

Existence of Real Analytic Surfaces with Hyperbolic Complex Tangent that Are Formally but Not Holomorphically Equivalent to Quadrics

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ABSTRACT. We shall prove the existence of a real analytic surface in \mathbb{C}^2 that is formally but not holomorphically equivalent to a quadratic surface with a hyperbolic complex tangent. The result is obtained through the pair of holomorphic involutions of Moser-Webster for a real analytic surface with a complex tangent of non-vanishing Bishop invariant. Our method is also used to show the existence of real analytic reversible maps of the real plane, defined near the origin, that are formally, but not real analytically, equivalent to a linear rotation.

1. INTRODUCTION

In this paper we shall prove the following result.

Theorem 1.1. *There exists a germ of real analytic surface M in \mathbb{C}^2 at the origin that is formally but not holomorphically equivalent to the quadric $Q_\gamma \subset \mathbb{C}^2$: $z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$ for some $\gamma \in (\frac{1}{2}, \infty)$; in fact, M is not contained in any smooth Levi-flat real analytic hypersurface in \mathbb{C}^2 .*

The main purpose of this paper is, of course, to show the existence of real analytic surfaces which are formally but not holomorphically equivalent to Q_γ . The difficulties in constructing such a real analytic surface come from the formal normal form of Moser-Webster. Namely, a real analytic surface

$$M \subset \mathbb{C}^2: z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O(3)$$

is formally equivalent to the surface in \mathbb{C}^2 defined by

$$x_2 = z_1\bar{z}_1 + \gamma(1 + \varepsilon x_2^2)(z_1^2 + \bar{z}_1^2), \quad y_2 = 0,$$

for $\gamma \in (\frac{1}{2}, \infty) \setminus E$, where $\varepsilon = \pm 1$ with s a positive integer, or $\varepsilon = 0$, and E is the set of γ such that the solutions λ to $\gamma\lambda^2 - \lambda + \gamma = 0$ are roots of unity. The divergence of the normalization is well understood when $\varepsilon = 1$ or -1 (see [11], [5], [7]), of which both cases are generic. The case $\varepsilon = 0$ is, however, non-generic, which corresponds to that of vanishing of infinitely many functional equations in coefficients of the defining function of M . Until now very little is known about such real surfaces, except that the formal equivalence of M and Q_γ implies the holomorphic equivalence, if $\gamma \in E$ or satisfies a certain Diophantine condition [5].

Our starting point is a recent observation [8] that there is a family of holomorphic mappings for which the formal linearization always holds. Namely, if a holomorphic map σ of \mathbb{C}^2 is formally equivalent to

$$\hat{\sigma}: \xi' = M(\xi\eta)\xi, \quad \eta' = M^{-1}(\xi\eta)\eta,$$

with $M(0) = \mu$ not a root of unity, and if σ has a meromorphic *eigenfunction* $m = f/g$ (i.e., a function m with $m\sigma = \nu m$ for some constant ν) with $(f, g) = \text{Id} + O(2)$, then $\hat{\sigma}$ must be linear. Note that the classical examples of maps which have $\hat{\sigma}$ as a formal normal form are reversible or area-preserving ones and their holomorphic counterparts. Here we say that a holomorphic map σ of \mathbb{C}^2 , defined near the origin with $\sigma(0) = 0$, is *reversible* if there is a holomorphic involution τ_1 ($\tau_1^2 = \text{Id}$), defined near the origin with $\tau_1(0) = 0$, such that $\sigma^{-1} = \tau_1\sigma\tau_1$. Examples of holomorphic maps that have a meromorphic eigenfunction are (non-constant) multiples of the linear map $\xi \rightarrow \mu\xi, \eta \rightarrow \mu^{-1}\eta$ (so they preserve $m(\xi, \eta) = \xi/\eta$, up to a constant multiple).

For the proof of Theorem 1.1 we shall use an intrinsic pair $\{\tau_1, \tau_2\}$ of holomorphic involutions which arises from a real analytic surface with a complex tangent (and with the non-vanishing Bishop invariant). This pair of involutions, developed by Moser-Webster [11], will be described in details in Section 2.

We should also mention that the proof of Theorem 1.1 does not yield any explicit example of real analytic surface with the desired properties. Nor do we know if there exists a real analytic surface *contained* in a real hyperplane, which is formally but not holomorphically equivalent to Q_γ ; instead, we have the following result.

Theorem 1.2. *Let M be a real analytic surface with $\gamma \in (\frac{1}{2}, \infty) \setminus E$, of which the corresponding pair of involutions is τ_1, τ_2 . Assume that $\sigma = \tau_1\tau_2$ has a non-constant meromorphic eigenfunction f/g with $f(\xi, \eta) = \xi + O(2)$ and $g(\xi, \eta) = \eta + O(2)$. If M is contained in a smooth Levi-flat real analytic hypersurface, then M is holomorphically equivalent to Q_γ .*

We would like to mention that the results of Moser-Webster [11] and Moser [10] say that a real analytic surface M in \mathbb{C}^2 of an elliptic complex tangent (i.e., of $0 \leq \gamma < \frac{1}{2}$) is holomorphically equivalent to Q_γ if M and Q_γ are formally equivalent. On the other hand, there are real analytic Lagrangian surfaces in \mathbb{C}^2

(with parabolic complex tangent) which are formally but not holomorphically equivalent to $Q_{1/2}$ (see [6]).

Our approach to the formally linearizable holomorphic reversible maps can also be easily adapted to obtain the following result.

Theorem 1.3. *There exists a reversible real analytic map $\sigma : z \rightarrow \mu z + O(|z|^2)$ of the real plane such that σ is formally but not real analytically linearizable.*

Theorem 1.3 is new. Also, to the best knowledge of the author, it is unknown if the analogous result holds for area-preserving maps, i.e., if there exists a real analytic area-preserving map of the real plane that is formally but not real analytically equivalent to a rotation $z \rightarrow \mu z$. Notice that, under the Bruno condition, all formally linearizable area-preserving real analytic map are indeed real analytically linearizable; see Rüssmann [12], [13]. Rüssmann's convergence result also implies that formally linearizable reversible real analytic maps are real analytically linearizable if the μ in Theorem 1.3 satisfies the Bruno condition.

Bedford [2] found explicit algebraic real surfaces in \mathbb{C}^2 with a hyperbolic complex tangent that are not contained in any smooth Levi-flat real analytic hypersurface. Moser and Webster [11] had explicit algebraic real surfaces with a hyperbolic complex tangent that are not contained in any Levi-flat real analytic hypersurface given by $\Re h = 0$, where h is a holomorphic function. Huang and Krantz [9] proved that if M is a real analytic surface in \mathbb{C}^2 with a Bishop invariant $\gamma = 0$, then M is contained in a smooth Levi-flat real analytic hypersurface.

Notice that Baouendi, Ebenfelt and Rothschild [1] proved that two germs of real analytic CR submanifolds M and M' in \mathbb{C}^N are always holomorphically equivalent, if they are formally equivalent and if M is of finite type and M' contains no non-trivial holomorphic variety.

2. A PAIR OF INVOLUTIONS

In this section we shall recall the Moser-Webster theory about a pair of involutions intrinsically associated with the real surfaces with a complex tangent [11]. For our purpose we shall only consider the case that the complex tangent is hyperbolic.

Consider a real analytic surface in \mathbb{C}^2 of the form

$$(2.1) \quad M: z_2 = z_1 \bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + H(z_1, \bar{z}_1),$$

where $H(z_1, \bar{z}_1)$ is a complex-valued real analytic function starting with terms of order 3 or higher, and $0 \leq \gamma < \infty$ is the Bishop invariant [3]. When the complex tangent is hyperbolic as we assume, we have $\gamma > \frac{1}{2}$. In (2.1), replacing (z, \bar{z}) with (z, w) yields a complex surface in \mathbb{C}^4

$$M^c : \begin{cases} z_2 = q(z_1, w_1) + H(z_1, w_1), \\ w_2 = q(z_1, w_1) + \bar{H}(w_1, z_1), \end{cases}$$

in which and in what follows, we put

$$\bar{H}(z_1, w_1) = \sum_{i,j} \bar{H}_{i,j} z_1^i w_1^j, \quad \text{for } H(z_1, w_1) = \sum_{i,j} H_{i,j} z_1^i w_1^j.$$

We shall use (z_1, w_1) as coordinates to identify M^c with \mathbb{C}^2 . The projections $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$, restricted to M^c , induce two double-sheeted branched coverings. Let $(z'_1, w'_1) = \tau_1(z_1, w_1)$ be the covering transformation for π_2 . Then we have

$$\tau_1 : \begin{cases} z'_1 = -z_1 - \frac{1}{y} w_1 + O(2), \\ w'_1 = w_1. \end{cases}$$

Let ρ be the restriction of the anti-holomorphic involution $(z, w) \mapsto (\bar{w}, \bar{z})$ to M^c . Then $\rho(z_1, w_1) = (\bar{w}_1, \bar{z}_1)$. Notice that M^c is invariant under ρ , and $\pi_2 = c \circ \pi_1 \circ \rho$ for $c(w) = \bar{w}$. Hence, the covering transformation τ_2 for π_1 satisfies the relation $\tau_2 = \rho \tau_1 \rho$. Since the complex tangent is hyperbolic one can introduce new linear coordinates such that

$$(2.2) \quad \tau_j : \begin{cases} \xi' = \lambda_j \eta + f_j(\xi, \eta), & \lambda_1 = \lambda = \bar{\lambda}_2, \\ \eta' = \bar{\lambda}_j \xi + g_j(\xi, \eta), & |\lambda| = 1, \end{cases}$$

in which f_j, g_j start with terms of order two or higher, and λ and y are related by $y\lambda^2 - \lambda + y = 0$. We say that $y \in (\frac{1}{2}, \infty)$ is *exceptional* if λ is a root of unity. The set of exceptional values is denoted by E . Meanwhile, the anti-holomorphic involution ρ and reality condition on τ_1 and τ_2 , under the new coordinates, are given by

$$(2.3) \quad \rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}), \quad \tau_2 = \rho \tau_1 \rho.$$

For a quadric Q_y with $\frac{1}{2} < y < \infty$, its pair of involutions τ_1^*, τ_2^* and $\sigma^* = \tau_1^* \tau_2^*$ are given by

$$\tau_j^* : \xi' = \lambda_j \eta, \quad \eta' = \bar{\lambda}_j \xi; \quad \sigma^* = \tau_1^* \tau_2^* : \xi' = \mu \xi, \quad \eta' = \bar{\mu} \eta.$$

The normalization of the pair τ_1, τ_2 is essentially determined by that of

$$\sigma = \tau_1 \tau_2 : \xi' = \mu \xi + O(2), \quad \eta' = \bar{\mu} \eta + O(2), \quad |\mu| = 1.$$

We shall need the following: First, two real analytic surfaces are equivalent through (local) biholomorphic mappings of \mathbb{C}^2 , if and only if their corresponding pairs of involutions are equivalent through a biholomorphic mapping f satisfying the

reality condition $\rho f = f\rho$. In fact, the pair $\{\tau_1, \tau_2\}$ is linearizable by mappings commuting with ρ , if and only if σ is linearizable by some holomorphic map (which may not satisfy the reality condition). Second, each pair of holomorphic involutions satisfying (2.2)-(2.3) arises from a real analytic surface with a complex tangent (see [11], pp. 263–264).

Therefore, the first half of Theorem 1.1 follows from the theorem below.

Theorem 2.1. *There exists a pair of holomorphic involutions $\{\tau_1, \tau_2\}$ of the form (2.2)-(2.3) that is formally but not holomorphically equivalent to the pair τ_1^*, τ_2^* ; moreover, $\tau_1\tau_2$ admits a meromorphic eigenfunction f/g with $f(\xi, \eta) = \xi + O(2)$ and $g(\xi, \eta) = \eta + O(2)$.*

The proof of Theorem 2.1 will be given in Section 3.

3. A SPECIAL FAMILY OF HOLOMORPHICALLY REVERSIBLE MAPS

In this section we shall prove Theorem 2.1.

We start with the following result.

Proposition 3.1. *Let τ_1, τ_2 be two holomorphic involutions given by (2.2). Assume that λ is not a root of unity and that $\sigma = \tau_1\tau_2$ admits a meromorphic eigenfunction $m(\xi, \eta) = f(\xi, \eta)/g(\xi, \eta)$ with $f(\xi, \eta) = \xi + O(2)$ and $g(\xi, \eta) = \eta + O(2)$. There exists a formal map $\Phi: \xi \rightarrow U(\xi, \eta), \eta \rightarrow V(\xi, \eta)$, which is tangent to the identity and satisfies the normalizing condition $U_{i+1,i} = V_{i,i+1} = 0$ for all $i > 0$, such that $\Phi^{-1}\sigma\Phi$ becomes linear and $m\Phi(\xi, \eta) = \xi\omega(\xi\eta)/\eta$.*

Proof. An analogous result is proved for holomorphic symplectic maps in [8].

From [11], we know that there is a unique normalized formal transformation Φ such that $\hat{\sigma} = \Phi^{-1}\sigma\Phi$ becomes

$$\xi' = M(\xi\eta)\xi, \quad \eta' = M^{-1}(\xi\eta)\eta, \quad M(\xi\eta) = \mu + O(1).$$

Now the proof of Theorem 4.6 in [8] implies that M is a constant and $\hat{m} = m \circ \Phi$ has the form $\xi\omega(\xi\eta)/\eta$. However, because of the advantage of dimension 2, we would like to give an alternative and simpler argument. We know that $\hat{m} = p/q$ is an eigenfunction of $\hat{\sigma}$, where p, q are formal power series in ξ, η . Notice that the coordinate axes are the only formal invariant curves of $\hat{\sigma}$, and that $\hat{\sigma}$ preserves the formal curves defined by $p = 0$ and by $q = 0$. Hence ξ divides p and η divides q . So we can write $\hat{m}(\xi, \eta) = \xi K(\xi, \eta)/\eta$. From $\hat{m}\hat{\sigma} = \mu^2\hat{m}$, we get

$$M^2(\xi\eta)K(M(\xi\eta)\xi, M^{-1}(\xi\eta)\eta) = \mu^2K(\xi, \eta).$$

Since μ is not a root of unity, one readily sees that K is a power series in the product $\xi\eta$, and that $M^2 = \mu^2$. □

As mentioned in Section 2, when $\sigma = \tau_1\tau_2$ is linearizable by some holomorphic map, the pair $\{\tau_1, \tau_2\}$ is linearizable by a holomorphic map satisfying the reality condition. Thus, a result in [5] yields the following theorem.

Theorem 3.2. *Let τ_1, τ_2 be the holomorphic involutions associated with a real analytic surface M of which the Bishop invariant $\gamma \in (\frac{1}{2}, \infty) \setminus E$. Assume that the eigenvalues $\mu, \bar{\mu}$ of $\sigma = \tau_1\tau_2$ satisfy the Diophantine condition*

$$|\mu^n - 1| > \frac{c_0}{n^{c_0}}, \quad n = 1, 2, \dots$$

for some positive constant c_0 . If σ has a meromorphic eigenfunction $m(\xi, \eta) = f(\xi, \eta)/g(\xi, \eta)$ with $(f, g) = \text{Id} + O(2)$, then M is holomorphically equivalent to Q_γ .

Of course the analogous result holds for real analytic surfaces with Bishop invariant $\gamma \in (0, \frac{1}{2})$, by the convergence of normalization obtained by Moser and Webster [11].

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1 — the first half. We shall consider the following pair of holomorphic involutions

$$(3.1) \quad \tau_1 = \psi^{-1}\tau_1^*\psi, \quad \tau_2 = \rho\tau_1\rho$$

with $\tau_1^*: \xi \rightarrow \lambda\eta, \eta \rightarrow \bar{\lambda}\xi, \rho: \xi \rightarrow \bar{\xi}, \eta \rightarrow \bar{\eta}$, and

$$\psi(\xi, \eta) = (\xi u(\xi), \eta u(\xi)), \quad u(0) = 1.$$

Note that ψ preserves the quotient ξ/η , and that $\psi^{-1}(\xi, \eta) = (\xi v(\xi), \eta v(\xi))$, in which v is related to u by

$$(3.2) \quad u(\xi)v(\xi u(\xi)) = 1.$$

Put $m(\xi, \eta) = \xi/\eta$. For τ_1, τ_2 given by (3.1) we have

$$m\psi^{-1}\tau_1^*\psi = \lambda^2 m^{-1}, \quad m\tau_2 = m\rho\tau_1\rho = \rho m\tau_1\rho = \bar{\lambda}^2 m^{-1}.$$

Hence m is an eigenfunction of $\sigma = \tau_1\tau_2$, and Proposition 3.1 says that σ is formally linearizable.

Return to (3.1). A simple computation shows that

$$\tau_1(\xi, \eta) = (\lambda\eta p(\xi, \eta), \bar{\lambda}\xi p(\xi, \eta))$$

with $p(\xi, \eta) = u(\xi)v(\lambda\eta u(\xi))$ and

$$p(\xi, 0) = u(\xi), \quad p(0, \eta) = v(\lambda\eta).$$

Now

$$\begin{aligned} \sigma(\xi, 0) &= \tau_1 \rho \tau_1 \rho(\xi, 0) = \tau_1 \rho \tau_1(\bar{\xi}, 0) = \tau_1 \rho(0, \bar{\lambda} \bar{\xi} p(\bar{\xi}, 0)) = \tau_1 \rho(0, \bar{\lambda} \bar{\xi} u(\bar{\xi})) \\ &= \tau_1(0, \lambda \xi \bar{u}(\xi)) = (\lambda^2 \xi \bar{u}(\xi) p(0, \lambda \xi \bar{u}(\xi)), 0) = (\lambda^2 \xi \bar{u}(\xi) v(\lambda^2 \xi \bar{u}(\xi)), 0). \end{aligned}$$

Setting $\sigma(\xi, 0) = (f(\xi), 0)$, we obtain

$$(3.3) \quad f(\xi) = \mu \xi \bar{u}(\xi) v(\mu \xi \bar{u}(\xi)).$$

We now assume that $\Phi^{-1} \sigma \Phi = \sigma^* : \xi' = \mu \xi, \eta' = \bar{\mu} \eta$. Without loss of generality, we may assume that Φ is tangent to the identity. One first notices that since μ is not a root of unity, a smooth holomorphic curve passing through the origin is invariant under σ^* , if and only if the curve is one of coordinates axes. This means that there are only two invariant holomorphic curves of σ that pass through the origin, namely the two coordinate axes. Hence Φ must preserve both axes and $\Phi(\xi, 0) = (g(\xi), 0)$. From $\Phi^{-1} \sigma \Phi = \sigma^*$ it follows that

$$g^{-1} f g(\xi) = \mu \xi,$$

that is, f is linearizable by a convergent map. We have reduced Theorem 2.1 to the following theorem.

Theorem 3.3. *There exists μ , which is not a root of unity, and a holomorphic function $u(\xi) = 1 + O(1)$ such that the holomorphic map $f(\xi) = \mu \xi + O(2)$ defined by (3.2) and (3.3) is not holomorphically linearizable.*

Proof. The proof is based on a standard application of small-divisors.

We first recall some basic facts about linearizing $f(\xi) = \mu \xi + \sum_{n \geq 2} f_n \xi^n$; see [14] for details. It is known that there is a unique formal map $g(\xi) = z + \sum_{n \geq 2} g_n \xi^n$ such that $g(\mu \xi) = f g(\xi)$ ($g(c \xi)$ with $c \neq 0$ are the rest of maps that linearize f also). In fact

$$(3.4) \quad g_n = \frac{1}{\mu^n - \mu} \left\{ f_n + G_n \left(\mu, \frac{1}{\mu^2 - \mu}, \dots, \frac{1}{\mu^{n-1} - \mu}, f_2, \dots, f_{n-1} \right) \right\},$$

where G_n is a polynomial in the specified quantities with integral coefficients. It is also a standard fact that there is μ , not a root of unity, such that

$$(3.5) \quad |\mu^n - \mu| < \frac{1}{n!}, \quad n = n_k,$$

with $n_k \rightarrow \infty$ as $k \rightarrow \infty$.

Returning to (3.2) we see that $v_n = -u_n + V_n(u_2, \dots, u_{n-1})$, where u_n, v_n are coefficients of power series $u(z)$ and $v(z)$, respectively, and V_n is a polynomial in the specified quantities. Now (3.3) implies that

$$f_n = \mu(\bar{u}_{n-1} - \mu^{n-1} u_{n-1}) + F_n(\mu, u_2, \dots, u_{n-2}, \bar{u}_2, \dots, \bar{u}_{n-2}).$$

Together with (3.4), we see that

$$g_n = \frac{\mu}{\mu^n - \mu} \left\{ (\bar{u}_{n-1} - \mu^{n-1} u_{n-1}) + \tilde{G}_n \left(\mu, \frac{1}{\mu^2 - \mu}, \dots, \frac{1}{\mu^{n-1} - \mu}, u_2, \dots, u_{n-2}, \bar{u}_2, \dots, \bar{u}_{n-2} \right) \right\}.$$

Fix $\mu = \lambda^2$ satisfying (3.5), and next we choose u_{n-1} . We put $u_{n-1} = 0$ for $n \neq n_k$, and choose u_{n-1} for $n = n_k$ inductively as follows: Put $u_{n-1} = i\varepsilon_{n-1} \bar{\lambda}^{n-1}$ if

$$\left| \tilde{G}_n \left(\mu, \frac{1}{\mu^2 - \mu}, \dots, \frac{1}{\mu^{n-1} - \mu}, u_2, \dots, u_{n-2}, \bar{u}_2, \dots, \bar{u}_{n-2} \right) \right| < 1,$$

and put $u_{n-1} = 0$ otherwise. Then for all $n = n_k$, we achieve $|g_n| \geq n!$. This shows that the unique $g(z) = z + O(2)$ that linearizes f is divergent, from which one knows that f is not linearizable by any convergent map. \square

Next we modify the above argument to give a proof for Theorem 1.3.

Consider a real analytic map of the real plane given by

$$\sigma: z' = \mu z + E(z, \bar{z}),$$

where E is a complex-valued convergent power series in z, \bar{z} . Recall that σ is *reversible* if there is a real analytic involution τ_1 ($\tau_1^2 = \text{Id}$), defined near the origin with $\tau_1(0) = 0$, such that $\sigma^{-1} = \tau_1 \sigma \tau_1$.

Proof of Theorem 1.3. We shall consider holomorphic maps on \mathbb{C}^2 , and then consider their restrictions on suitable totally real subspace \mathbb{R}^2 . Start with the holomorphic map

$$\psi: \xi \rightarrow \xi u(\xi, \eta), \quad \eta \rightarrow \eta u(\xi, \eta), \quad u(\xi, \eta) = \bar{u}(\eta, \xi), \quad u(0) = 1.$$

Then $m\psi = m$ for $m(\xi, \eta) = \xi/\eta$, and ψ sends $\mathbb{R}^2: \eta = \bar{\xi}$ into itself. Note that \mathbb{R}^2 is fixed pointwise by

$$\rho_0: \xi \rightarrow \bar{\eta}, \quad \eta \rightarrow \bar{\xi}.$$

We also have

$$\psi^{-1}: \xi \rightarrow \xi v(\xi, \eta), \quad \eta \rightarrow \eta v(\xi, \eta), \quad v(\xi, \eta) = \bar{v}(\eta, \xi).$$

Put $\tau_1(\xi, \eta) = (\eta, \xi)$ and

$$\sigma = \sigma^* \tau_1 \psi^{-1} \tau_1 \psi, \quad \sigma^*(\xi, \eta) = (\mu \xi, \bar{\mu} \eta), \quad |\mu| = 1.$$

One readily sees that σ preserves \mathbb{R}^2 also, and that σ is reversible with respect to $\sigma^*\tau_1$. Note that $\sigma^*\tau_1$ is an involution preserving \mathbb{R}^2 . From [11] we have a unique formal map

$$\Phi: \xi \rightarrow U(\xi, \eta) = \xi + O(2), \quad \eta \rightarrow V(\xi, \eta) = \eta + O(2)$$

satisfying the normalizing condition

$$(3.6) \quad U_{j+1,j} = V_{j,j+1} = 0, \quad j > 0,$$

such that $\hat{\sigma} = \Phi^{-1}\sigma\Phi$ has the form $\xi \rightarrow M(\xi\eta)\xi, \eta \rightarrow M^{-1}(\xi\eta)\eta$. Since $m\sigma = \mu^2m$ Proposition 3.1 says that σ is linearizable by Φ . From $\rho_0\sigma\rho_0 = \sigma$, one sees easily that the unique transformation must satisfy $\rho_0\Phi\rho_0 = \Phi$, that is, Φ preserves \mathbb{R}^2 . Therefore, σ , as a reversible map of \mathbb{R}^2 , is linearizable by a formal map of \mathbb{R}^2 .

To finish the proof of the theorem, we need to choose some $u(\xi, \eta)$ so that σ is not linearizable by any holomorphic map. We have

$$\begin{aligned} \sigma^*\tau_1\psi^{-1}\tau_1\psi(\xi, 0) &= \sigma^*\tau_1\psi^{-1}\tau_1(\xi u(\xi, 0), 0) = \sigma^*\tau_1\psi^{-1}(0, \xi u(\xi, 0)) \\ &= \sigma^*\tau_1(0, \xi u(\xi, 0)v(0, \xi u(\xi, 0))) = (\mu\xi u(\xi, 0)v(0, \xi u(\xi, 0)), 0). \end{aligned}$$

Setting $\sigma(\xi, 0) = (f(\xi), 0)$, we obtain

$$f(\xi) = \mu\xi u(\xi, 0)\bar{v}(\xi u(\xi, 0), 0).$$

We also have $u(\xi, \eta)v(\xi u(\xi, \eta), \eta u(\xi, \eta)) = 1$. Hence

$$u_{\alpha\beta} = -u_{\alpha\beta} + V_{\alpha\beta}(\{u_{\alpha'\beta'} : 1 < \alpha' + \beta' < \alpha + \beta\});$$

and consequently,

$$f_n = 2i\mu\mathfrak{I}u_{n-1,0} + \mu F_n(\{u_{\alpha\beta}, \bar{u}_{\alpha\beta} : \alpha + \beta < n - 1\}),$$

where $V_{\alpha,\beta}$ and F_n are polynomials in specified quantities.

As in the proof of Theorem 3.3, we assume that $\Phi^{-1}\sigma\Phi = \sigma^*$. Without loss of generality we may assume that Φ is tangent to the identity. We also know that Φ preserves both coordinate axes. Write $\Phi(\xi, 0) = (g(\xi), 0)$. Then $g^{-1}fg(\xi) = \mu\xi$. Comparing coefficients yields

$$g_n = \frac{1}{\mu^n - \mu} \{2i\mu\mathfrak{I}u_{n-1,0} + G_n\},$$

where G_n is a polynomial of integral coefficients in quantities

$$\mu, \frac{1}{\mu^2 - \mu}, \dots, \frac{1}{\mu^{n-1} - \mu}, u_{\alpha,\beta}, \bar{u}_{\alpha,\beta}, \quad 1 < \alpha + \beta < n - 1.$$

Again, we fix μ , which is not a root of unity, such that $|\mu^n - 1| < 1/n!$ for a sequence $n = n_k \rightarrow \infty$. Choose $\varepsilon_n = 1$ or 0 for $n = n_k - 1$ and $\varepsilon_n = 0$ for $n \neq n_k - 1$ such that for

$$u(\xi, \eta) = 1 + \sum_{n=1}^{\infty} i\varepsilon_n(\xi^n - \eta^n),$$

we have $|g_n| \geq n!$ when $n = n_k - 1$. This shows that the corresponding σ is not linearizable by any convergent transformation. \square

4. REVERSIBLE MAPS WITH ONE HOLOMORPHIC FIRST-INTEGRAL AND ONE MEROMORPHIC EIGENFUNCTION

In Section 3, we have proved that there exists a pair of holomorphic involutions τ_1, τ_2 such that $\sigma = \tau_1\tau_2: \xi \rightarrow \mu\xi + O(2), \eta \rightarrow \mu^{-1}\eta + O(2)$, with μ not a root of unity, is formally but not holomorphically equivalent to a pair of linear involutions. We actually constructed τ_1, τ_2 satisfying $\tau_2 = \rho\tau_1\rho$ (see (2.2)-(2.3)), so that the pair τ_1, τ_2 yields a real analytic surface M with the Bishop invariant $\gamma \in (\frac{1}{2}, \infty) \setminus E$; consequently M is formally but not holomorphically equivalent to the quadric Q_γ . It remains to show that the M is not contained in any smooth Levi-flat real analytic hypersurface in \mathbb{C}^2 . In fact, M is not contained in any (singular Levi-flat real analytic) real hypersurface defined by $\Im\{h(z)\} = 0$, where h with $h(0) = 0$ is a nonconstant holomorphic function. (For singular Levi-flat real analytic hypersurfaces see [4].) Recall from [11] that if $\Im\{h(z)\}|_M = 0$, then $h(z)$ is equal to $\bar{h}(w)$ on M^c and hence its restriction on M^c is invariant under both τ_1 and τ_2 .

Therefore, Theorem 1.2 and the last assertion in Theorem 1.1 follow from the theorem below.

Theorem 4.1. *Let $\sigma: \xi' = \mu\xi + O(2), \eta' = \mu^{-1}\eta + O(2)$ be a reversible holomorphic map of \mathbb{C}^2 . Assume that μ is not a root of unity. Assume also that there exist a non-constant holomorphic function h and a meromorphic function $m = f/g$, $(f(\xi, \eta), g(\xi, \eta)) = (\xi, \eta) + O(2)$, such that $m\sigma = \mu^2\sigma$ and $h\sigma = h$. Then σ is holomorphically linearizable.*

Proof. We first put $\psi_0 = (f, g)$. Let $\sigma_0 = \psi_0\sigma\psi_0^{-1}$. Then σ_0 is still reversible with respect to some holomorphic involution. Also $m_0(\xi, \eta) = m\psi_0^{-1}(\xi, \eta) = \xi/\eta$ and $h_0(\xi, \eta) = \xi\eta + O(3)$. Replacing ψ, m, σ by ψ_0, m_0, σ_0 and still denoting them by ψ, m, σ , Proposition 3.1 says that σ is formally linearizable. By Lemma 3.2 in [11] there exists a unique formal map

$$\Phi: \xi' = a(\xi, \eta), \quad \eta' = b(\xi, \eta),$$

satisfying the normalizing condition

$$a_{j+1,j} = 0, \quad b_{j,j+1} = 0, \quad j > 0, \quad a_{1,0} = b_{0,1} = 1, \quad a_{0,1} = b_{1,0} = 0,$$

such that $\Phi^{-1}\sigma\Phi = \sigma^*$. Since the coordinate axes are the two unique smooth formal curves that are invariant by σ^* , and since Φ is tangent to the identity, then Φ preserves two coordinate axes also. So we can write

$$\Phi: \xi' = \xi(1 + u(\xi, \eta)), \quad \eta' = \eta(1 + v(\xi, \eta)),$$

with

$$(4.1) \quad u(0) = u_{jj} = v(0) = v_{jj} = 0, \quad j > 0.$$

By Proposition 3.1, we know that $\hat{m}(\xi, \eta) = m\Phi(\xi, \eta) = \xi(1 + \omega(\xi\eta))/\eta$ with $\omega(0) = 0$, i.e., that u, v satisfy

$$u(\xi, \eta) - v(\xi, \eta) - \omega(\xi\eta) = \omega(\xi\eta)v(\xi, \eta),$$

from which and (4.1) we obtain

$$\omega = 0, \quad u = v.$$

We also know that $\hat{h}(\xi, \eta) = h\Phi(\xi, \eta)$ is invariant by σ^* . Since μ is not a root of unity, then $\hat{h}(\xi, \eta)$ is a power series in the product $\xi\eta$. We may assume that $h(0) = 0$ and write $\hat{h}(\xi, \eta) = (\xi\eta)^t \hat{h}_1(\xi\eta)$ with t a positive integer and $\hat{h}_1(0) \neq 0$. We see that $h(\xi, \eta) = (\xi\eta)^t h_1(\xi, \eta)$ with $h_1(0) \neq 0$. Replacing h with a constant multiple of $h^{1/t}$, we may assume that $h(\xi, \eta) = \xi\eta(1 + g(\xi, \eta))$ with $g(0) = 0$. Then $\hat{h}(\xi, \eta) = \xi\eta(1 + \kappa(\xi\eta))$ with $\kappa(0) = 0$. Now $h\Phi(\xi, \eta) = \xi\eta(1 + \kappa(\xi\eta))$ becomes

$$(4.2a) \quad 2u(\xi, \eta) - \kappa(\xi\eta) = B(\xi, \eta, u(\xi, \eta)),$$

$$(4.2b) \quad B(\xi, \eta, u) = -u^2 - (1 + u)^2 \cdot g(\xi(1 + u), \eta(1 + u)).$$

We need to show that (4.1)-(4.2) has a unique convergent solution u, κ . That uniqueness assertion also implies the convergence of the original map Φ (satisfying the normalizing condition (4.1)). We first introduce some notations. For a power series $f(\xi, \eta)$, denote

$$\begin{aligned} [f]_s(\xi, \eta) &= [f(\xi, \eta)]_s = \sum_{i-j=s} f_{i,j} \xi^i \eta^j, \quad s = 0, \pm 1, \dots, \\ f^*(\xi, \eta) &= \sum_{i,j} |f_{i,j}| \xi^i \eta^j, \\ f < g & \quad \text{if } |f_{i,j}| \leq g_{i,j}. \end{aligned}$$

Now the normalizing condition (4.1) means that $[u]_0 = 0$, and from (4.2) we first get

$$\kappa(\xi\eta) = -[B(\xi, \eta, u(\xi, \eta))]_0.$$

For $s \neq 0$, (4.2) yields

$$u_s(\xi, \eta) = \frac{1}{2}[B(\xi, \eta, u(\xi, \eta))]_s.$$

Put

$$W(\xi, \eta) = \kappa^*(\xi\eta) + \sum_{|s|>0} u_s^*(\xi, \eta).$$

Then we get

$$W(\xi, \eta) < \hat{B}(\xi, \eta, W(\xi, \eta))$$

for

$$\hat{B}(\xi, \eta, W) = W^2 + (1 + W)^2 g^*(\xi(1 + W), \eta(1 + W)).$$

Putting the above together, we see that $W < \hat{W}$ if $\hat{W}(0) = 0$, and

$$\hat{W}(\xi, \eta) = \hat{B}(\xi, \eta, \hat{W}).$$

By the implicit function theorem, \hat{W} is convergent. Therefore, u, v are convergent. \square

We also have the following result.

Corollary 4.2. *Let $\sigma: z' = \mu z + O(2)$ be a reversible real analytic map of the real plane. Assume that μ is not a root of unity. Assume further that σ has a non-constant invariant real analytic function and an eigenfunction $p(z, \bar{z})/q(z, \bar{z})$, where $p(z, \bar{z}) = z + O(2)$ and $q(z, \bar{z}) = \bar{z} + O(2)$ are convergent power series. Then σ is real analytically linearizable.*

Proof. Let τ be a real analytic involution such that $\sigma^{-1} = \tau\sigma\tau$. Write $\sigma(z) = f(z, \bar{z})$ and $\tau(z) = g(z, \bar{z})$, where f, g are convergent power series in z, \bar{z} . Consider the complexifications $\sigma^c: \xi' = f(\xi, \eta), \eta' = \bar{f}(\eta, \xi)$ and $\tau^c: \xi' = g(\xi, \eta), \eta' = \bar{g}(\eta, \xi)$. Then τ^c is a holomorphic involution of \mathbb{C}^2 and σ^c is reversible with respect to τ^c . From the proof of Theorem 4.1, we see that σ^c is linearizable by a holomorphic map Φ satisfying the normalizing condition (3.6). As seen in the proof of Theorem 1.3, the map $\varphi: z \rightarrow \Phi(z, \bar{z})$ is a real analytic map from \mathbb{R}^2 into itself, and it linearizes σ . \square

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