

For φ given by (4.1), define $\mathcal{N}\varphi$ by $\xi_1 = \xi \mathcal{N} f(\xi, \eta)$, $\eta_1 = \eta \mathcal{N} g(\xi, \eta)$.

Lemma 4.1. *Let $\mathcal{F}_j, \mathcal{H}_j$ be defined as above. For $p \in \mathcal{F}_j$ and $\varphi \in \mathcal{H}_0$, one has $p \circ \varphi \in \mathcal{F}_j$. Also, $\mathcal{H}_0, \mathcal{H}_1$ are subgroups of $\mathcal{H}$ with $\mathcal{N}(\varphi_1 \circ \varphi_1) = (\mathcal{N}\varphi_1) \circ (\mathcal{N}\varphi_2)$ for $\varphi_1, \varphi_2 \in \mathcal{H}_0$.*

Proof. Dropping the restriction $f(0)g(0) \neq 0$, we let $\hat{\mathcal{H}}_0$ be the set of mappings (4.1) with $f, g \in \mathcal{F}_0$. We shall first prove that $\mathcal{F}_0, \mathcal{F}_1$ are closed under the action of $\hat{\mathcal{H}}_0$. Evidently, $\mathcal{F}_0$ contains $\mathbf{C}$ and is closed under addition and multiplication. Hence, if $p(z_1, \ldots, z_k)$ is a power series and $f_1, \ldots, f_k$ are elements of $\mathcal{F}_0$ with all $f_j(0) = 0$, then $p(f_1, \ldots, f_k)$ remains in $\mathcal{F}_0$. Now take $\varphi \in \hat{\mathcal{H}}_0$ and $p \in \mathcal{F}_0$. Then

$$p \circ \varphi(\xi, \eta) = p(0) + \sum_{i \geq j} p_{ij}\xi^i\eta^j f^i(\xi, \eta)g^j(\xi, \eta).$$

Since $\xi^i\eta^j$ are in $\mathcal{F}_0$ for $i \geq j$ and $f^i g^j$ are in $\mathcal{F}_0$, then $p \circ \varphi$ is in $\mathcal{F}_0$ also.

Next, we want to show that $\hat{\mathcal{H}}_0$ (and hence $\mathcal{H}_0$) is closed under composition. Take $\varphi_j \in \hat{\mathcal{H}}_0$ $(j = 1, 2)$, for which $f_j, g_j \in \mathcal{F}_j$ are power series corresponding to f, g in (4.1). Then $\varphi_1 \circ \varphi_2$ is still of the form (4.1) with

$$f = f_1 \circ \varphi_2 \cdot f_2, \quad g = g_1 \circ \varphi_2 \cdot g_2,$$

from which one sees that $\varphi_1 \circ \varphi_2$ remains in $\hat{\mathcal{H}}_0$. Note that the above two identities also say that $\mathcal{N}(\varphi_1 \circ \varphi_1) = (\mathcal{N}\varphi_1) \circ (\mathcal{N}\varphi_2)$ holds for $\varphi_1, \varphi_2 \in \mathcal{H}_0$.

Now, we want to show that $\mathcal{H}_0$ is closed under inversion. Take $\varphi \in \mathcal{H}_0$. Denote by ϕ_1 the inverse of the linear part of φ. It is clear that ϕ is in $\mathcal{H}_0$. Write $\phi_1 \circ \varphi = \mathrm{Id} + \widetilde{\varphi}_1$. Then $\widetilde{\varphi}_1 = O(2)$ is in $\hat{\mathcal{H}}_0$. Hence, $\phi_2 = \mathrm{Id} - \widetilde{\varphi}_1$ is in $\mathcal{H}_0$ also. Put $\phi_2 \circ \phi_1 \circ \varphi = \mathrm{Id} + \widetilde{\varphi}_2$. Recursively, one obtains

$$\phi_n \circ \cdots \circ \phi_1 \circ \varphi = \mathrm{Id} + \widetilde{\varphi}_n$$

with $\phi_n \in \mathcal{H}_0$, $\widetilde{\varphi}_n \in \hat{\mathcal{H}}_0$, and $\widetilde{\varphi}_n = O(2^{n-1} + 1)$. Thus $\varphi^{-1} = \ldots \phi_n \circ \cdots \circ \phi_1$ is in $\mathcal{H}_0$.

A similar argument shows that $\mathcal{H}_1$ is a subgroup of $\mathcal{H}_0$. □

We shall denote by $N_j(n) > 1$, $N_j'(n) > 1$, etc., constants depending only on n and $m_1, \ldots, m_n$, in which $m_1, \ldots, m_n$ are positive integers given as in Theorem 1.3. Put

$$\delta_n = \delta_n(\lambda) = \min\{1/2, |\lambda^k - 1| \colon 1 \leq k \leq n - 1\}.$$

For a convergent power series $f(\xi, \eta)$, define

$$M_f = \sup\{|f_{ij}|^{\frac{1}{i+j-1}} : i + j > 0\}.$$

For a holomorphic mapping $F = (f_1, f_2)$ defined by two power series $f_j(\xi, \eta)$ convergent near the origin, put

$$M_F = \max\{M_{f_1}, M_{f_2}\}.$$

It is straightforward that if L_c $(c \neq 0)$ is the dilation $(\xi, \eta) \to (c\xi, c\eta)$, then

$$M_{cf \circ L_c^{-1}} = c^{-1} M_f.$$

Given two formal power series f, g in the same multivariable. We say that f is *majorized* by g, denoted by $f \prec g$, if coefficients f_α and g_α satisfy $|f_\alpha| \leq g_\alpha$ for all α. We need the following

Lemma 4.2. *Let φ be a holomorphic transformation with $\varphi(\xi, \eta) = (\lambda\xi, \overline{\lambda}\eta) + O(2)$. There exists constant c_1 such that if $|\lambda| = 1$ and n is an integer, then*

$$\varphi^n \colon B(r) \to B(2r), \quad r < 1/(c_1|n|M_\varphi). \tag{4.2}$$

Proof. The lemma is trivial if $M_\varphi = 0$. We now consider the case $M_\varphi \neq 0$. Replacing φ with $L_{M_\varphi}\varphi L_{M_\varphi}^{-1}$ if necessary, one may assume that $M_\varphi = 1$. Now φ is majorized by

$$\widehat{\varphi}\colon \xi' = \xi + \frac{(\xi+\eta)^2}{1-\xi-\eta}, \quad \eta' = \eta + \frac{(\xi+\eta)^2}{1-\xi-\eta}.$$

Obviously, $\widehat{\varphi}$ sends $B(r)$ into $B(r(1+8r))$ for $r < 1/4$. Note that the mapping $r \to r(1+8r)$ is majorized by the time-1 mapping of the vector field $\dot{r} = 8r^2$ of which the flow is $r \to r/(1-8tr)$. For a positive integer n one obtains

$$\varphi^n\colon B(r) \to B(2r), \quad r < 1/(16n).$$

To obtain (4.2) for the negative integer n, it suffices to show $M_{\varphi^{-1}} \le c_1' M_\varphi$ for some constant c_1'. Write

$$\varphi^{-1} = \phi_1 + \phi_2$$

with ϕ_1 being the linear part of φ^{-1}. From $\varphi\varphi^{-1} = \mathrm{Id}$, one gets $\phi_1^{-1}\phi_2 = (\phi_1^{-1} - \varphi) \circ (\phi_1 + \phi_2)$. Thus, $\phi_2 \prec (u, u)$ with $u(\xi, \eta)$ being determined by

$$u = \frac{(\xi+\eta+2u)^2}{1-\xi-\eta-2u}, \quad u(0) = 0.$$

It is clear that u is convergent and $M_{\varphi^{-1}} \le M_u \equiv c_1'$. The proof of the lemma is complete. □

Lemma 4.3. *For each holomorphic function f with $f(0) = \lambda$ and $|\lambda| = 1$, there exists a unique holomorphic function $g(\xi, \eta)$ such that* (4.1) *defines a symplectic mapping φ_f.*

(a) There exists a polynomial $Q_{ij}f$ in $\lambda, \overline{\lambda}$ and $f_{i'j'}$ with $0 < i' + j' < i + j$ such that

$$g_{ij} = -\frac{i+1}{j+1}\lambda^{-2} f_{ij} + Q_{ij}(f). \tag{4.3}$$

If $f_{i'j'} = 0$ for $i' + j' < i + j$, then

$$Q_{ij}f = 0.$$

(b) $M_g \le c_2 M_f$ for some constant $c_2 > 1$.

(c) Assume that f is in $\mathcal{F}_j$. Then φ_f is in $\mathcal{H}_j$ and $\mathcal{N}\varphi_f$ is the symplectic mapping

$$\varphi_{\mathcal{N}f}\colon \xi' = \xi\,\mathcal{N}f(\xi, \eta), \quad \eta' = \eta/\mathcal{N}f(\xi, \eta).$$

Proof. Let φ be given by (4.1). Then φ is symplectic if and only if g satisfies

$$g(\xi,\eta)+\eta g_\eta(\xi,\eta)=\frac{1+[\eta f_\eta(\xi,\eta)][\xi g_\xi(\xi,\eta)]}{f(\xi,\eta)+\xi f_\xi(\xi,\eta)}. \tag{4.4}$$

Collecting the coefficients of $\xi^i\eta^j$ on both sides of (4.4), one sees that

$$(1+j)g_{ij}=-\lambda^{-2}(i+1)f_{ij}+\dots,$$

where the omitted term is a polynomial in $\overline{\lambda}$, λ and $f_{i'j'}$, $g_{i'j'}$ with $0<i'+j'<i+j$. In particular, g_{ij} are uniquely determined by f, and (a) holds.

To see the convergence of $g(\xi,\eta)$, note that $g(0)=\overline{\lambda}$, and that $g(\xi,\eta)-g(0)$ is majorized by G satisfying

$$G(\xi,\eta)+\eta G_\eta(\xi,\eta)=\frac{1+\big[\xi K_\eta(\xi,\eta)\big]\cdot\big[\eta G_\xi(\xi,\eta)\big]}{1-K(\xi,\eta)-\xi K_\xi(\xi,\eta)}$$

for $K(\xi,\eta)=\sum_{i+j>0}M_f^{i+j}\xi^i\eta^j$. In fact, $\eta G(\xi,\eta)=\tilde{G}$ is the unique solution to the Cauchy problem

$$\tilde{G}_\eta=\frac{1}{1-K(\xi,\eta)-\xi K_\xi(\xi,\eta)}\Big(1+\big[\eta K_\eta(\xi,\eta)\big]\cdot\big[\tilde{G}_\xi(\xi,\eta)\big]\Big)$$

with the initial data $\tilde{G}|_{\eta=0}=0$. This implies that $\tilde{G}$, and hence G, is convergent. This establishes the convergence of g, and the proof of the first statement in the lemma is complete.

Note that the above argument also yields $M_g\le c_2$, if $M_f=1$ or 0. The proof of (b) for the case $M_f\ne 0$ is reduced to the case $M_f=1$, by replacing f, φ_f with $f\circ L_{M_f}^{-1}$ and $L_{M_f}\varphi_f L_{M_f}^{-1}$, respectively.

To prove (c), we return to (4.4). For $f\in\mathcal{F}_j$ with $f(0)=\lambda$, we have $\frac{1}{f+\xi f_\xi}$, $\eta f_\eta\in\mathcal{F}_j$. It is trivial that the linear term in ξg_ξ is in $\mathcal{F}_j$. By induction, one can verify that the homogeneous terms of g are in $\mathcal{F}_j$. For $f,g\in\mathcal{F}_0$ one readily sees that

$$\begin{aligned}1&=\mathcal{N}\left\{\frac{\partial(\xi f(\xi,\eta))}{\partial\xi}\frac{\partial(\eta g(\xi,\eta))}{\partial\eta}-\frac{\partial(\xi f(\xi,\eta))}{\partial\eta}\frac{\partial(\eta g(\xi,\eta))}{\partial\xi}\right\}\\&=\left\{\frac{\partial(\xi\mathcal{N}f(\xi,\eta))}{\partial\xi}\frac{\partial(\eta\mathcal{N}g(\xi,\eta))}{\partial\eta}-\frac{\partial(\xi\mathcal{N}f(\xi,\eta))}{\partial\eta}\frac{\partial(\eta\mathcal{N}g(\xi,\eta))}{\partial\xi}\right\}.\end{aligned}$$

Thus, $\mathcal{N}\varphi$ is also symplectic. A straightforward computation shows that the mapping $\xi\to\xi\mathcal{N}f,\eta\to\eta/\mathcal{N}f$ is symplectic. Now the uniqueness of g implies that $\mathcal{N}\varphi_f=\varphi_{\mathcal{N}f}$ and $\mathcal{N}g=1/\mathcal{N}f$. The proof of the lemma is complete.

Proposition 4.4. *Let φ_f be a holomorphic symplectic mapping of the form*

$$\xi_1 = \xi f(\xi,\eta), \quad \eta_1 = \eta g(\xi,\eta), \quad f(0) = \lambda = \overline{g(0)}.$$

Assume that λ is not a root of unity and $f \in \mathcal{F}_0$. For each positive integer n, there exists a unique polynomial $u_n \in \mathcal{F}_1$ of the form

$$u_n(\xi,\eta) = 1 + \sum_{0 \le k < j < n,\, j+k \le n} u_{jk}\xi^j\eta^k$$

such that $\varphi_{u_n}\varphi_f\varphi_{u_n}^{-1} = \varphi_{f_n}$ with $f_n \in \mathcal{F}_0$ and

$$f_n(\xi,\eta) = \mathcal{N} f_n(\xi,\eta) + f_{n;n,0}\xi^n + O(n+1). \tag{4.5}$$

Furthermore, the following hold

(a) $f_{n;n,0} = f_{n0} + Q_n(f)$.
(b) $Q_n(f)$ *and coefficients of* $u_n(\xi,\eta)$ *are polynomials in quantities*

$$\lambda,\ \overline{\lambda},\ f_{ij}\ (i+j \le n,\, j \le i < n),\ \frac{1}{1-\lambda^k}\ (1 \le k \le n-1). \tag{4.6}$$

(c) If $M_f \le 1$, *there exists a constant* $N_0(n) > 1$ *such that*

$$M_{u_n} \le \delta_n^{-N_0(n)}, \quad M_{f_n} \le \delta_n^{-N_0(n)}. \tag{4.7}$$

Proof. Let us first discuss the effect of a holomorphic symplectic transformation φ_u on φ_f, where $u(\xi,\eta)$, $f(\xi,\eta)$ are holomorphic functions with $u(0) = 1$. Lemma 4.1 says that $\varphi_u\varphi_f\varphi_u^{-1} = \varphi_{f_n}$ for some holomorphic function $f_n(\xi,\eta)$.

We assume that $u(\xi,\eta) - 1$ is a homogeneous polynomial of degree $d \le n$. Write

$$\varphi_u \circ \varphi_f = \varphi_{f_n} \circ \varphi_u.$$

The first components on both sides of the above identity give us

$$f(\xi,\eta)\cdot u\left(\varphi_f(\xi,\eta)\right) = u(\xi,\eta)\cdot f_n\left(\varphi_u(\xi,\eta)\right).$$

Comparing terms of order less than d yields $f_n(\xi,\eta) - f(\xi,\eta) = O(d)$, while terms of order d tell us

$$\lambda u(\lambda\xi, \lambda^{-1}\eta) + [f]_d(\xi,\eta) = \lambda u(\xi,\eta) + [f_n]_d(\xi,\eta).$$

Note that $f(0) = \lambda$ is not a root of unity. For a fixed j with $0 \le j < d/2$ one puts

$$u_{d-j,j} = \frac{\lambda^{-1}}{1-\lambda^{d-2j}} f_{d-j,j}, \quad u_{i',j'} = 0, \quad (i',j') \ne (d-j,j).$$

Then $f_{n;d-j,j} = 0$ and $f_{n;i'j'} = f_{i'j'}$ for $i'+j' \le d$ and $(i',j') \ne (d-j,j)$. Recursively, one determines u_{ij} for all $i+j < n$ and $u_{n-k,k}$ for $0 \le k < n/2$

($u_{ij} = 0$ for all $i \le j$ and $i+j > 0$) such that for $u_n = \sum_{j<i<n, i+j\le n} u_{ij}\xi^i\eta^j$ one has $\varphi_{u_n}\varphi_f\varphi_{u_n}^{-1} = \varphi_{f_n}$ with $f_{n;ij} = 0$ for $j < i < n$ and $i+j \le n$. Inductively, one can verify that the coefficients of u_n and $f_{n;n0} - f_{n0}$ are polynomial in quantities (4.6).

Since f is in $\mathcal{F}_0$ by the assumption and u_n is in $\mathcal{F}_1 \subset \mathcal{F}_0$, Lemma 4.1 says that f_n is in $\mathcal{F}_0$ also. Therefore, we actually have $f_{n;ij} = 0$ for all $i < j$ and achieve (4.5). The proof of (a) and (b) is complete.

For the proof of (c), we note that the coefficients of the polynomial u_n have absolute values bounded by $\delta_n^{-N_0'(n)}$. Thus $M_{u_n} \le \delta_n^{-N_0'(n)}$. By Lemma 4.2, there is a constant c_3 such that both φ_{u_n} and $\varphi_{u_n}^{-1}$ send $B(r)$ into $B(2r)$ for $r < \delta_n^{N_0'(n)}/c_3$. Also φ_f sends $B(r)$ into $B(2r)$ for some $r < 1/c_3'$ with $c_3' > c_1$. Hence $\varphi_{f_n} = \varphi_{u_n}\varphi\varphi_{u_n}^{-1}$ sends $B(\delta_n^{N_0'(n)}/c_3'')$ into $B(8\delta_n^{N_0'(n)}/c_3'')$ for $c_3' > \max\{4c_3, 2c_3'\}$. By the Cauchy inequalities, one obtains $M_{f_n} \le 8\sqrt{2}c_3\delta_n^{-N_0'(n)}$. Therefore, there exists $N_0(n) > N_0'(n)$ such that (4.7) holds. □

Remark. The purpose of Proposition 4.4 is to use transformations involving small divisors $|\lambda - 1|, \dots, |\lambda^{n-1} - 1|$ to transform φ_f into φ_{f_n} such that the terms of the smallest order in the power series expansion of $f_n(\xi,\eta) - \mathcal{N}f_n(\xi,\eta)$ contains only a single monomial term. The presence of such a single (non-vanishing) term is to be essential to the proof of Theorem 1.3.

Next, we consider the n-th iterate $\varphi_{f_n}^n$. We need to find the coefficients of ξ^n for the iterates. To simplify the computation, we shall absorb terms involving $\lambda^n - 1$ into terms of higher order. This will be justified later on by choosing λ such that for a sequence of positive integers n, $|\lambda^n - 1|$ is much smaller than all $|\lambda^j - 1|$ for $j = 1, \dots, n-1$.

Proposition 4.5. *Let m_n be a positive integer. Let $f_n \in \mathcal{F}_0$ be given by*

$$f_n(\xi,\eta) = \Lambda(\xi\eta) + f_{n;n,0}\xi^n + p_n(\xi,\eta)$$

with $\Lambda(0) = \lambda$, $|\lambda| = 1$, $p_n(\xi,\eta) = O(n+1)$, and $\mathcal{N}p_n = 0$. Assume that $M_\Lambda \le 1$ and $M_{f_n} \le \delta_n^{-N_0(n)}$. Then there exists $N_1(n) > N_0(n)$, dependent of m_n, such that $\varphi_{f_n}^j$ has the form

$$\xi_j = \xi(\Lambda^j(\xi\eta) + a_j\xi^n + \tilde{p}_j(\xi,\eta)), \tag{4.8}$$

$$\eta_j = \eta(\Lambda^{-j}(\xi\eta) + b_j\xi^n + \tilde{q}_j(\xi,\eta)) \tag{4.9}$$

with

$$a_j = j\lambda^{j-1}f_{n;n,0}, \quad b_j = -(n+1)j\lambda^{j-3}f_{n;n,0}, \tag{4.10}$$

$$|\tilde{p}_j(\xi,\eta)| + |\tilde{q}_j(\xi,\eta)| \le \delta_n^{-N_1(n)}(|\lambda^n - 1| + |\xi| + |\eta|)(|\xi|^n + |\eta|^n) \tag{4.11}$$

for $(\xi,\eta) \in B(\delta_n^{N_1(n)})$ and $1 \le j \le m_n$.

Proof. We first prove (4.8)-(4.10) by induction on j. We shall seek

$$\begin{aligned} \tilde{p}_j(\xi,\eta) &= u_j(\lambda^n-1)f_{n;n,0}\xi^n + p_j^*(\xi,\eta), \quad p_j^*(\xi,\eta) = O(n+1),\\ \tilde{q}_j(\xi,\eta) &= v_j(\lambda^n-1)f_{n;n,0}\xi^n + q_j^*(\xi,\eta), \quad q_j^*(\xi,\eta) = O(n+1), \end{aligned} \tag{4.12}$$

in which u_j, v_j are polynomials in λ with constant coefficients. Start with $j=1$. We put $a_1 = f_{n;n,0}$, $u_1 = v_1 = 0$, $p_1^* = p_n$, and $q_1^* = q_n$. By (4.3), we also have $b_1 = -(n+1)\lambda^{-2}f_{n;n,0}$. Thus (4.8)-(4.10) and (4.12) hold for $j=1$. Assuming that (4.8)-(4.10) and (4.12) hold, we want to verify that they are still valid when j is replaced by $j+1$. The first component of $\varphi_{f_n}(\varphi_{f_n}^j(\xi,\eta))$ has the form

$$\xi(\Lambda^j(\xi\eta) + a_j\xi^n + u_j(\lambda^n-1)\xi^n)(\Lambda(\xi\eta) + f_{n;n,0}\lambda^{nj}\xi^n) + O(n+2).$$

Taking

$$a_{j+1} = \lambda a_j + \lambda^j f_{n;n,0}, \quad u_{j+1} = \lambda u_j + \lambda^j\frac{\lambda^{nj}-1}{\lambda^n-1},$$

we get

$$a_{j+1} = (j+1)\lambda^j f_{n;n,0}, \quad |u_{j+1}| \le j(j+1)/2. \tag{4.13}$$

Applying Lemma 4.3, the second component of $\varphi_{f_n}^{j+1}$ is of the form (4.9), in which b_{j+1} is given as in (4.10) and

$$\tilde{q}_{j+1}(\xi,\eta) = -(n+1)\lambda^{-2}u_{j+1}(\lambda^n-1)\xi^n + O(n+1).$$

Therefore, we have proved (4.8)-(4.10) and (4.12) by induction.

To obtain estimate (4.11), write

$$\xi p_j^*(\xi,\eta) = \xi_j - \xi\Lambda^j(\xi\eta) - a_j\xi^{n+1} - u_j(\lambda^n-1)f_{n;n,0}\xi^{n+1}, \tag{4.14}$$

in which ξ_j is the first component of $\varphi_{f_n}^j(\xi,\eta)$. Since $M_{f_n} \le \delta_n^{-N_0(n)}$, one sees from Lemma 4.2 that

$$\varphi_{f_n}^j : B_r \to B_{2r}, \quad r < \delta_n^{N_0(n)}/(c_1|j|).$$

We have $|\xi_j| < 1$ for $(\xi,\eta) \in B(\delta_n^{N_0(n)}/(c_1|j|))$, $|\Lambda^j(\xi,\eta)| < 2^{m_n}$ for $(\xi,\eta) \in B(1/2)$, and $|a_j| < m_n\delta_n^{-m_nN_0(n)}$ by (4.7) and (4.13). By the maximum principle, we obtain from (4.14) that

$$|p_j^*(\xi,\eta)| \le \delta_n^{-N_1(n)}, \quad (\xi,\eta) \in B(\delta_n^{N_1(n)}).$$

Since $p_j^* = O(n+1)$, applying the maximum principle again yields

$$|p_j^*(\xi,\eta)| \le \delta_n^{-2N_1(n)}|(\xi,\eta)|^{n+1}, \quad (\xi,\eta) \in B(\delta_n^{N_1(n)}).$$

One can also get a similar estimate for $\tilde{q}_j$. Together with (4.12) and (4.13), we obtain (4.11) for a possibly larger $N_1(n)$. The proof of the proposition is complete. □

We conclude the section with the following

Lemma 4.6. *For* $m = 1, 2, \dots$, *let* $\phi_m(t_1, \dots, t_{m-1})$ *be a positive continuous function defined on* $(0, 2]^{m-1}$. *Let* E *be the set of values* λ *satisfying* $|\lambda| = 1$, $\lambda^k \neq 1$ *for all positive integers* k, *and*

$$|\lambda^n - 1| < \phi_n(|\lambda - 1|, \dots, |\lambda^{n-1} - 1|) \tag{4.15}$$

for a sequence of positive integers $n = n_k \to \infty$ *with* $\operatorname{Im} \lambda^{n_k} > 0$ *and for a sequence of positive integers* $n = n'_k \to \infty$ *with* $\operatorname{Im} \lambda^{n'_k} < 0$. *Then* E *is dense on the unit circle* S^1.

Proof. Put

$$\tilde{\phi}_n(t_1, \dots, t_{n-1}) = \min\{t_1, \dots, t_{n-1}, \phi_n(t_1, \dots, t_{n-1})\}.$$

Fix $\epsilon \in (0, 1)$ and $\lambda_0 = e^{2\pi i \alpha_0}$ for some $0 \leq \alpha_0 < 1$. Choose a positive integer k_0 and an odd integer p_0, such that $|\alpha_0 - \frac{p_0}{2^{k_0}}| < \epsilon/(4\pi)$. Put

$$\beta_0 = \frac{p_0}{2^{k_0}}.$$

Note that $e^{2k\pi i\beta_0} - 1$ does not vanish for $1 \leq k < 2^{k_0}$, but vanishes for $k = 2^{k_0}$. Thus

$$|\lambda^{2^{k_0}} - 1| < \tilde{\phi}_{2^{k_0}}(|\lambda - 1|, \dots, |\lambda^{2^{k_0}-1} - 1|)$$

for $\lambda = e^{2\pi i \beta_0}$. Since $\tilde{\phi}_n$ is continuous, there exists k_1 such that the above inequality remains true if $\lambda = e^{2\pi i \beta}$ satisfies

$$|\beta - \beta_0| < \frac{1}{2^{k_1 - 1}}.$$

We may also choose k_1 so large that

$$\frac{1}{2^{k_1}} < \epsilon/(8\pi).$$

Put $\beta_1 = \beta_0 - \frac{1}{2^{k_1}}$. Recursively, one can find $k_0 < k_1 < k_2 < \dots$ and $\beta_j = \beta_{j-1} + (-1)^j \frac{1}{2^{k_j}}$ such that

$$|\lambda^{2^{k_j}} - 1| < \tilde{\phi}_{2^{k_j}}(|\lambda - 1|, \dots, |\lambda^{2^{k_j}-1} - 1|), \tag{4.16}$$

if $\lambda = e^{2\pi i \beta}$ satisfies $|\beta - \beta_{2^{k_j}}| < \frac{1}{2^{k_{j+1}-1}}$. Put

$$\beta_* = \frac{p_0}{2^{k_0}} + \sum_{j>0} \frac{(-1)^j}{2^{k_j}}. \tag{4.17}$$

One readily sees that $|\beta_* - \beta_{2^{k_j}}| < \frac{1}{2^{k_{j+1}-1}}$; hence, (4.16) holds for $\lambda = e^{2\pi i \beta_*}$. Note that $|\lambda - \lambda_0| < \epsilon$, and that (4.16) implies that λ is not a root of unity. From (4.17), it is clear that $(-1)^j \operatorname{Im} \lambda^{2^{k_j}} < 0$. Taking $n_j = 2^{k_{2j+1}}$ and $n'_j = 2^{k_{2j}}$, the proof of the lemma is complete. □

5 An example and proof of Theorem 1.3

First, we would like to summarize the results which have been proved. Roughly speaking, we have shown that near *each* curve of periodic points of the twist mapping $\xi \to \lambda\xi e^{(\xi\eta)^s}$, $\eta \to \overline{\lambda}\eta e^{-(\xi\eta)^s}$, one can find periodic points for a perturbed mapping if the perturbed mapping, in real and complex cases, is reversible, or if it preserves the real plane $\eta = \sqrt[2s]{-1}\,\overline{\xi}$ and is area-preserving. Those results are formulated in Theorem 1.1 and Theorem 3.2 and are proved through the Birkhoff curves. Despite the existence of Birkhoff curves (Proposition 3.1), we shall show that periodic points of the twist map can be destroyed through a perturbation of holomorphic symplectic mapping, if they are too close to the origin in relative to the period. This is clearly demonstrated by the example below.

Consider a family of holomorphic symplectic mappings

$$\varphi_a\colon (\xi, \eta) \to (\xi', \eta')$$

given by

$$\xi' = \lambda\xi(1 + \xi\eta') + a\beta\xi^{\alpha}\eta'^{\beta-1}, \quad \eta = \lambda\eta'(1 + \xi\eta') + a\alpha\xi^{\alpha-1}\eta'^{\beta}, \quad \alpha \neq \beta$$

with $a \in \mathbf{C}$ and $\alpha + \beta > 4$. By a straightforward computation, one readily sees that if $\lambda \neq 1$ and α, β are not zero, then φ_a has no fixed point other than the origin in a ball centered at the origin of fixed radius if $a \neq 0$. On the other hand, φ_0 has a curve consisting of fixed points in the ball $B(c\sqrt{|\lambda - 1|})$ for some constant c independent of λ, if $|\lambda - 1|$ is small. Nevertheless, φ_a has a Birkhoff holomorphic curve of order one with respect to the family of complex lines $l_w\colon \eta = w^2\xi$ $(1/2 < |w| < 2)$ when $|\lambda - 1|/|a|$ is sufficiently small.

However, we do not know if φ_a $(a \neq 0)$ has periodic points accumulating at the origin.

We now turn to the proof of our main theorem.

Proof of Theorem 1.3. We are given a holomorphic function $a(\xi\eta) = a_s(\xi\eta)^s + \ldots$ $(a_s \neq 0)$ and a sequence of positive integers m_n. (Note that the case $a(\xi\eta) \equiv 0$ is trivial.) Put $\Lambda(\xi\eta) = \lambda e^{a(\xi\eta)}$. We need to find a holomorphic symplectic mapping

$$\varphi_f\colon \xi_1 = \xi f(\xi, \eta), \quad \eta_1 = \eta g(\xi, \eta)$$

with $\mathcal{N} f(\xi, \eta) = \Lambda(\xi\eta)$ such that φ_f^k $(1 \leq k \leq m_n)$ have no fixed point in $B^*(d_n\kappa_n(\lambda))$ for some $n = n_k \to \infty$ and $d_n \to \infty$. One may assume that $M_\Lambda \leq 1$, by replacing $\Lambda(\xi\eta)$ with $\Lambda(K^{-2}\xi\eta)$ and φ_f with $L_K\varphi_f L_K^{-1}$, where $K = \sup\{\sqrt[2j]{|\Lambda_j|}\}$ for $\Lambda(\xi\eta) = \sum \Lambda_j(\xi\eta)^j$ and $L_K(\xi, \eta) = (K\xi, K\eta)$. We may also assume that $m_n > n$.

For $f \in \mathcal{F}_0$ with $M_f \leq 1$, Proposition 4.4 says that there exists a unique polynomial $u_n(\xi, \eta) = 1 + \sum_{k<j<n, j+k\leq n} u_{jk}\xi^j\eta^k$ such that $\varphi_{u_n}\varphi_f\varphi_{u_n}^{-1} = \varphi_{f_n}$ and

$$f_n(\xi, \eta) = \mathcal{N} f(\xi, \eta) + f_{n;n,0}\xi^n + O(n + 1).$$

Let F_Λ be the set of holomorphic functions $f \in \mathcal{F}_0$ satisfying $M_f \leq 1$, $\mathcal{N} f(\xi, \eta) = \Lambda(\xi\eta)$ and

$$|f_{n;n,0}| \geq 1/2, \quad n = 2, 3, \dots . \tag{5.1}$$

We first want to show that for each λ that is not a root of unity, the set F_Λ is non-empty. To this end, we need to find an element f in F_Λ of the form

$$f(\xi, \eta) = \Lambda(\xi\eta) + \sum_{n>1} f_{n,0}\xi^n$$

by determining f_{n0} as follows: Taking $f_{2,0} = 1$, we have $f_{2;2,0} = f_{2,0}$. It is clear that (5.1) holds for $n = 2$. Assuming that $f_{l,0}$ has been determined such that (5.1) is valid for $n < l$. By Proposition 4.4 (a), we know that

$$f_{l;l,0} = f_{l,0} + Q_l(f),$$

in which $Q_l(f)$ depends only on f_{jk} with $j + k < l$. Put

$$f_{l0} = \begin{cases} 0 & |Q_l(f)| > 1/2, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that (5.1) valid for $n = l$ and remains valid for any choice of higher order terms f, as f_{jk} $(j + k \leq l)$ are fixed; by induction, we know that F_Λ is non-empty.

Return to the decomposition $\varphi_{f_n} = \varphi_{u_n}\varphi_f\varphi_{u_n}^{-1}$. Assume that $f \in \mathcal{F}_0$ and $M_f \leq 1$. From Proposition 4.4 (c) we know that $M_{u_n} \leq \delta_n^{-N_1(n)}$ and $M_{f_n} \leq \delta_n^{-N_1(n)}$. Now Lemma 4.2 says that φ_{u_n}, $\varphi_{u_n}^{-1}$, and $\varphi_{f_n}^k$ send $B(r)$ into $B(2r)$ for $r < \delta_n^{N_1(n)}/(c_2 m_n)$ and $1 \leq k \leq m_n$. In particular, φ_f^k sends $B(r)$ to $B(8r)$ for $r < \delta_n^{N_1(n)}/(4c_2 m_n)$ and $1 \leq k \leq m_n$; furthermore, if φ_f has a periodic point of period $\leq m_n$ in $B^*(\delta_n^{N_1(n)}/(4c_2 m_n))$, then φ_{f_n} has a periodic point in $B^*(\delta_n^{N_1(n)}/(2c_2 m_n))$ of period $\leq m_n$.

We now restrict ourselves to $f \in F_\Lambda$ and assume that φ_f^k has a fixed point in $B^*(d_n\kappa_n(\lambda))$ for some $1 \leq k \leq m_n$. We shall derive some inequalities that $\kappa_n(\lambda)$ must satisfy, under the additional conditions

$$d_n\kappa_n(\lambda) \leq \delta_n^{N_1(n)}/(4c_2 m_n), \quad d_n \geq 1, \quad |\lambda^n - 1| < 1/m_n. \tag{5.2}$$

Put $(\xi_j, \eta_j) = \varphi_{f_n}^j(\xi, \eta)$. Then we have

$$\xi_k = \xi, \quad \eta_k = \eta \tag{5.3}$$

for some k and (ξ, η) with $1 \leq k \leq m_n$ and

$$0 \neq |(\xi, \eta)| < 2d_n\kappa_n(\lambda). \tag{5.4}$$

As before we shall denote by c_j constants independent of $m_1, \ldots, m_n$ and by $N_j(n)$, $N_j'(n)$ constants dependent of $m_1, \ldots, m_n$. From $\Lambda(\xi\eta) = \lambda(1+a_s(\xi\eta)^s)+O(|\xi\eta|^{s+1})$, we get

$$|\Lambda^k(\xi\eta) - \lambda^k| \leq c_3^{m_n}|\xi\eta|^s. \tag{5.5}$$

We shall consider three cases.

(a) $\xi\eta \neq 0$ and n divides k. Since $m_n > n$, the last inequality in (5.2) implies that $|\lambda^{nj} - 1| < 1$ for $|j| \leq m_n$. Hence

$$|\lambda^n - 1| \leq |\lambda^k - 1| \leq m_n|\lambda^n - 1|. \tag{5.6}$$

Now (5.5) yields

$$|\lambda^n - 1| - c_3^{m_n}|\xi\eta|^s \leq |\Lambda^k(\xi\eta) - 1| \leq m_n|\lambda^n - 1| + c_3^{m_n}|\xi\eta|^s. \tag{5.7}$$

Using Proposition 4.5, rewrite (5.3) as

$$1 = \Lambda^k(\xi\eta) + a_k\xi^n + \tilde{p}_k(\xi, \eta), \tag{5.8}$$

$$1 = \Lambda^{-k}(\xi\eta) + b_k\xi^n + \tilde{q}_k(\xi, \eta), \tag{5.9}$$

in which a_j, b_j, $\tilde{p}_j$ and $\tilde{q}_j$ satisfy (4.10)-(4.11). Multiplying (5.9) by $\Lambda^k(\xi\eta)$ and adding it to (5.8) yields

$$(a_k + b_k)\xi^n = (1 - \Lambda^k(\xi\eta))b_k\xi^n - \Lambda^k(\xi\eta)\tilde{q}_k(\xi, \eta) - \tilde{p}_k(\xi, \eta).$$

By (4.10) we have

$$a_k + b_k = k\lambda^{k-3}(\lambda^2 - n - 1)f_{n;n,0}.$$

Combining (4.11), (5.1), (5.7), and $|\lambda^n - 1| = n\kappa_n^{2s}(\lambda)$, one obtains

$$\begin{aligned} |\xi|^n &\leq \delta_n^{-N_2(n)}(|\lambda^n - 1| + |\xi| + |\eta|)(|\xi|^n + |\eta|^n) \\ &\leq 6n2^{n+1}\delta_n^{-N_2(n)}(d_n\kappa_n(\lambda))^{n+1}. \end{aligned}$$

Hence

$$|\xi| \leq c_4\delta_n^{-N_2(n)}(d_n\kappa_n(\lambda))^{1+\frac{1}{n}}.$$

Returning to (5.8), one also gets from (4.10)-(4.11) that

$$|\Lambda^k(\xi\eta) - 1| \leq c_5\delta_n^{-N_1(n)}\left(|\xi|^n + (|\lambda^n - 1| + |\xi| + |\eta|)|\eta|^n\right).$$

Now (5.7) and the last two inequalities yields

$$\begin{aligned} |\lambda^n - 1| &\leq c_3^{m_n}|\xi\eta|^s + c_5\delta_n^{-N_1(n)}\left(|\xi|^n + (|\lambda^n - 1| + |\xi| + |\eta|)|\eta|^n\right) \\ &\leq \delta_n^{-N_3(n)}(d_n\kappa_n(\lambda))^{2s+\frac{s}{n}} \end{aligned}$$

for $n \geq 2s$. Using $|\lambda^n - 1| = n\kappa_n^{2s}(\lambda)$, one gets

$$d_n^{2n+1}\kappa_n(\lambda) > n^n \delta_n^{nN_3(n)}. \tag{5.10}$$

(b) $\xi\eta = 0$ and n divides k. Since λ is not a root of unity and $\tilde{p}_k(\xi, \eta) = 0$ when $\xi = 0$, then $\xi \neq 0$ and $\eta = 0$. Thus one still has (5.8), which can now be rewritten as

$$\lambda^k - 1 = -a_k\xi^n - \tilde{p}_k(\xi, 0).$$

Combining (4.11) and (5.6), one obtains

$$\begin{aligned} |\lambda^n - 1| &< \delta_n^{-N_1(n)}(|\xi|^n + (|\lambda^n - 1| + |\xi| + |\eta|)(|\xi|^n + |\eta|^n)) \\ &\leq \delta_n^{-N_4(n)} d_n^{n+1} \kappa_n^n(\lambda). \end{aligned}$$

Thus, one has

$$d_n^{n+1}\kappa_n(\lambda) \geq \delta_n^{N_4(n)}, \quad n \geq 2s+1. \tag{5.11}$$

(c) $k = m'n + r$ with $0 < r < k$. As in (b), we still have $\xi \neq 0$ and (5.8). The latter and (4.11) imply that

$$|\Lambda^k(\xi\eta) - 1| \leq c_6 \delta_n^{-N_1(n)} d_n^{n+1} \kappa_n^n(\lambda).$$

Together with (5.5), we obtain

$$\begin{aligned} \delta_n &\leq |\lambda^r - 1| \leq |\lambda^k - 1| + |\lambda^{m'n} - 1| \\ &\leq |\Lambda^k(\xi\eta) - 1| + |\Lambda^k(\xi\eta) - \lambda^k| + m_n|\lambda^n - 1| \\ &\leq c_6\delta_n^{-N_1(n)} d_n^{n+1}\kappa_n^n(\lambda) + c_3^{m_n}(d_n\kappa_n(\lambda))^{2s} + m_n n\kappa_n^{2s}(\lambda) < \delta_n^{-N_5(n)} d_n^{n+1}\kappa_n(\lambda) \end{aligned}$$

for $n \geq 2s$. Together with (5.10) and (5.11), we obtain

$$d_n^{2n+1}\kappa_n(\lambda) > \delta_n^{N_6(n)}, \quad n > 2s.$$

We now put

$$d_n = \kappa_n^{-\frac{1}{2n+2}}(\lambda). \tag{5.12}$$

Then $\kappa_n(\lambda) > \delta_n^{(2n+2)N_6(n)}(\lambda)$ for $n > 2s$, i.e.,

$$|\lambda^n - 1| > n\delta_n^{(n+1)N_6(n)/s}(\lambda), \quad n > 2s. \tag{5.13}$$

In summary, (5.13) would hold, if φ_f $(f \in F_\Lambda)$ had a periodic point in $B^*(d_n\kappa_n(\lambda))$ with period $\leq m_n$ and if (5.2) and (5.12) were true; in particular, by eliminating d_n in (5.2) through (5.12) and substituting $\sqrt[2s]{|\lambda^n - 1|/n}$ for $\kappa_n(\lambda)$, one sees that

(5.13) would hold, if some φ_f with $f \in F_\Lambda$ had a periodic point in $B^*(d_n\kappa_n(\lambda))$ of period $\leq m_n$, and if

$$|\lambda^n - 1| < \min\left\{\frac{1}{m_n}, n\left(\frac{\delta_n^{N_1(n)}(\lambda)}{4c_2m_n}\right)^{\frac{n+1}{s(2n+1)}}\right\}. \tag{5.14}$$

To continue our construction, we put

$$\phi_n(t_1,\dots,t_{n-1}) = \min\left\{\frac{1}{m_n}, n\left(\frac{\delta_n^{*N_1(n)}}{4c_2m_n}\right)^{\frac{n+1}{s(2n+1)}}, n\delta_n^{*(n+1)N_6(n)/s}\right\}$$

with

$$\delta_n^* = \min\{1/2, t_1, \dots, t_{n-1}\}.$$

For each n, ϕ_n is a positive continuous function on $(0,2]^{n-1}$. By Lemma 4.6, we can find λ, not a root of unity, such that

$$|\lambda^n - 1| < \phi_n(|\lambda - 1|, \dots, |\lambda^{n-1} - 1|), \quad \text{for } n = n_k,$$

However, this implies that, for $n = n_k$, (5.14) is valid and (5.13) does not hold, which contradicts to the conclusion summarized at the end of last paragraph. Therefore, we conclude that for all $f \in F_\Lambda$ the mapping φ_f has no periodic point in $B^*(d_n\kappa_n(\lambda))$ with period $\leq m_n$, if $n = n_k$. From (5.14), one also has $d_n \to \infty$ as $n = n_k \to \infty$. Replacing d_n by n for $n \neq n_k$, one gets $d_n \to \infty$ as $n \to \infty$. This completes the proof of the theorem. □

6 Totally real invariant submanifolds

The main results of this section conclude that there exists a holomorphic symplectic mapping having infinitely many invariant totally real formal surfaces, but none of the formal surfaces is C^1.

Given a formal power series $f(\xi,\eta,\overline{\xi},\overline{\eta})$ of complex coefficients with $f(0) = 0$ and $d\,\mathrm{Re}\, f \wedge d\,\mathrm{Im}\, f(0) \neq 0$, we shall call the set $\{uf\}$ a real *formal surface*, in which u runs over the set of formal power series of complex coefficients with non-vanishing constant term; the formal surface is totally real if the vanishing of the linear part of f defines a totally real plane in $\mathbf{C}^2$. To study the existence of invariant totally real formal surfaces of holomorphic symplectic mappings, let us recall the Birkhoff formal normal form: Let $\varphi\colon (\xi,\eta) \to (\lambda\xi, \lambda^{-1}\eta) + \dots$ ($\lambda \in \mathbf{C}^*$) be a symplectic holomorphic mapping. We assume that λ is not a root of unity. Then under symplectic formal transformations, φ is equivalent to (1.3), in which $a(\xi\eta)$ is a formal power series with $a(0) = 0$. Since in the following discussion we are only interested in invariant formal totally real surfaces, we can further normalize $\hat{\varphi}$ by using possibly non-symplectic coordinates. Assume that $a(\xi\eta) \not\equiv 0$. Choose $u(t) = 1 + \dots$ such that $a(u^2(t)t) = t^s$ and put

$\Phi\colon \xi' = u(\xi\eta)\xi, \eta' = u(\xi\eta)\eta$. Then Φ^{-1} is given by $\xi' = \tilde{u}(\xi\eta)\xi, \eta' = \tilde{u}(\xi\eta)\eta$ with $\tilde{u}(tu^2(t))u(t) = 1 = u(t\tilde{u}^2(t))\tilde{u}(t)$. Thus, one readily sees that

$$T_c = \Phi^{-1}\hat{\varphi}\Phi\colon \xi \to \lambda\xi e^{(\xi\eta)^s}, \quad \eta \to \lambda^{-1}\eta e^{-(\xi\eta)^s}. \tag{6.1}$$

Therefore, if φ has an invariant totally real formal surfaces, so does T_c, and vice versa.

Proposition 6.1. *Let T_c be given by* (6.1) *with $\lambda \in \mathbf{C}^*$.*

(a) A totally real plane $M \subset \mathbf{C}^2$ is invariant under $\xi \to \lambda\xi, \eta \to \lambda^{-1}\eta$, if and only if $\lambda^2 = 1$ and M is any totally real plane, or $\overline{\lambda} = \lambda \neq \pm 1$ and M is defined by $\xi = \mu_1\overline{\xi}, \eta = \mu_2\overline{\eta}$ with $|\mu_j| = 1$, or $|\lambda| = 1 \neq \lambda^2$ and M is defined by $\xi = \mu_3\overline{\eta}$ with $\mu_3 \in \mathbf{C}^*$.

(b) Assume that $\lambda = \overline{\lambda}$. Let $b(t)$ be a formal power of real coefficients. Then

$$\overline{\xi} = \xi e^{ib(\xi\eta)}, \quad \overline{\eta} = \eta e^{-ib(\xi\eta)} \tag{6.2}$$

define a totally real and formal surface invariant under T_c.

(c) Assume that $|\lambda| = 1$. Then

$$\eta = h(\xi\overline{\xi})\overline{\xi} \tag{6.3}$$

defines a totally real and formal surface invariant under T_c, in which $h(t)$ with $h(0) \neq 0$ is a formal power series such that $(h(t)t)^s$ has pure-imaginary coefficients.

Proof. For (a), it is obvious that if λ and M are among the list, then M is invariant under the corresponding linear mapping. Now assume that the linear mapping $\varphi\colon \xi \to \lambda\xi, \eta \to \lambda^{-1}\eta$ with $\lambda^2 \neq 1$ admits a totally real plane M. Since M is totally real, then M is the fixed-point set of a unique anti-holomorphic involution ρ. Note that $\varphi^{-1}\rho\varphi$ is also an anti-holomorphic involution fixing M pointwise. Thus, $\varphi^{-1}\rho\varphi = \rho$, i.e, $\varphi\rho = \rho\varphi$. Letting $\rho_0(\xi, \eta) = (\overline{\xi}, \overline{\eta})$, we see that $(\rho\rho_0)^{-1}\varphi\rho\rho_0 = \rho_0\varphi\rho_0$. Since $\rho\rho_0$ is holomorphic, then $\{\lambda, \lambda^{-1}\} = \{\overline{\lambda}, \overline{\lambda}^{-1}\}$. Hence, either $\lambda = \overline{\lambda}$ or $\lambda = \overline{\lambda}^{-1}$. For the former case, we know that $\rho\rho_0$ is of the form $\xi \to \mu_1\xi, \eta \to \mu_2\eta$. Hence, ρ is given by $\xi \to \mu_1\overline{\xi}, \eta \to \mu_2\overline{\eta}$. Since ρ is an involution, then $|\mu_j| = 1$ and M is given by $\xi = \mu_1\overline{\xi}, \eta = \mu_2\overline{\eta}$. For the latter case, we know that $\rho\rho_0$ is given by $\xi \to \mu_3\eta, \eta \to \mu_3'\xi$. Since ρ is an involution, then $\mu_3'\overline{\mu}_3 = 1$; hence, ρ is given by $\xi \to \mu_3\overline{\eta}, \eta \to \overline{\mu}_3^{-1}\overline{\xi}$ and M is defined by $\xi = \mu_3\overline{\eta}$.

For (b), one notices that when b is of real coefficients, the formal anti-holomorphic mapping

$$\rho\colon \xi' = e^{-ib(\overline{\xi}\overline{\eta})}\overline{\xi}, \quad \eta' = e^{ib(\overline{\xi}\overline{\eta})}\overline{\eta}$$

is an involution, and that (6.2) is the set of fixed points of ρ. Therefore, (6.2) defines a totally real formal surface. We now want to show that the surface (6.2) is invariant under T_c. Assume that ξ, η satisfy (6.2), and let $(\xi', \eta') = T_c(\xi, \eta)$. Then $\overline{\xi'\eta'} = \xi\eta$ and $\xi'\eta' = \xi\eta$. In (6.2), multiplying both sides of the first equation by $e^{(\xi\eta)^s}$ and that of the second equation by $e^{-(\xi\eta)^s}$, one sees that ξ', η' satisfy (6.2) also. The proof of (b) is complete.

For (c), it is trivial that (6.3) defines a totally real formal surface. To see that the set defined by (6.3) is invariant under T_c, assume that ξ, η satisfy (6.3). Then $(\overline{\xi\eta})^s = -(\xi\eta)^s$ and $(\xi', \eta') = T_c(\xi, \eta)$ satisfies $\xi'\overline{\xi}' = \xi\overline{\xi}$ and $\eta'\overline{\eta}' = \eta\overline{\eta}$. Multiplying both sides of (6.3) by $e^{-(\xi\eta)^s}$, one readily sees that ξ', η' satisfy (6.3). The proof of the proposition is complete. □

It is a theorem of Moser [6] that if a holomorphic symplectic map $\varphi\colon \xi \to \lambda\xi + O(2), \eta \to \lambda^{-1}\eta + O(2)$ is *hyperbolic* (i.e., $|\lambda| \neq 1$), then φ can be transformed into its Birkhoff normal form (1.3) by convergent transformation. (Moser proved the theorem for real case. However, the proof given by Siegel [9] is valid for both real and complex cases.) Combining Moser's theorem with Proposition 6.1, we have the following.

Corollary 6.2. *A holomorphic symplectic mapping of* $\mathbf{C}^2$ *admits an invariant totally real and real analytic surface passing through a hyperbolic fixed point, if and only if the eigenvalues of the fixed point are real.*

Proof of Corollary 1.5. By Theorem 1.3, we know that there exists a holomorphic symplectic mapping

$$\varphi\colon \xi' = \lambda\xi e^{(\xi\eta)^s} + O(2s+2), \quad \eta' = \overline{\lambda}e^{-(\xi\eta)^s} + O(2s+2)$$

with λ not a root of unity, such that φ has no periodic points of period $\leq n$ in $B^*(d_n\kappa_n(\lambda))$ for two sequences $n = n_k \to +\infty$ and $n = n'_k \to +\infty$, in which $d_n \to \infty$, $\operatorname{Im}\lambda^{n_k} > 0$ and $\operatorname{Im}\lambda^{n'_k} < 0$.

Assume for the sake of contradiction that there is a totally real and C^1 surface M invariant under φ which passes through the origin. By Proposition 6.1 (a), we know that M is given by $\eta = a\overline{\xi} + o(|\xi|)$ with $a \neq 0$. Choose $\operatorname{Re}\xi, \operatorname{Im}\xi$ as coordinates of M. Then

$$\varphi|_M\colon \xi \to \lambda\xi e^{a^s(\xi\overline{\xi})^s} + p(\xi)$$

is C^1 with $|p_\xi(\xi)| + |p_{\overline{\xi}}(\xi)| = o(|\xi|^{2s})$. Now $d\xi \wedge d\eta|_M = b(\xi)d\xi \wedge d\overline{\xi}$. Since φ preserves $b(\xi)d\xi \wedge d\overline{\xi}$, it also preserves the real 2-form $i|b(\xi)|d\xi \wedge d\overline{\xi}$. Note that $|b(\xi)|$ is C^0 and positive near the origin. Applying the Birkhoff fixed-point theorem (Theorem 3.2), one sees that φ, when restricted to $M \equiv \mathbf{R}^2$, has periodic points of period dividing n in the punctured disk $D^*(c\kappa_n(\lambda, a^s))$ for some constant c and for all large n, which contradicts to the earlier assumption since we have either $\kappa_n(\lambda, a^s) = \kappa_n(\lambda)$ for $\operatorname{Im} a^s > 0$ and $n = n'_k$, or $\kappa_n(\lambda, a^s) = \kappa_n(\lambda)$ for $\operatorname{Im} a^s < 0$ and $n = n_k$. The proof is complete. □

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