

SINGULAR LEVI-FLAT REAL ANALYTIC HYPERSURFACES

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Abstract. We initiate a systematic local study of singular Levi-flat real analytic hypersurfaces, concentrating on the simplest nontrivial case of quadratic singularities. We classify the possible tangent cones to such hypersurfaces and prove the existence and convergence of a rigid normal form in the case of generic (Morse) singularities. We also characterize when such a hypersurface is defined by the vanishing of the real part of a holomorphic function. The main technique is to control the behavior of the homorphic Segre varieties contained in such a hypersurface. Finally, we show that not every such singular hypersurface can be defined by the vanishing of the real part of a holomorphic or meromorphic function, and give a necessary condition for such a hypersurface to be equivalent to an algebraic one.

1. Introduction. A well-known theorem of E. Cartan says that a real analytic smooth hypersurface M in \mathbf{C}^n has no local holomorphic invariant, if M is Levi-flat, i.e., it is foliated by smooth holomorphic hypersurfaces of \mathbf{C}^n . In suitable local coordinates such a hypersurface is given by $\Re z_1 = 0$. On the other hand, if the Levi-form is nondegenerate, the invariants of M are given by the theory of Cartan [5], Chern-Moser [6] and Tanaka [11]. In this paper we are concerned with real analytic hypersurfaces in \mathbf{C}^n with singularity. Such a real analytic hypersurface M in \mathbf{C}^n is decomposed into M^* and M_s , where M^* is a smooth real analytic hypersurface and M_s , the singular locus, is contained in a proper analytic subvariety of lower dimension. We say that a real analytic hypersurface M with singularity is Levi-flat if its smooth locus M^* is Levi-flat. An interesting class of Levi-flat hypersurfaces are those real analytic varieties defined by the vanishing of the real part of a meromorphic function. Not every Levi-flat hypersurface, however, can be realized in this way.

Singular Levi-flat real analytic sets occur naturally as invariant sets of integrable holomorphic Hamiltonian systems. One of the motivations of the current paper is the relationship between the invariant sets of a Hamiltonian system and the convergence of the normalization for the Birkhoff normal form of that system. The techniques developed in this paper have been applied by the second author to the study of holomorphic Hamiltonian systems [8].

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Let M be a real analytic hypersurface defined by $r = 0$. The Segre varieties associated to M are the complex varieties in \mathbf{C}^n defined by

$$Q_w := \{z \in \mathbf{C}^n \mid r(z, \bar{w}) = 0\}.$$

Segre varieties, used first by B. Segre [10] and developed by S. M. Webster [13], Diederich-Fornaess [7] and others, have been a powerful tool in dealing with smooth real analytic hypersurfaces. In this paper, we need to understand the Segre variety at a singular point of a Levi-flat hypersurface.

The Segre varieties at singular points are important invariants for Levi-flat hypersurfaces, in addition to the usual invariants of such singular points such as the tangent cones. The simplest singular Levi-flat hypersurface is the *real cone*

$$(1.1) \quad \Re\{z_1^2 + z_2^2 + \cdots + z_n^2\} = 0,$$

in which 0 is an isolated singular point. In this example the Segre variety Q_0 is contained in the cone. However, in $\mathbf{C}^n (n \geq 2)$ another example is the *complex cone*

$$(1.2) \quad z_1 \bar{z}_1 - z_2 \bar{z}_2 = 0,$$

for which the singular locus is \mathbf{C}^{n-2} , and the Segre variety Q_0 is the whole space \mathbf{C}^n . This paper will show how the Segre variety Q_0 provides the essential information in constructing a normal form for certain Levi-flat hypersurfaces. The hypersurfaces to which most of our results apply are those with defining function r with quadratic leading terms. Fortunately, the possible tangent cones which can occur for such hypersurfaces can be classified and understood. As in many similar areas of study in complex analysis, we can analyze completely some of these possible cases, but for hypersurfaces with some of the more degenerate tangent cones our results are incomplete.

The main results are as follows. We consider a real analytic hypersurface in \mathbf{C}^n given by

$$(1.3) \quad \Re\{z_1^2 + z_2^2 + \cdots + z_n^2\} + H(z, \bar{z}) = 0$$

with

$$(1.4) \quad H(z, \bar{z}) = O(|z|^3), \quad H(z, \bar{z}) = \overline{H(\bar{z}, z)}.$$

We have the following.

THEOREM 1.1. *Let M be a real analytic hypersurface defined by (1.3) and (1.4). Assume that M is Levi-flat away from the origin of \mathbf{C}^n . Then there is a biholomorphic mapping defined near 0 which transforms M into the real cone (1.1).*

We next characterize Levi-flat hypersurfaces which are equivalent to the complex cone (1.2) as follows.

THEOREM 1.2. *Let $M: r = 0$ be a Levi-flat real analytic hypersurface in \mathbf{C}^n with $r = q(z, \bar{z}) + O(|z|^3)$. Assume that the quadratic form q is positive definite on a complex line and its Levi-form has rank at least 2. Then M is biholomorphically equivalent to the complex cone (1.2).*

In very general terms, a Levi-flat hypersurface, even a singular one, is a family of complex hypersurfaces dependent on one real parameter t . The difficulties encountered in the proofs of these theorems result from needing to control carefully the analytic properties of this parameter, in particular, extending it when possible to a holomorphic function of complex values of t .

The contents of the paper in the various sections are as follows. In Section 2 we shall give a complete classification of Levi-flat quadrics in \mathbf{C}^n , that is, Levi-flat hypersurfaces defined by the vanishing of a homogeneous quadratic polynomial. (We return to the case of a homogeneous quadratic polynomial whose zero set is smaller dimensional in Section 3.) We shall also show that the tangent cone of a singular Levi-flat hypersurface remains Levi-flat, if the cone is a hypersurface. Combining Theorem 1.1 and Theorem 1.2, we shall see that a Levi-flat hypersurface in \mathbf{C}^n , defined by a real analytic function with a nondegenerate critical point, is biholomorphically equivalent to the real cone (1.1), or possibly to the complex cone (1.2) when $n = 2$. Section 3 is devoted to the study of the Segre varieties at singular points of Levi-flat hypersurfaces. We shall prove that the smooth locus of a real analytic Levi-flat hypersurface must be connected if the dimension of its singular locus is not too large. In Section 3 we shall focus on the question when a Levi-flat hypersurface is defined by the real part of a holomorphic function. We shall prove a theorem stronger than Theorem 1.1, which allows some degeneracy at singular points. The proof of Theorem 1.2 will be presented in Section 4. At the end of Section 4 we summarize what we know about irreducible Levi-flat hypersurfaces with quadratic singularities according to their possible tangent cones. In Section 5, we shall study Levi-flat hypersurfaces generated by meromorphic functions, and we give a sufficient condition for a Levi-flat real analytic hypersurface to be equivalent to an algebraic hypersurface.

Acknowledgments. Singular Levi-flat hypersurfaces have previously been studied by E. Bedford [3] in relation to continuation of holomorphic functions defined on open sets bounded by such singular hypersurfaces. His hypersurfaces are assumed to have singularities contained in a complex subvariety of codimension at least two. We thank him for calling [3] to our attention.

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2. Levi-flat quadrics. In this section we shall first give a complete classification of Levi-flat quadrics. We shall also see that quadratic tangent cones of Levi-flat hypersurfaces are Levi-flat, if the cones are of dimension $2n - 1$. The quadratic tangent cones of smaller dimension will be discussed in section 3 by using Segre varieties.

Recall that a germ of real analytic variety V at $0 \in \mathbf{R}^k$ is the zero set of a finite number of real analytic functions. A germ of irreducible real analytic variety is always defined by an irreducible real analytic function. Throughout this paper, we shall denote by V^* the set of points $x \in V$ such that near x , V is a real analytic submanifold of \mathbf{R}^k of dimension $d = \dim V$. Then $V_s = V \setminus V^*$ is contained in a germ of real analytic variety of dimension less than d . We shall denote by $\dim V_s$ the dimension of the smallest germ of real analytic variety at 0 which contains V_s .

For a holomorphic function $f(z)$, we shall define the holomorphic function $\bar{f}(z) = \overline{f(\bar{z})}$. We need the following elementary fact.

LEMMA 2.1. *Let $r(x)$ be an irreducible germ of real analytic function defined at $0 \in \mathbf{R}^k$. Assume that $V: r = 0$ has dimension $k - 1$. Then r is irreducible as a germ of holomorphic function at $0 \in \mathbf{C}^k$. Moreover, for any ball $B_\epsilon = \{x \in \mathbf{R}^k \mid |x| < \epsilon\}$ there exists an open set $U \subset B_\epsilon$, containing 0 , such that for any real power series $R(x)$ which is convergent on B_ϵ and vanishes on a nonempty open subset of $U \cap V^*$, r divides R on U .*

Proof. Let $r = f_1^{d_1} \cdots f_m^{d_m}$, where $f_j(x)$ are irreducible holomorphic functions in x with $f_j(0) = 0$, and assume for the sake of contradiction that $m > 1$, that is, that r is reducible as a holomorphic function at 0 in \mathbf{C}^n . Since r is real, then rearranging the order of f_j gives $\bar{f}_1 = \mu f_2$, where μ is a nonvanishing holomorphic function. One sees that $\mu f_1 f_2$ is real on \mathbf{R}^k , and it divides r . The irreducibility of r then implies that $m = 2$ and $d_j = 1$. Therefore, $f_j|_V \equiv 0$ for $j = 1, 2$, i.e., $\{f_1 = 0\}$ and $\{f_2 = 0\}$ are the same complex variety because V is of dimension $k - 1$. This contradicts that f_1 and f_2 are relatively prime. Hence, r is irreducible as a germ of holomorphic function. Choose an open set $\tilde{U} \subset \{z \in \mathbf{C}^k \mid |z| < \epsilon\}$ containing the origin such that the smooth locus $(V^c)^*$ of $V^c \equiv \tilde{U} \cap \{z \in \mathbf{C}^k \mid r(z) = 0\}$ is connected, and such that any holomorphic function on \tilde{U} which vanishes on $(V^c)^*$ is divisible by r on \tilde{U} . Let $U = \tilde{U} \cap \mathbf{R}^k$. Assume that $R(x)$ is a power series convergent on B_ϵ and vanishes on a nonempty open subset of $U \cap V^*$. Then R vanishes on V^c . Therefore, r divides R on \tilde{U} . \square

LEMMA 2.2. *Let M be a real analytic hypersurface in \mathbf{C}^n defined by $r = 0$. Suppose that r is irreducible. Then there is an open set $U \subset \mathbf{C}^n$ containing 0 such that $M \cap U$ is Levi-flat if and only if one of the connected components of $M^* \cap U$ is Levi-flat.*

Proof. Fix $\epsilon > 0$ such that $r(z, \bar{z})$ converges on $B_\epsilon \subset \mathbf{C}^n$. By Lemma 2.1 there exists an open subset U of B_ϵ such that all real power series convergent on B_ϵ and vanishing on a nonempty open subset of $M^* \cap U$ are divisible by r .

To study the Levi-form of r , consider the following Levi-matrix of r

$$(2.1) \quad L = L^r = \begin{pmatrix} 0 & r_{\bar{z}} \\ r_z & r_{z\bar{z}} \end{pmatrix}.$$

From [13], one knows that for $p \in M^*$ with $dr(p) \neq 0$, M is Levi-flat near p if and only if the rank of L is 2. Fix a connected component K of $M^* \cap U$. We want to show that dr is not identically zero on K . Otherwise, all $r_{z_j}, r_{\bar{z}_j}$ vanish on K . Lemma 2.1 implies that $r_{z_j} = a_j r$. We now have

$$\sum z_j r_{z_j} + \bar{z}_j r_{\bar{z}_j} = \sum (a_j z_j + \bar{a}_j \bar{z}_j) r.$$

The vanishing order of the left-hand side is the same as that of r , while the right-hand side has a higher order of vanishing. This contradiction shows that r_z does not vanish identically on K . We now take K to be a Levi-flat component of $M^* \cap U$. Let Δ be the determinant of any 3×3 submatrix of L . Then $\Delta|_K \equiv 0$, and Lemma 2.1 says that r divides Δ , i.e., $\Delta|_{M \cap U} = 0$. This proves that $M^* \cap U$ is Levi-flat. \square

We now turn to the classification of Levi-flat real quadrics in \mathbf{C}^n .

Table 2.1 is a list of Levi-flat quadrics in \mathbf{C}^n accompanied by the defining functions for both the quadrics and the corresponding Segre varieties Q_0 .

Table 2.1. Levi-flat quadrics.

<i>type</i>	<i>normal form</i>	<i>singular set</i>	Q_0
$Q_{0,2k}$	$\Re\{z_1^2 + \cdots + z_k^2\}$	\mathbf{C}^{n-k}	$z_1^2 + \cdots + z_k^2$
$Q_{1,1}$	$z_1^2 + 2z_1\bar{z}_1 + \bar{z}_1^2$	empty	z_1^2
$Q_{1,2}^\lambda, \lambda \in (0, 1)$	$z_1^2 + 2\lambda z_1\bar{z}_1 + \bar{z}_1^2$	\mathbf{C}^{n-1}	z_1^2
$Q_{2,2}$	$(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)$	$\mathbf{R}^2 \times \mathbf{C}^{n-2}$	$z_1 z_2$
$Q_{2,4}$	$z_1\bar{z}_1 - z_2\bar{z}_2$	\mathbf{C}^{n-2}	0

THEOREM 2.3. *The above table is a complete list of holomorphic equivalence classes of Levi-flat quadratic real hypersurfaces in \mathbf{C}^n .*

Proof. In the table, $Q_{i,j}$ stands for a quadric defined by a quadratic form q such that the Levi-form of q has rank i , and as a real quadratic form, q has rank j . This shows that all quadrics in Table 2.1, except the family $Q_{1,2}^\lambda$, are not biholomorphically equivalent. The singular set of $Q_{1,2}^\lambda$, defined by $z_1 = 0$, is invariant under a complex linear transformation mapping from $Q_{1,2}^\lambda$ to $Q_{1,2}^{\lambda'}$, i.e., the transformation must be of the form $z_1 \rightarrow cz_1$. Therefore, we may restrict ourselves to the case $n = 1$ to distinguish the family $Q_{1,2}^\lambda$. However, each $Q_{1,2}^\lambda$

consists of two lines in \mathbf{C} intersecting at the origin with the angle $\arccos \lambda$. Hence, all $\mathcal{Q}_{1,2}^\lambda$ are not equivalent.

Next, we want to prove that a Levi-flat quadric is always equivalent to one of the quadrics in Table 2.1. Let $q(z, \bar{z})$ be a real quadratic form, which defines a quadric \mathcal{Q} in \mathbf{C}^n of real dimension $2n - 1$. We first consider the case that q is reducible with multiplicity two. In this case, $q(z, \bar{z})$ is obviously equivalent, up to a possible change of sign, to $(z_1 + \bar{z}_1)^2$ through a \mathbf{C} -linear transformation. Assume now that $q = q_1 q_2$, where q_1, q_2 are linearly independent \mathbf{R} -linear functions. There are two cases to be considered: (i) $q_1(z, 0), q_2(z, 0)$ are \mathbf{C} -linearly independent; (ii) $q_1(z, 0) = \mu q_2(z, 0)$ with $\mu \in \mathbf{C} \setminus \mathbf{R}$. For case (i), one can introduce \mathbf{C} -linear coordinates with $z'_j = q_j(z, 0) (j = 1, 2)$. Thus, \mathcal{Q} becomes $\mathcal{Q}_{2,2}$. For case (ii), one may assume that $|\mu| = 1$. One then obtains $\Re \mu \geq 0$ by changing the sign of q , and $\Im \mu \geq 0$ by interchanging q_1 and q_2 . Thus, \mathcal{Q} is given by $(z_1 + \bar{z}_1)(\mu z_1 + \bar{\mu} \bar{z}_1) = 0$. When μ is pure imaginary, \mathcal{Q} becomes $\mathcal{Q}_{0,2}$; otherwise, \mathcal{Q} is equivalent to $\mathcal{Q}_{1,2}^\lambda$ with $\lambda = \Re \mu$.

We now assume that q is irreducible. Write

$$q(z, \bar{z}) = 2\Re \sum A_{ij} z_i \bar{z}_j + \sum B_{\bar{i}\bar{j}} z_i \bar{z}_j$$

with $A_{ij} = A_{ji}, B_{\bar{i}\bar{j}} = \overline{B_{\bar{j}\bar{i}}}$. Consider

$$(2.2) \quad L^q = \begin{pmatrix} 0 & q_{\bar{z}} \\ q_z & q_{z\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & \overline{A_{\bar{z}} + B_{\bar{z}}} \\ A_z + B_z & B_{z\bar{z}} \end{pmatrix}.$$

Since q is irreducible, then $q_z \not\equiv 0$ on the smooth locus of \mathcal{Q} . From [13], one then knows that at the smooth points of \mathcal{Q} , the rank of L is 2. Thus, the rank of $B_{z\bar{z}}$ is at most two.

We first consider the case $B(z, \bar{z}) \equiv 0$. Here, it is easy to see that \mathcal{Q} is equivalent to $\mathcal{Q}_{0,2k}$ with k the rank of the complex quadratic form $A(z)$. Next, we assume that $\text{rank } B_{z\bar{z}} = 1$. Replacing r by $-r$ if necessary, one may further assume that $B(z, \bar{z}) = z_1 \bar{z}_1$. Among the 3×3 submatrices of L^q , we consider

$$(2.3) \quad \begin{pmatrix} 0 & \overline{A_{z_1} + z_1} & \overline{A_{z_j}} \\ A_{z_1} + \bar{z}_1 & 1 & 0 \\ A_{z_j} & 0 & 0 \end{pmatrix}, \quad j > 1.$$

Notice that the determinant $-|A_{z_j}|^2$ of (2.3) contains no harmonic terms. Since L is of rank at most 2, then $|A_{z_j}|^2$ vanishes identically at the smooth points of \mathcal{Q} . Hence, $|A_{z_j}|^2 = c_j q(z, \bar{z})$, where c_j is a real constant. If $A_{z_j} \not\equiv 0$, then this would imply that in suitable coordinates, $q(z, \bar{z}) = |z_1|^2$. This contradicts that $\dim \mathcal{Q} = 2n - 1$. Therefore, $A(z)$ is independent of z_j for $j > 1$, which leads to another contradiction since q is irreducible. We now turn to the case $\text{rank } B_{z\bar{z}} = 2$.

One may assume that $B(z, \bar{z}) = z_1 \bar{z}_1 + \epsilon z_2 \bar{z}_2$ with $\epsilon = \pm 1$. If $\epsilon = +1$ then the complex Hessian $(r_{z_i \bar{z}_j})$ would have rank at least 1 on the leaf of the Levi foliation of \mathcal{Q} through any smooth point sufficiently close to 0. But this contradicts the fact that $r \equiv 0$ along such a leaf. Thus $\epsilon = -1$. The above argument shows that A is independent of z_j for $j > 2$. Now the determinant of the first 3×3 submatrix of (2.1) gives us

$$(2.4) \quad -1 \cdot |A_{z_1} + \bar{z}_1|^2 + |A_{z_2} - \bar{z}_2|^2 = Cq(z, \bar{z}).$$

Comparing the holomorphic terms in A and in (2.4) gives us

$$(2.5) \quad -z_1 A_{z_1} - z_2 A_{z_2} = CA(z).$$

The left-hand side of (2.5) is precisely $-2A(z)$, which implies that either $A = 0$, or $C = -2$. For the former case, \mathcal{Q} becomes $\mathcal{Q}_{2,4}$. In the case $C = -2$ with $A \neq 0$, (2.4) reads

$$|A_{z_1}(z)|^2 - |A_{z_2}(z)|^2 = |z_1|^2 - |z_2|^2.$$

Put $A(z) = az_1^2 + 2bz_1z_2 + cz_2^2$. One may assume that a, c are nonnegative. The above identity then gives us $a\bar{b} = bc$ and $4(a^2 - |b|^2) = 4(c^2 - |b|^2) = 1$. Hence, $a = c > 0$ and b is also real, and

$$2A(z) = ZU(t)Z^t = ZU(t/2)(ZU(t/2))^t,$$

where $Z = (z_1, z_2)$ and

$$2 \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \equiv U(t).$$

Notice that $U(t)$ preserves $|z_1|^2 - |z_2|^2$. Therefore, $q(z)$ is equivalent to $\Re\{z_1^2 + z_2^2\} + |z_1|^2 - |z_2|^2$. This shows that q is reducible, which is a contradiction. The proof of the theorem is complete. \square

PROPOSITION 2.4. *Let M be an irreducible real analytic hypersurface defined by $r = 0$, where $r = q(z, \bar{z}) + O(|z|^{k+1})$ and q is a homogeneous polynomial of degree k . Assume that $\{q = 0\}$ is of dimension $2n - 1$, and that q is reduced if $k > 2$. If M is Levi-flat, so is the tangent cone $q = 0$.*

Proof. Let $r^\epsilon(z, \bar{z}) = r(\epsilon z, \epsilon \bar{z})/\epsilon^k$. Fix indices $I = (i_1, i_2, i_3), J = (j_1, j_2, j_3)$ with $1 \leq i_l, j_l \leq n + 1$. Denote by L_{IJ}^r the 3×3 submatrix consisting of entries in rows i_1, i_2, i_3 and in columns j_1, j_2, j_3 of (2.2). Since M is Levi-flat and r is irreducible, then

$$\det L_{IJ}^r(z, \bar{z}) = p_{IJ}(z, \bar{z})r(z, \bar{z}),$$

where p_{IJ} is a convergent power series. Hence,

$$\epsilon^d \det L_{IJ}^{r\epsilon}(z, \bar{z}) = p_{IJ}(\epsilon z, \epsilon \bar{z}) r(\epsilon z, \epsilon \bar{z}),$$

where d equals $3k - 4$ for $i_1 + j_1 = 2$, $3k - 5$ for $i_1 = 1 < j_1$ or $j_1 = 1 < i_1$, and $3k - 6$ for $i_1, j_1 \geq 2$, respectively. Obviously, ϵ^{d-k} divides $p_{IJ}(\epsilon z, \epsilon \bar{z})$, if $d - k > 0$. Letting $\epsilon \rightarrow 0$, we see that for $d \geq k$, $\det L_{IJ}^q(z, \bar{z}) = \tilde{p}_{IJ}(z, \bar{z}) q(z, \bar{z})$, where \tilde{p}_{IJ} is the sum of homogeneous terms of p_{IJ} of degree $d - k$. Also, we have $L_{IJ}^q \equiv 0$ for $d < k$. In both cases, we obtain that L_{IJ}^q vanishes on $q = 0$. This shows that the Levi-matrix of q has rank at most 2 on $q = 0$, i.e., $\{q = 0\} \setminus \{dq = 0\}$ is Levi-flat.

On the other hand, by the assumptions we have a decomposition $q = q_1 \dots q_k$, where q_j are irreducible. Let V_j be the vanishing set of q_j . From Lemma 2.1 it follows that $\dim(V_i \cap V_j) < 2n - 1$ for $i \neq j$. From the proof of Lemma 2.2, one sees that $\dim(\{dq_j = 0\} \cap V_j) < 2n - 1$. Hence, $\dim(\{dq = 0\} \cap \{q = 0\}) < 2n - 1$. Therefore, $q = 0$ is Levi-flat. \square

Let us record here the following result about the case of r with generic quadratic singularity, anticipating the proofs of Theorems 1.1 and 1.2 in Sections 3 and 4.

COROLLARY 2.5. *Let M be a smooth Levi-flat real analytic hypersurface in \mathbf{C}^n defined by $r = 0$. Suppose that 0 is a nondegenerate critical point of r . Then M is biholomorphically equivalent to the real cone (1.1), or to the complex cone (1.2) with $n = 2$.*

Proof. By Proposition 2.4, the quadratic term q of r defines a Levi-flat quadric with isolated singularity at 0 . Theorem 2.3 says that q is equivalent to $\Re(z_1^2 + z_2^2 + \dots + z_n^2)$, or $z_1 \bar{z}_1 - z_2 \bar{z}_2$ with $n = 2$. Now Corollary 2.5 follows from Theorem 1.1 and Theorem 1.2. \square

3. Nondegenerate Segre variety at a singular point. In this section, we shall first study the Segre variety Q_0 , where 0 is a singular point at which the germ of a Levi-flat real analytic hypersurface M is defined. According to a theorem of J. E. Fornaess (see [9], Theorem 6.23), such a Levi-flat hypersurface M always contains a complex variety of dimension $n - 1$ which passes through the origin. (The theorem is stated for M being the boundary of a domain with smooth and real analytic boundary. However, the proof is valid without any change if M is a germ of real analytic hypersurface with singularities.) We shall use this fact to classify the quadratic tangent cones of Levi-flat hypersurfaces which are of dimension less than $2n - 1$. The main result of this section is to determine when a Levi-flat hypersurface is defined by the real part of a holomorphic function.

We shall denote by Δ_ϵ the disc in \mathbf{C} of radius ϵ , and by Δ_ϵ^k the product of k discs Δ_ϵ . We first need the following lemma.

LEMMA 3.1. *Let M be a smooth Levi-flat real analytic hypersurface in \mathbf{C}^n defined by $r = 0$. Assume that $r(z, 0) \not\equiv 0$ and M is of dimension $2n - 1$ at 0 . One branch Q'_0 of Q_0 is contained in M . Furthermore, Q'_0 is smooth, and it is the unique germ of complex variety of pure dimension $n - 1$ at 0 which is contained in M .*

Proof. Choose holomorphic coordinates such that near 0 , M is given by $\Re z_1 = 0$. Then M contains the hyperplane $z_1 = 0$. Since $r \equiv 0$ on M , then $r(z, \bar{z}) = a(z, \bar{z})\Re z_1$. Obviously, $z_1 = 0$ is a branch of Q_0 . For the uniqueness, assume that a germ of complex variety V of pure dimension $n - 1$ at 0 is inside M . This means that z_1 is pure imaginary on V . Since $0 \in V$, then $z_1 \equiv 0$ on V . Therefore, V is the hyperplane $z_1 = 0$. \square

Given a real analytic hypersurface M defined by $r = 0$, we shall denote by Q_w the Segre variety defined by $r(z, \bar{w}) = 0$, and by Q_w^* the smooth locus of Q_w . We shall denote by Q'_w the union of all branches of Q_w which are contained in M . Q'_w could be an empty set when $w \in M_s$; otherwise, it is a complex variety of pure dimension $n - 1$. Lemma 3.1 implies that if Q'_w is nonempty, then $Q'_w \cap M^*$ is smooth.

The following result is essentially due to Fornaess [9]. We present a proof here, since some arguments will be needed later on.

LEMMA 3.2. *Let M be a Levi-flat hypersurface defined by $r(z, \bar{z}) = 0$ with $r(z, 0) \not\equiv 0$.*

(a) *Q_0 is contained in M , if $Q_{p_j} \cap \Delta_{\epsilon_0}^n$ is contained in M for a sequence $p_j \in M^*$ with $p_j \rightarrow 0$ and for a fixed $\epsilon_0 > 0$.*

(b) *Q'_0 is nonempty. The closure of each topological component of M^* containing 0 contains a component of Q'_0 provided $\dim M_s \leq 2n - 4$.*

(c) *If $\dim M_s < 2n - 4$ ($n > 2$), Q'_0 is irreducible and M^* has a unique connected component of which the closure contains 0 . Furthermore, the dimension of the singular locus of Q'_0 is less than $n - 2$, and M contains no germ of complex variety of pure dimension $n - 1$ at 0 other than Q'_0 .*

Proof. Since $r(z, 0) \not\equiv 0$, one can introduce new coordinates such that $r(z_1, 0, 0) = z_1^d$. Choose small ϵ, δ such that

$$(3.1) \quad \pi_w: Q_w \equiv Q_w \cap \Delta_\epsilon \times \Delta_\delta^{n-1} \rightarrow \Delta_\delta^{n-1}, \quad \text{for } w \in \Delta_\delta^n$$

is a proper d -to-1 branched covering, where π_w is the restriction of the projection $\pi: (z_1, z') \mapsto z'$ to Q_w . Let $\sigma_1, \dots, \sigma_d$ be the elementary symmetric functions on the symmetric power space $\mathbf{C}_{sym}^d := \mathbf{C}^d / \mathfrak{S}_d$, where \mathfrak{S}_d is the permutation group operating on \mathbf{C}^d . Then $\sigma = (\sigma_1, \dots, \sigma_d): \mathbf{C}_{sym}^d \rightarrow \mathbf{C}^d$ is a homeomorphism (see [14]). Counted with multiplicity, $\pi_w^{-1}(z')$ is a set of d points in $\Delta_\epsilon \times \Delta_\delta^{n-1}$, of which the first coordinate of these d points form a point $\varphi(z', \bar{w}) \in \mathbf{C}_{sym}^d$. Notice that the components of $\sigma \circ \varphi(z', \bar{w})$ are precisely the coefficients of the Weierstrass polynomial of $r(z, \bar{w})$ in z_1 , which depend on z', \bar{w} holomorphically. In particular,

$\varphi(z', \bar{p}_j) \rightarrow \varphi(z', 0)$ in \mathbf{C}_{sym}^d as $p_j \rightarrow 0$. This implies that $\pi_{p_j}^{-1}(z') \rightarrow \pi_0^{-1}(z')$ as $p_j \rightarrow 0$. More precisely, $\pi_0^{-1}(z') = \bigcap_{i \geq 1} \bigcup_{j \geq i} \overline{\pi_{p_j}^{-1}(z')}$.

For the proof of (a), assume that $Q_{p_j} \cap \Delta_{\epsilon_0}$ is contained in M . Then in the original coordinates, Q'_{p_j} is equal to $Q_{p_j} \cap \Delta_{\epsilon_0}$. In particular, $\pi_{p_j}^{-1}(z')$ is contained in M . Letting $p_j \rightarrow 0$ shows that $\pi_0^{-1}(z') \subset M$ for $z' \in \Delta_\delta^{n-1}$. Therefore, $Q_0 \cap \Delta_\epsilon \times \Delta_\delta^{n-1}$ is contained in M .

For the proof of (b), we take any sequence $p_j \in M^*$ with $p_j \rightarrow 0$. Fix z' and take a point $p'_j \in \pi_{p_j}^{-1}(z') \cap Q'_{p_j}$, where Q'_{p_j} is now the branch of $Q_{p_j} \cap \Delta_\epsilon \times \Delta_\delta^{n-1}$ contained in M . One may assume that p'_j is convergent as $j \rightarrow \infty$, of which the limit is in $\pi_0^{-1}(z')$. Hence, $\pi_0^{-1}(z') \cap M$ is nonempty for all $z' \in \Delta_\epsilon^{n-1}$. Therefore, at least one branch of Q_0 is contained in M . We now assume further that $\dim M_s \leq 2n - 4$ and K is a connected component of M^* with $0 \in \bar{K}$. By choosing all p_j in K , the above argument shows that there exists a branch Q' of Q_0 such that $Q' \cap \bar{K}$ is nonempty. Since the codimension of $M_s \cap Q'$ in Q' is at least 2, $Q' \setminus M_s$ is connected; in particular, it is contained in K . Therefore, \bar{K} contains Q' .

(c) follows from (b) and Lemma 3.1 immediately. \square

An immediate corollary of Lemma 3.2 (b) is the following.

COROLLARY 3.3. *Let M_1, M_2 be two germs of Levi-flat real analytic hypersurface at $0 \in \mathbf{C}^n$ with no common component of their regular points. Then the singular set of $M_1 \cup M_2$ is of dimension at least $2n - 4$.*

The following gives all possible (quadratic) tangent cones and $\dim M_s$ when M is the union of two smooth Levi-flat hypersurfaces. The notation is as in Table 2.1 in Section 2.

PROPOSITION 3.4. *Let M be a reducible Levi-flat real analytic hypersurface defined by $r = 0$, where r starts with nonvanishing quadratic terms. Then M_s is a real analytic variety in \mathbf{C}^n of codimension 2 and the quadratic tangent cone of M is equivalent to one of $Q_{0,2}$, $Q_{1,1}$, $Q_{1,2}^\lambda$, $Q_{2,2}$.*

Proof. Since M is reducible and r starts with nonvanishing quadratic form, Lemma 2.1 implies that r is reducible. Hence, M is the union of two smooth real hypersurfaces. Obviously the quadratic tangent cone of M , denoted by C , is equivalent to one of $Q_{0,2}$, $Q_{1,2}^\lambda$, $Q_{2,2}$, and $Q_{1,1}$, since r is reducible. All these quadrics, except $Q_{1,1}$, are the union of two transversal hyperplanes, and hence M is also the union of two transversal smooth hypersurfaces, from which the proposition follows. For $C = Q_{1,1}$, one may assume

$$M: \Re z_1 \cdot \Re(z_1 + h(z)) = 0,$$

where $h(z) = O(|z|^2) \not\equiv 0$ is a holomorphic function. M_s is then the intersection of $x_1 = 0$ with $\Re h(iy_1, z') = 0$. If $h(0, z') \equiv 0$, M_s contains $z_1 = 0$. Otherwise,

$\Re h(0, z') = 0$ defines a germ of Levi-flat hypersurface at $0 \in \mathbf{C}^{n-1}$ (see Proposition 5.1). In fact, fixing any small y_1 , $\Re h(iy_1, z') = 0$ defines a germ of Levi-flat hypersurface at $0 \in \mathbf{C}^{n-1}$. Therefore, $\Re h(iy_1, z') = 0$ is a germ of real analytic variety of codimension one in $\mathbf{R} \times \mathbf{C}^{n-1}$. The proof of the proposition is complete. \square

We now consider quadratic tangent cones of Levi-flat hypersurfaces which are of small dimension, a case left open in Section 2 above.

PROPOSITION 3.5. *Let M be a Levi-flat real analytic hypersurface defined by $r(z, \bar{z}) = q(z, \bar{z}) + O(|z|^3) = 0$, where q is a real quadratic form. Assume that $q = 0$ is a real analytic set C of dimension less than $2n - 1$. Then after a possible complex linear transformation, and up to a possible change of sign, q is equivalent to $z_1 \bar{z}_1 + \lambda z_1^2 + \lambda \bar{z}_1^2$, where $0 \leq \lambda < 1/2$.*

Proof. By a theorem of Fornaess (see [9], p. 114), there is a germ of complex hypersurface V satisfying $0 \in V \subset M$. Let $C(V)$ be the tangent cone of V at 0 (see [14]). Given $p \in C(V) \setminus \{0\}$, there exists a sequence $p_j \rightarrow 0$ such that $p_j \in V \setminus \{0\}$ and $p_j/|p_j| \rightarrow p/|p|$ as $j \rightarrow \infty$. Hence, we have $0 = r(p_j, \bar{p}_j)/|p_j|^2 \rightarrow q(p, \bar{p})/|p|^2$ as $j \rightarrow \infty$, i.e., $C(V) \subset C$. Since $C(V)$ is a complex hypersurface, we obtain that $\dim C = 2n - 2$, and q is a semi-definite real quadratic form of rank 2. Let us assume that $q \geq 0$. Then the eigenvalues of the Levi-form of q are nonnegative. It is clear that the Levi-form of q is not identically zero. On the other hand, the Levi-form cannot have more than one positive eigenvalue. Otherwise, the Levi-form is positive definite on a complex subvariety of V of positive dimension, which contradicts the fact that q vanishes on $C(V)$. Therefore, we may assume that $q(z, \bar{z}) = z_1 \bar{z}_1 + \Re A(z)$, where $A(z)$ is a complex quadratic form. For a fixed z_1 , q is a nonnegative pluriharmonic function on \mathbf{C}^{n-1} . Hence, q is independent of z_2, \dots, z_n . Now, it is easy to see that C is given by $z_1 = 0$, and q is equivalent to $z_1 \bar{z}_1 + \lambda z_1^2 + \lambda \bar{z}_1^2$ with $0 \leq \lambda < 1/2$. \square

We consider a more general Levi-flat real analytic hypersurface defined by

$$(3.2) \quad r(z, \bar{z}) = q(z, \bar{z}) + O(|z|^3) = 0, \quad \text{rank } q(z, 0) > 0.$$

As a use of Segre varieties, we shall prove the following result on the connectivity of the smooth locus of M . The result will however not be used in this paper.

PROPOSITION 3.6. *Let M be a real analytic hypersurface in \mathbf{C}^n defined by (3.2). Assume that $\dim M_s \leq 2n - 4$. Then M^* is connected.*

Proof. More precisely, we would like to prove that any neighborhood U of the origin contains an open subset U' with $0 \in U'$ such that $U' \cap M^*$ is connected. To this end note that Proposition 3.4 implies that r is irreducible. From Lemma 2.1 it follows that M is irreducible. Now a theorem of Bruhat and Cartan [4] says

that there exists U' such that $U' \cap M^* = U_1 \cup \cdots \cup U_k$, where U_j are connected open sets with $0 \in \overline{U_j}$. We need to show that $k = 1$.

One may assume that $q(z_1, 0, 0) = z_1^2$. With the notations in the proof of Lemma 3.2, the projection $\pi_w: Q_w \rightarrow \Delta_\epsilon^{n-1}$, given by (3.1), is 2-to-1 and proper. We also choose ϵ, δ so small that M_s is closed in $\Delta_\epsilon \times \Delta_\delta^{n-1}$.

Assume for contradiction that $M^* \equiv M^* \cap U'$ is not connected. Then Lemma 3.2 (b) says that M^* has exactly two components M', M'' , and Q_0 has two branches Q'_0, Q''_0 with $Q'_0 \subset \overline{M'}$ and $Q''_0 \subset \overline{M''}$. Take a sequence of points $p_j \rightarrow 0$ in $M' \setminus (Q_0 \cup M_s)$. In view of Lemma 3.2 (a), one may further assume that each Q_{p_j} is reducible. By Lemma 3.1, Q_{p_j} contains a unique branch Q'_{p_j} which is inside M' and contains p_j . Let Q'_0, Q''_0 , and Q'_{p_j} be the graphs of holomorphic functions f'_0, f''_0 , and $f'_j(z')$, respectively. Passing to a subsequence if necessary, we know that f'_j converges to f'_0 . Since $0 \in Q'_0 \cap Q''_0$, we can find a complex line, say the z_2 -axis, which goes through the origin of Δ_δ^{n-1} but is not contained in the complex variety $\pi(Q'_0 \cap Q''_0)$. By Rouché's theorem, $f'_j - f''_0$ has zeros on the z_2 -axis for large j , and these zeros can be arbitrarily close to the origin as $p_j \rightarrow 0$. Hence, $Q'_{p_j} \cap Q''_0$ is a nonempty complex variety of dimension $n-2$, and it contains points arbitrarily close to the origin as $p_j \rightarrow 0$. In particular, $\dim M_s$ is $2n-4$.

Next, we want to show that $Q'_{p_j} \cap Q''_0$ actually contains the origin for large j . Since $\dim M_s \leq 2n-4$, we may assume that $M_s \subset V_1 \cup \cdots \cup V_k$, where each V_i is a germ of irreducible real analytic variety of dimension at most $2n-4$ at the origin. Let V'_i be the set of points in V_i where V_i is a real analytic manifold of dimension $2n-4$. Then there is a neighborhood U_i of the origin such that $V'_i \cap U_i$ has only a finite number of connected components A_{ij} [4]. Furthermore, A_{ij} is either empty when $\dim V_i < 2n-4$, or $0 \in \overline{A_{ij}}$. Choose a large j such that $Q = Q'_{p_j} \cap Q''_0$ intersects all of U_1, \dots, U_k . It is clear that intersection of Q with one of A_{ij} 's has interior points in Q . Consequently, Q contains one of A_{ij} ; hence, $0 \in \overline{A_{ij}} \subset Q_{p_j}$. Notice the reality property of Segre varieties; namely, $z \in Q_w$ if and only if $w \in Q_z$. Hence, we have $p_j \in Q_0$, which is a contradiction. The proof of the proposition is complete. \square

We return to the main line of argument to show Theorem 1.1.

LEMMA 3.7. *Let M be defined as in (3.2). Suppose that one branch of Q_0 is not contained in M . Then there exist $\epsilon, \delta > 0$ such that for $w \in \Delta_\delta^n$, Q'_w is given by $z_1 = f(z', \overline{w})$, where f is holomorphic in $z' \in \Delta_\delta^{n-1}$ for each fixed $w \in \Delta_\delta^n \cap M^*$. Furthermore, $f(z', \overline{w})$ tends to $f(z')$ uniformly on Δ_δ^{n-1} as $w \rightarrow 0$ in M^* , where $z_1 = f(z')$ defines Q'_0 .*

Proof. Choose ϵ, δ as in the proof of Proposition 3.6 so that the projection π_w given by (3.1) is a 2-to-1 branched covering for $w \in \Delta_\delta^n$. Since Q_0 is not contained in M , then Lemma 3.2 (a) says that for $w \in M^* \cap \Delta_\delta^n$ close to the origin, Q_w consists of two branches, of which both are graphs of holomorphic

functions over Δ_δ^{n-1} . Let Q'_w be given by $z_1 = f(z', \bar{w})$, and let the other branch Q''_w of Q_w be given by $z_1 = g(z', \bar{w})$.

To see the uniform convergence of $f(\cdot, \bar{w})$, we first notice that all the partial derivatives of $f(\cdot, \bar{w})$ are uniformly bounded on each compact subset of Δ_δ^{n-1} . Hence, it suffices to show that $f(\cdot, \bar{w})$ is pointwise convergent on a dense subset of Δ_δ^{n-1} . Since Q''_0 is not contained in M , then $E = \pi(Q'_0 \cap Q''_0)$ is nowhere dense in Δ_δ^{n-1} . Fix $z' \in \Delta_\delta^{n-1} \setminus E$. Then $(g(z', 0), z')$ has a positive distance d to M . From the Rouché theorem (or the elementary symmetric functions argument in the proof of Lemma 3.2), it follows that $|f(z', w) - g(z', 0)| > d/2$ for w sufficiently close to the origin; hence, $|f(z', w) - f(z', 0)| \rightarrow 0$ and $|g(z', w) - g(z', 0)| \rightarrow 0$, as $w \rightarrow 0$. \square

LEMMA 3.8. *Let $p(z)$ be an irreducible germ of holomorphic function at 0. Assume that $r(z, \bar{z})$ is a real analytic function vanishing on $V: p = 0$. Then $r = ap + \bar{a}\bar{p}$ for some convergent power series $a(z, \bar{z})$.*

Proof. One may assume that $p(z)$ is a Weierstrass polynomial $z_1^d + \sum_{j=0}^{d-1} p_j(z')z_1^j$. Since p is reduced, there is a complex variety B in \mathbf{C}^{n-1} such that $p(z_1, z') = 0$ has d distinct roots for each fixed $z' \notin B$. Consider the complex variety

$$V^c \subset \mathbf{C}^n \times \mathbf{C}^n: p(z) = 0, \quad \bar{p}(w) = 0.$$

It is clear that V^c is irreducible and of codimension 2 in \mathbf{C}^{2n} . We identify V with the set $V^c \cap \{w = \bar{z}\}$. Then V^* is a totally real submanifold in $(V^c)^*$ of maximal dimension. Notice that $r(z, w)$ vanishes on V^* , hence also on V^c .

Applying the Weierstrass division theorem, one gets

$$r(z, w) = a_d(z, w)p(z) + \sum_{j=0}^{d-1} a_j(z', w)z_1^j,$$

and

$$a_j(z', w) = b_{j,d}(z', w)\bar{p}(w) + \sum_{k=0}^{d-1} b_{j,k}(z', w')w_1^k$$

for $0 \leq j \leq d-1$. On V^c , one has

$$(3.3) \quad \sum_{j,k=0}^{d-1} b_{j,k}(z', w')z_1^j w_1^k = 0.$$

Fix $(z', w') \notin B \times \mathbf{C}^{n-1} \cup \mathbf{C}^{n-1} \times \bar{B}$, where $\bar{B} = \{\bar{z}' \mid z' \in B\}$. Then there are d distinct zeros $z_{1,1}, \dots, z_{1,d}$ of $p(z_1, z')$, and d distinct zeros $w_{1,1}, \dots, w_{1,d}$ of $\bar{p}(w_1, w')$. In (3.3), replace z_1 by $z_{1,i}$ with a fixed i , and w_1 by $w_{1,1}, \dots, w_{1,d}$, respectively. This gives us d linear equations in terms of $\sum_j b_{j,k}(z', w')z_{1,i}^j$ for

$0 \leq k \leq d - 1$. The coefficient matrix of the d linear equations is a nonsingular Vandermonde matrix $(w_{1,\alpha}^{\beta-1})_{1 \leq \alpha, \beta \leq d}$. Thus, we get

$$\sum_{j=0}^{d-1} b_{j,k}(z', w') z_{1,i}^j = 0, \quad 0 \leq k \leq d - 1.$$

Now, fix k and vary i . The above are d linear equations in terms of $b_{0,k}(z', w'), \dots, b_{d-1,k}(z', w')$; one readily sees that $b_{j,k}(z', w') = 0$ for $0 \leq j, k \leq d - 1$. Therefore, we obtain a decomposition $r = a_0 p + b_0 \bar{p}$ for some convergent power series $a_0(z, \bar{z}), b_0(z, \bar{z})$. Replacing $a_0(z, \bar{z})$ by the average $(a_0(z, \bar{z}) + \bar{b}_0(\bar{z}, z))/2$ completes the proof of the lemma. \square

THEOREM 3.9. *Let M be a Levi-flat real analytic hypersurface defined by (3.2) with $\dim M_s \leq 2n - 3$. Assume further that Q_0 is not a double hypersurface. Then M is given by $\Re h(z) = 0$ with h a holomorphic function.*

Proof. Before we proceed to the details, we shall explain how the Segre varieties will be used in the proof. Roughly speaking, h is constructed as follows: We pick one branch of Q'_0 . Then take a curve $\gamma(t)$ in M^* which is transverse to that branch of Q'_0 at $\gamma(0)$. One would hope that for any point $z \in M$ close to the origin, one branch of Q_z intersects γ at a unique point $\gamma(t)$; this is achieved essentially due to the fact that Q_0 contains no component of multiplicity 2. Then, $-ti$ will be the value of h at z . Now the question becomes whether h extends to a holomorphic function defined in a neighborhood of 0. To deal with this problem, we shall substitute \bar{z} for $\bar{\gamma}(t)$ in the complexified function $r(z, \bar{z})$. Solving for t from $r(z, \bar{\gamma}(t)) = 0$ would yield $t = ih(z)$. We shall obtain the convergence of $h(z)$ by choosing the curve γ carefully to cope with the singularities of M .

The proof will be accomplished in two steps. The first step is to show that M contains Q_0 . The second step is to show the existence of the holomorphic function h stated in the theorem.

Step 1. Assume for contradiction that Q_0 is not contained in M . In this case, Q_0 consists of two smooth hypersurfaces, and only one branch Q'_0 is in M .

We have $r(z, 0) = q(z, 0) + O(|z|^3)$. Since $q(z, 0) \not\equiv 0$, then, by a parametric Morse lemma [2], one can find holomorphic coordinates such that $r(z, 0) = z_1^2 + \tilde{a}(z')$. Since Q_0 are two distinct smooth hypersurfaces, then $\tilde{a}(z') = \tilde{b}^2(z') \not\equiv 0$. Assume that $\{z_1 + i\tilde{b}(z') = 0\}$ is inside M . Replacing z_1 by $z_1 + i\tilde{b}(z')$ yields $r(z, 0) = z_1(z_1 + p(z'))$ with $p = -2i\tilde{b}$ and $Q'_0: z_1 = 0$. Since M contains $z_1 = 0$, by Lemma 3.8 M is now given by

$$M: \Re\{z_1(z_1 + a(z, \bar{z}))\} = 0.$$

It is convenient to decompose

$$a(z, \bar{z}) = p(z') + b(z', \bar{z}') + c(z, \bar{z})$$

with

$$p(z') \not\equiv 0, \quad b(z', 0) \equiv c(z, 0) \equiv 0, \quad c(0, z', 0, \bar{z}') \equiv 0.$$

Let k be the vanishing order of $p(z')$, and l the vanishing order of $p(z') + b(z', \bar{z}')$. By the assumptions, we have $1 \leq l \leq k < \infty$. Denote by p_k, b_l the homogeneous terms of p, b with degree k and l , respectively. Fix z'_0 such that $(0, z'_0) \in \mathcal{Q}'_0 \setminus M_s$, and such that

$$(3.4) \quad p_k(z'_0) \neq 0, \quad p_l(z'_0) + b_l(z'_0, \bar{z}'_0) \neq 0.$$

A linear dilation $(z_1, z') \rightarrow (\epsilon^{l+1} z_1, \epsilon z')$ transforms M into

$$(3.5) \quad M^\epsilon: r^\epsilon(z, \bar{z}) = \Re\{z_1[p(z', \epsilon) + b(z', \bar{z}', \epsilon) + \epsilon(z_1 + c(z, \bar{z}, \epsilon))]\} = 0$$

with $c(z, 0, \epsilon) \equiv c(0, z', 0, \bar{z}', \epsilon) \equiv 0$ and

$$(3.6) \quad p(z', 0) + b(z', \bar{z}', 0) = p_l(z') + b_l(z', \bar{z}').$$

Throughout the rest of the proof, we shall replace r by the equivalent r^ϵ , and M by the equivalent M^ϵ , unless stated otherwise.

We shall find a real curve in M^* given by

$$(3.7) \quad \gamma = \gamma^\epsilon: z_1 = i\mu(\epsilon)(t + i\alpha(t, \epsilon)), \quad z' = z'_0,$$

where $\alpha(0, \epsilon) = 0, \bar{\alpha}(t, \epsilon) = \alpha(t, \epsilon)$, and $\mu(\epsilon)$ is determined by

$$p(z'_0, \epsilon) + b(z'_0, \bar{z}'_0, \epsilon) = \rho(\epsilon)\bar{\mu}(\epsilon), \quad \rho(\epsilon) = |p(z'_0, \epsilon) + b(z'_0, \bar{z}'_0, \epsilon)|.$$

Substituting (3.7) into (3.5), one sees that γ is contained in M , if and only if

$$(3.8) \quad -\rho(\epsilon)\alpha(t) + \epsilon\Re\{z_1(z_1 + c(z_1, z'_0, \bar{z}'_0, \epsilon))\} = 0,$$

in which z_1 is replaced by $i\mu(\epsilon)(t + i\alpha(t))$. From (3.4), it follows that $\rho(\epsilon)$ is a real analytic function bounded from below by a positive constant. Using the implicit function theorem, we get a unique real analytic solution $\alpha = \alpha(t, \epsilon)$ with

$$(3.9) \quad \alpha(t, \epsilon) = O(t^2), \quad \alpha(t, 0) \equiv 0.$$

The Segre varieties $Q_{\gamma(t)}$ are given by $r(z, \bar{\gamma}(t)) = 0$, from which we want to solve for t as follows. Regard $r(z, \bar{\gamma}(t))$ as a power series in t, z , and ϵ . From (3.5), (3.7) and (3.9), we get the following expansion:

$$r(z, \bar{\gamma}(t)) = -i\bar{\mu}(\epsilon)\bar{p}(z'_0, \epsilon)t + O(|z_1| + |tz'| + t^2),$$

in which, in view of (3.4), the coefficient of t does not vanish. From the implicit function theorem, it follows that $r(z, \gamma(t)) = 0$ has a unique holomorphic solution $t = ih(z, \epsilon)$ such that $h = 0$ when $z_1 = 0$ and $|z'|$ is small. Therefore, the uniqueness of the solution implies that the value of $h(z, \epsilon)$ is uniquely determined under the condition $|h(z, \epsilon)| < \epsilon_0$ for $z \in \Delta_{\delta_0}^n$, where ϵ_0, δ_0 are sufficiently small positive numbers. Here we remark that $\Delta_{\delta_0}^n$ might be too small to contain the curve γ .

Next, we want to show that $ih(z, \epsilon)$ is real-valued on M . Notice that $\gamma(0) \in M^*$, and γ is transverse to Q'_0 . Hence, the union of $Q_{\gamma(t)}$ for $-\epsilon_0 < t < \epsilon_0$ contains a neighborhood D of $\gamma(0)$ in M^* . In view of Lemma 3.7, we can choose a possibly smaller δ_0 such that $(f(z'_0, \bar{z}), z'_0) \in D$ for $z \in \Delta_{\delta_0}^n$, where f is given in Lemma 3.7. In other words, for each $z \in M^* \cap \Delta_{\delta_0}^n$, there is a real t with $|t| < \epsilon_0$ such that Q'_z intersects γ at $\gamma(t)$. Notice the reality property of Segre varieties; namely, $z \in Q_w$ if and only if $w \in Q_z$. Also, passing through each point in M^* there is only one complex hypersurface in M . Hence, we obtain $z \in Q'_{\gamma(t)}$. Since $t = ih(z, \epsilon)$ is the unique solution to $r(z, \bar{\gamma}(t)) = 0$, then $ih(z, \epsilon) = t$ is real for $z \in M^* \cap \Delta_{\delta_0}^n$. Since $\dim M_s \leq 2n - 3$, by Proposition 3.4 r is irreducible. Now Lemma 2.1 says that r divides $\Re h(z, \epsilon)$. Thus, $h(z, \epsilon)$ starts with terms of order at least 2 for each fixed ϵ . Expand $r(z, \bar{\gamma}(t))$ as a power series in t and z . One first sees that in that expansion, the linear terms in z must be zero since the coefficient of t is nonzero and h starts with terms of order at least 2. The linear terms in t and the quadratic terms in z are given by

$$-i\bar{\mu}(\epsilon)\bar{p}(z'_0, \epsilon)t + (\epsilon + O(\epsilon^2))z_1^2 + z_1 \sum_{j \geq 2} b_{z_j}(0, \bar{z}'_0, \epsilon)z_j.$$

Thus, the quadratic form of $h(z, \epsilon)$ contains z_1^2 when $\epsilon \neq 0$; in particular, the order of $h(z, \epsilon)$ is 2. Fixing a small nonzero ϵ , we obtain that $\Re h(z, \epsilon) = u(z, \bar{z})r(z, \bar{z})$ with $u(0) \neq 0$. In particular, Q_0 , given by $h = 0$, is contained in M . This contradicts our assumption. Therefore, Q_0 is contained in M .

Notice that the above argument is valid if Q_0 is reducible and the set of points $z \in M^*$ with $Q_z \subset M$ contains an open subset U with $0 \in \bar{U}$. Assuming that Q_0 is reducible, we want to show such a set U always exists. Otherwise, choose a sequence $z^j \rightarrow 0$ in M^* such that $z^j \notin \cup_{i < j} Q_{z^i}$ and $Q_{z^j} = Q'_{z^j} \cup Q''_{z^j}$ with $Q''_{z^j} \not\subset M$. Without loss of generality, one may assume that $Q'_{z^j} \subset M$ approaches to one branch Q'_0 of Q_0 as $j \rightarrow \infty$. With the above set-up, one finds a real curve γ such that Q'_{z^j} intersects $\gamma(t)$ for some small real t as $z^j \rightarrow 0$. One also has a unique (complex-valued) solution $t = ih(z)$ to $r(z, \bar{\gamma}(t)) = 0$, where $h(z)$ is

holomorphic on $\Delta_{\delta_0}^n$. Now $h(z)$ is pure-imaginary on $Q'_{z_j} \cap \Delta_{\delta_0}^n$. Hence, $\Re h = 0$ contains infinitely many complex hypersurfaces $Q'_{z_j} \subset M$. Since $z^j \in Q'_{z_j}$ and $z^j \rightarrow 0$, the real analytic set $\{\Re h = 0\} \cap M$ has dimension $2n - 1$ at 0. Therefore, r divides $\Re h$, a contradiction. This gives us the conclusion of the theorem when Q_0 is reducible.

Step 2. We now consider the case that Q_0 is irreducible. As seen in the beginning of Step 1, one may change coordinates such that $r(z, 0) = z_1^2 + z_2^k + p(z')$ with

$$p(z') = O(|z'|^k), \quad p(z_2, 0) \equiv 0.$$

In view of Lemma 3.8, we rewrite

$$M: r(z, \bar{z}) = \Re\{(z_1^2 + z_2^k + p(z'))(1 + a(z, \bar{z}))\} = 0.$$

Rotating the z_1 and z_2 axes and dividing r by $|1 + a(0)|$, one may achieve $a(0) = 0$.

A linear transformation

$$z_1^* = \epsilon^k z_1, \quad z_2^* = \epsilon z_2, \quad z_j^* = \epsilon^2 z_j, \quad j > 2$$

transforms M into M^ϵ given by

$$(3.10) \quad r^\epsilon(z, \bar{z}) = \Re\{z_1^2 + z_2^k\} + H(z, \bar{z}, \epsilon) = 0, \quad \epsilon \neq 0,$$

where $H(z, \bar{z}, \epsilon)$ is a real analytic function in z, \bar{z}, ϵ with

$$(3.11) \quad H(z, \bar{z}, 0) \equiv 0.$$

Again, we shall denote r^ϵ by r , and M^ϵ by M .

We now parameterize the Segre varieties of M by a real curve

$$(3.12) \quad \gamma : z_1 = \sqrt{1 + it} + \alpha(t, \epsilon), \quad z_2 = i^{2/k}, \quad z_j = 0, \quad j > 2$$

with $\bar{\alpha}(t, \epsilon) = \alpha(t, \epsilon)$ and $\alpha(0, \epsilon) = \alpha(t, 0) = 0$. This amounts to solving for α in the equation

$$(3.13) \quad \alpha = E(t, \alpha), \quad E(t, \alpha) = -\frac{1}{2\Re\sqrt{1 + it}} \left(\alpha^2 + H(\gamma(t), \bar{\gamma}(t), \epsilon) \right).$$

Obviously, E is a convergent power series in t, α . $E(t, \alpha)$ is also real-valued when t, α are real. From (3.11), it follows that $E_\alpha = 0$ for $\epsilon, \alpha = 0$. By the implicit

function theorem, (3.13) has a real analytic solution $\alpha = \alpha(t, \epsilon)$ with

$$(3.14) \quad \alpha(t, 0) \equiv 0.$$

In (3.10), substitute (3.12) for \bar{z} . Then the Segre varieties $Q_{\gamma(t)}$ are given by

$$(3.15) \quad \begin{aligned} T(t, z, \epsilon) &= it - 2\sqrt{1-it}\alpha(t, \epsilon) - \alpha^2(t, \epsilon) \\ &\quad - z_1^2 - z_2^k - H(z, \bar{\gamma}(t, \epsilon), \epsilon) = 0. \end{aligned}$$

Now, (3.11) and (3.14) yield $T_t = i$ for $\epsilon = 0$. By the fixed-point theorem, (3.15) admits a unique holomorphic solution

$$(3.16) \quad t = ih(z, \epsilon)$$

on a domain

$$D_s = \{z \mid |z_j| < 2, 1 \leq j \leq n; |z_1^2 + z_2^k| < s\}$$

for a fixed small s . It is clear that when ϵ is small relative to s , $\gamma = \gamma^\epsilon \subset D_s$, and $Q_0 = \{r^\epsilon(z, 0) = 0\} \cap \Delta_\epsilon \times \Delta_\delta^{n-1} \subset D_s$.

Return to the Segre varieties $Q_{\gamma(t)}$ determined by (3.15). The solution (3.16) means that $h(z, \epsilon)$ is pure imaginary on each $Q_{\gamma(t)}$ for real t . It is clear that γ is not contained in Q_0 . Thus, $Q'_{\gamma(t)}$ sweep out an open subset of M , on which $h(z, \epsilon)$ is pure imaginary. For any small neighborhood $U \subset \mathbf{C}^n$ of 0, $Q'_{\gamma(t)}$ remains in M and intersects U as $t \rightarrow 0$. As in step one, we conclude by Lemma 2.1 that $\Re h(z, \epsilon) = u(z, \bar{z})r(z, \bar{z})$ near 0. Obviously, $\Re h(0, \epsilon) = 0$, and the origin is a critical point of $h(z, \epsilon)$. Notice that $h(z, 0) = z_1^2 + z_2^k$. Thus, the quadratic form of $h(z, \epsilon)$ is not identically zero for small ϵ . Therefore, u is nonvanishing near 0, which shows that M^ϵ is the zero set of $\Re h(z, \epsilon)$ for small $\epsilon \neq 0$. Since the original hypersurface M is biholomorphically equivalent to M^ϵ , this completes the proof of the theorem. \square

Now Theorem 1.1 follows from the following.

THEOREM 3.10. *Let M be a real analytic Levi-flat hypersurface defined by (3.2). Assume that the complex quadratic form $q(z, 0)$ is of rank $k \geq 2$, and the real quadratic form $q(z, \bar{z})$ of rank at least 3. Then the Levi-form of q vanishes, and M is equivalent to a real analytic hypersurface defined by $\Re\{z_1^2 + \cdots + z_k^2 + p(z_{k+1}, \dots, z_n)\} = 0$, where p is a holomorphic function starting with terms of order at least 3.*

Proof. Since the rank of $q(z, 0)$ is greater than 1, then $q(z, 0)$ is nondegenerate. This implies that $Q_0: q(z, 0) + O(|z|^3) = 0$ is not a double hypersurface. M_s is contained in the set defined by $r_{z_j}(z, \bar{z}) = 0$ for $1 \leq j \leq n$, which has codimension at least 3 in \mathbf{C}^n . Hence, $\dim M_s \leq 2n - 3$. Applying Theorem 3.9, we can find a

holomorphic function $h(z) = O(|z|^2)$ such that $r(z, \bar{z}) = u(z, \bar{z})\Re h(z)$. This implies that the Levi-form of q vanishes, and the rank of the quadratic form of $h(z)$ is k . A parametric Morse lemma [2] says that h is equivalent to $z_1^2 + \cdots + z_k^2 + p(z_{k+1}, \dots, z_n)$ for some holomorphic function p . This completes the proof of the theorem. \square

4. Rigidity of quadrics. The purpose of this section is to show the rigidity property of quadrics $\mathcal{Q}_{2,2}$, $\mathcal{Q}_{2,4}$, and $\mathcal{Q}_{1,2}^\lambda$. At the end of the section we summarize the relationship as we understand it up to now between irreducible Levi-flat hypersurfaces with quadratic singularities and their possible tangent cones.

The following proposition gives a sufficient condition for \mathcal{Q}_0 of a Levi-flat hypersurface to be completely degenerate, i.e. $\mathcal{Q}_0 = \mathbf{C}^n$. This proposition also shows that the invariant λ in Proposition 3.5 vanishes.

PROPOSITION 4.1. *Let M be a Levi-flat real analytic hypersurface in \mathbf{C}^n with defining function $r(z, \bar{z}) = q(z, \bar{z}) + O(|z|^3)$. Assume that the quadratic form q is positive definite on a 1-dimensional complex linear subspace of \mathbf{C}^n . Then $r(z, 0) \equiv 0$, and \mathcal{Q}'_z is a smooth complex hypersurface passing through the origin for $z \in M^*$. Moreover, one of the following occurs:*

- (a) *The rank of the Levi-form of r at 0 is 1, and M_s contains a complex variety of codimension 2.*
- (b) *The rank of the Levi-form of r at 0 is 2, and M_s is a complex submanifold of codimension 2.*

In the latter case, M can be transformed into a hypersurface defined by

$$(4.1) \quad \Re\{z_1\bar{z}_2(1 + a(z, \bar{z})) + z_1z_2b(z, \bar{z}')\} = 0,$$

where $z'' = (z_3, \dots, z_n)$, and a, b are power series satisfying

$$(4.2) \quad a(z, 0) \equiv a(0, \bar{z}) \equiv b(z, 0) \equiv 0.$$

Proof. Without loss of generality, we assume that q is positive definite on the z_1 -axis. Then a linear change of coordinates gives us

$$q(z_1, 0, \bar{z}_1, 0) = z_1\bar{z}_1 + \lambda(z_1^2 + \bar{z}_1^2) + O(|z_1|^3), \quad 0 \leq \lambda < 1/2.$$

Obviously,

$$(4.3) \quad r(z, \bar{z}) \geq c_1|z_1|^2$$

for $|z_1| > |z'|/c_2$, where c_1, c_2 are small positive numbers. Hence, for small δ , $\pi: M \rightarrow \Delta_\delta^{n-1}$ is a proper mapping, where π is the projection $(z_1, z') \rightarrow z'$. Thus,

$$(4.4) \quad \pi_w: \mathcal{Q}'_w \equiv \mathcal{Q}'_w \cap (\mathbf{C} \times \Delta_\delta^{n-1}) \rightarrow \Delta_\delta^{n-1}$$

is a branched covering. In particular, $Q'_w \cap \{z' = 0\}$ is nonempty; hence, it follows from (4.3) that $0 \in Q'_w$. This shows that Q'_0 contains infinitely many complex hypersurfaces, i.e., $r(z, 0) \equiv 0$.

We now know that

$$(4.5) \quad r(z, \bar{z}) = \Re\{z_1 \bar{a}_1(\bar{z})\} + \Re \sum_{j \geq 2} z_j \bar{a}_j(\bar{z}') + \tilde{b}(z, \bar{z}),$$

where $a_1(z_1, 0) = z_1 + O(|z_1|^2)$, and $\tilde{b}(z, \bar{z})$ contains no term of the form $z^\alpha \bar{z}^\beta$ with $|\alpha| \leq 1$ or $|\beta| \leq 1$. In particular, we have

$$(4.6) \quad r_{z_1}(0, \bar{w}) = \bar{a}_1(\bar{w}) + \bar{w}_1 \neq 0.$$

Since all branches of Q'_w go through the origin, then Q'_w can have only one branch, and it is the graph of a holomorphic function when $w \in M \setminus \{w_1 + a_1(w) = 0\}$. Another consequence of (4.6) is that

$$(4.7) \quad M_s \subset \{z_1 + a_1(z) = 0\}.$$

Note that the latter is transverse to the z_1 -axis on which q is positive definite. In particular, the singular set of M is exactly a complex variety of \mathbf{C}^n of codimension 2, if there exist two complex lines so that q is definite on each of them.

For the proof of (b), we now assume that the Levi-form of r at 0 is of rank $k > 1$. In view of (a), we may further assume that $q(z, \bar{z}) = z_1 \bar{z}_1 \pm \dots \pm z_k \bar{z}_k$. This means that $a_j(z) = \pm z_j + O(|z|^2)$ for $2 \leq j \leq k$, where a_j is given by (4.5). We have

$$\nabla_z r(z, \bar{w})|_{z=0} = (\bar{w}_1, \dots, \pm \bar{w}_k, 0, \dots, 0) + O(|w|^2).$$

Obviously, one can find two points on M^* such that the corresponding two Segre varieties are transverse to each other at the origin. By mapping these two Segre varieties onto $z_1 = 0$ and $z_2 = 0$ respectively, we see that r vanishes on $z_1 = 0$, and on $z_2 = 0$. Hence,

$$(4.8) \quad r(z, \bar{z}) = \Re\{z_1 \bar{z}_2 p(z, \bar{z}) + z_1 z_2 b(z, \bar{z}'')\}, \quad p(0) = 1.$$

This shows that $k = 2$ and M_s is a complex variety of dimension $n - 2$.

To eliminate the terms of $p(z, \bar{z}) - 1$ which are purely holomorphic in z , or in \bar{z} , we shall seek a holomorphic transformation

$$\varphi: z'_1 = z_1 f_1(z), \quad z'_2 = z_2 f_2(z), \quad z'_j = z_j, \quad j > 2.$$

We shall restrict ourselves to $f_1(0) = f_2(0) = 1$ and then solve the equation

$$f_1 p(z_1 f_1, z_2 f_2, z'', 0) = 1, \quad f_2 \bar{p}(0, z_1 f_1, z_2 f_2, z'') = 1.$$

By the implicit function theorem, there is a unique solution (f_1, f_2) with $f_1(0) = f_2(0) = 1$. Now, $\varphi^{-1}(M)$ is given by (4.1) and (4.2). \square

To continue the proof of Theorem 1.2, we now assume that M is a Levi-flat hypersurface defined by (4.1) and (4.2). We shall prove that M is actually the complex cone $\Re\{z_1 \bar{z}_2\} = 0$ in \mathbf{C}^n .

Without loss of generality, we may assume that a, b in (4.1) and (4.2) are small functions defined on $|z| < 2$. Let γ be the intersection of M with the complex line $z_1 = 1, z_j = 0$ for $j > 1$. We shall seek a parameterization of γ as follows

$$(4.9) \quad \gamma: z_1 = 1, \quad z_2 = it + \alpha(t), \quad z_j = 0, \quad j \geq 2,$$

where $\alpha(t)$ is real-valued for real t . Substituting (4.9) into (4.1), we see that $\alpha = \alpha(t)$ must satisfy

$$(4.10) \quad \alpha + \Re\{(it + \alpha)a(1, it + \alpha, 0, 1, -it + \alpha, 0)\} = 0.$$

By the implicit function theorem, there is a unique solution $\alpha = \alpha(t)$ to (4.10) with $\alpha(0) = 0$.

The Segre variety $\mathcal{Q}_{\gamma(t)}$ is implicitly defined by

$$(4.11) \quad z_2 (1 + \bar{a}(\bar{\gamma}(t), z)) + z_1 (-it + \alpha(t))(1 + a(z, \bar{\gamma}(t))) \\ + (-it + \alpha(t))\bar{b}(\bar{\gamma}(t), z'') = 0.$$

Using the implicit function theorem, we solve (4.11) for $z_2 = h(t, z_1, z'')$, where h is a convergent power series in t, z_1 , and z'' . It is clear that

$$h(t, 0, 0) \equiv 0.$$

Expand

$$h(t, z_1, z'') = \sum_{k=0}^{\infty} h_k(t, z'') z_1^k.$$

In (4.1), substitute $h(t, z_1, z'')$ for z_2 . This gives us $r(z_1, h, z'', \bar{z}_1, \bar{h}, \bar{z}'')$, which, as a power series in $z_1, z'', \bar{z}_1, \bar{z}''$ and t , is identically zero. In view of (4.2), terms of the form $\bar{z}_1(z_1, z'')^\alpha t^j$ in $r(z_1, h, z'', \bar{z}_1, \bar{h}, \bar{z}'')$ give us

$$z_1 \overline{h_{z_1}(t, 0, 0)} + h(t, z_1, z'') \equiv 0.$$

In particular, $h(t, z_1, z'')$ does not depend on z'' , and

$$(4.12) \quad h_0(t, z'') \equiv \Re h_1(t, 0) \equiv 0, \quad h_j(t, z'') \equiv 0, \quad j \geq 2.$$

To compute $h_1(t, 0)$, we set $z'' = 0$ in (4.11) and collect terms which are linear in z_1 and z_2 , which yields

$$h_1(t, 0) = \frac{1 + a(0, \bar{\gamma}(t))}{1 + \bar{a}(\bar{\gamma}(t), 0)}(it - \alpha(t)).$$

Thus, we have

$$h_{1,t}(0, 0) = i - \alpha'(0) \neq 0.$$

Combining this with (4.12), we obtain

$$Q_{\gamma(t)}: z_2 = i m(t) z_1,$$

where $m(t)$ is a real power series with $m'(0) \neq 0$. On the other hand, $z_2 = i m(t) z_1$ is also contained in the complex cone $\Re\{z_1 \bar{z}_2\} = 0$. Therefore, M coincides with a portion of the complex cone. Since M and the complex cone are irreducible, then M is actually the complex cone $\Re\{z_1 \bar{z}_2\} = 0$. The proof of Theorem 1.2 is complete.

We now want to show the rigidity of the degenerate quadric $Q_{2,2}$. Consider a real analytic hypersurface in \mathbf{C}^n given by

$$(4.13) \quad r(z, \bar{z}) = (z_1 + \bar{z}_1)(z_2 + \bar{z}_2) + H(z, \bar{z}) = 0,$$

where $H(z, \bar{z}) = O(|z|^3)$ is a real analytic function.

THEOREM 4.2. *Let M be a Levi-flat real analytic hypersurface in \mathbf{C}^n defined by (4.13). Then M is biholomorphically equivalent to $Q_{2,2}$.*

Proof. If Q_0 is irreducible, then it follows from Lemma 3.2 (b) that Q_0 is contained in M . Now, Lemma 3.8 implies that the Levi-form of r is zero, which is a contradiction. Therefore, Q_0 is reducible. It is clear that Q_0 consists of two smooth hypersurfaces intersecting transversally at the origin. By (b) of Lemma 3.2, one of the hypersurfaces is contained in M . Change the coordinates so that Q_0 consists of $z_1 = 0$ and $z_2 = 0$ with the latter being contained in M . This means that M is given by

$$(z_1 + \bar{z}_1)(z_2 + \bar{z}_2) + H(z, \bar{z}) = 0$$

with

$$(4.14) \quad H(z, \bar{z})|_{z_2=0} = 0.$$

We may assume that H is a small function defined on $|z| < 3$.

Consider the parameterization

$$\gamma: z_1 = 1, \quad z_2 = it + \alpha(t), \quad z_j = 0, \quad j \geq 3$$

with $\alpha = \alpha(t)$ satisfying

$$4\alpha(t) + H(\gamma(t), \bar{\gamma}(t)) = 0, \quad \bar{\alpha}(t) = \alpha(t).$$

Since H is a small function vanishing for $z_2 = 0$, there exists a unique solution α with $\alpha(0) = 0$ and $\alpha'(0)$ small. We now consider parameterized branches $\mathcal{Q}'_{\gamma(t)}$ determined by

$$(4.15) \quad (1 + z_1)(z_2 - it + \alpha(t)) + H(z, \bar{\gamma}(t)) = 0.$$

Rewrite the above equation in the form

$$it - \alpha(t) = z_2 + \frac{H(z, \bar{\gamma}(t))}{1 + z_1},$$

which, by the fixed-point theorem, has a unique solution

$$t = ih(z), \quad \text{for } |z_1 - 1| < 3/2, \quad |z'| < 1.$$

From (4.14), one sees that $h|_{z_2=0} \equiv 0$. Hence,

$$h(z) = cz_2(1 + O(|z|)), \quad c \neq 0.$$

In particular, $\mathcal{Q}'_{\gamma(t)} \subset \{|z_1 - 1| < 3/2\} \times \{|z'| < 1\}$ is given by

$$z_2 = -it/c + O(|(t, z_1, z'')|^2),$$

where t is small and real, $|z_1 - 1| < 3/2$, and $|z''| < 1$. Obviously, the above expression shows that as part of M , $\mathcal{Q}'_{\gamma(t)}$ sweep out a smooth hypersurface M' containing the origin of \mathbf{C}^n . Hence, r is reducible. Write $r = r_1 r_2$, where r_1, r_2 are real-valued analytic functions. Since M is Levi-flat, then each r_j defines a smooth Levi-flat real analytic hypersurface. Therefore, Cartan's theorem says that there are holomorphic functions φ_j such that $r_j = u_j \Re \varphi_j$, $\varphi_j(0) = 0$. By the uniqueness of factorization for quadratic forms, one may assume that $r_j(z, \bar{z}) = z_j + \bar{z}_j + O(|z|^2)$. Hence, $\varphi_j(z) = c_j z_j + O(|z|^2)$ for some nonvanishing real constants c_j ; the inverse of $z \rightarrow (\varphi_1(z), \varphi_2(z), z'')$ then transforms M into $\mathcal{Q}_{2,2}$. The proof of Theorem 4.2 is complete. \square

As a consequence of Theorem 3.9, we know that there is no Levi-flat real analytic hypersurface M with $\dim M_s \leq 2n - 3$ such that M is a higher order perturbation of $\mathcal{Q}_{1,2}^\lambda$ for $0 < \lambda \leq 1$, provided \mathcal{Q}_0 is not a double hypersurface. We now turn to the case that \mathcal{Q}_0 is a double hypersurface.

THEOREM 4.3. *Let M be a Levi-flat real analytic hypersurface in \mathbf{C}^n with a defining function*

$$r(z, \bar{z}) = z_1^2 + 2\lambda z_1 \bar{z}_1 + \bar{z}_1^2 + O(|z|^3), \quad 0 \leq \lambda < 1.$$

Assume that the Segre variety \mathcal{Q}_0 of M is a double hypersurface. Then the dimension of M_s is at least $2n - 3$.

Proof. We will seek a contradiction under the hypothesis that M is smooth, or $\dim M_s \leq 2n - 4$. By the assumption, we have

$$r(z, 0) = (z_1 + O(|z|^2))^2.$$

Hence, one may find new coordinates such that $r(z, 0) = z_1^2$, while the quadratic form of r remains unchanged. M is then given by

$$M: r(z, \bar{z}) = \Re\{z_1(z_1 + \lambda \bar{z}_1 + a(z, \bar{z}))\} = 0, \quad a(z, 0) \equiv 0,$$

where $a(z, \bar{z}) = O(|z|^2)$. The intersection of $q(z, \bar{z}) = 0$ with the z_1 -axis consists of two real lines, of which one is parameterized by

$$z_1 = \mu t, \quad z' = 0$$

with

$$(4.16) \quad \mu^2 = -\lambda + i\sqrt{1 - \lambda^2}, \quad \sqrt{1 - \lambda^2} > 0, \quad \Re \mu > 0.$$

We need to find a real analytic curve in M of the form

$$\gamma: z_1 = \mu t(1 + i\alpha(t)), \quad z' = 0, \quad \bar{\alpha}(t) = \alpha(t) = O(|t|).$$

This leads to the equation

$$(4.17) \quad \frac{1}{t^2} r(\gamma(t), \bar{\gamma}(t)) = i(\mu^2 - \bar{\mu}^2)\alpha(t) + \frac{1}{t^2} \Re\{\gamma_1(t)a(\gamma(t), \bar{\gamma}(t))\} = 0.$$

Obviously, $r(\gamma(t), \bar{\gamma}(t))/t^2 \equiv F$ is a real analytic function in t, α with

$$F_\alpha(0, 0) = i(\mu^2 - \bar{\mu}^2) \neq 0.$$

By the implicit function theorem, (4.17) has a unique real analytic solution α . Furthermore,

$$r_{z_1}(\gamma(t), \bar{\gamma}(t)) = (2\mu + 2\lambda\bar{\mu})t + O(t^2).$$

Hence, $\gamma \setminus \{0\}$ is contained in M^* .

Notice that

$$(4.18) \quad r(z, \bar{\gamma}(t))|_{t=0} = z_1^2.$$

The Weierstrass preparation theorem then gives us

$$(4.19) \quad \begin{aligned} r(z, \bar{\gamma}(t)) &= z_1^2 + 2\lambda\bar{\mu}t(1 - i\alpha(t))z_1 + \bar{\mu}^2t^2(1 - i\alpha(t))^2 \\ &\quad + z_1a(z, \bar{\gamma}(t)) + \bar{\mu}t(1 - i\alpha(t))\bar{a}(\bar{\gamma}(t), z) \\ &= u(z, t)(z_1^2 + 2b(z', t)z_1 + c(z', t)). \end{aligned}$$

We need to compute the discriminant $\Delta = b^2 - c$ of the Weierstrass polynomial. From (4.18), it is clear that

$$(4.20) \quad b(z', t) = tb_1(z', t), \quad c(z', t) = tc_1(z', t), \quad u(z, 0) = 1$$

and

$$(4.21) \quad \Delta(z', t) = t^2b_1^2(z', t) - tc_1(z', t) \equiv t(td_1(z', t) - d_0(z'))$$

with $d_0(z') = c_1(z', 0)$. Setting $z_1 = 0$ in (4.19) yields

$$\bar{\mu}^2t^2(1 - i\alpha(t))^2 + \bar{\mu}t(1 - i\alpha(t))\bar{a}(\bar{\gamma}(t), 0, z') = u(0, z', t)c_1(z', t)t.$$

Expanding both sides as power series in t , the linear terms give us

$$(4.22) \quad \bar{a}(0, 0, z') = \mu u(0, z', 0)d_0(z').$$

In particular,

$$(4.23) \quad c_1(0) = 0.$$

Expanding (4.19) as power series in t, z_1, z' and collecting the coefficients of z_1t only, one has

$$2\lambda\bar{\mu} = 2b_1(0)u(0) + c_1(0)u_{z_1}(0).$$

Now, it follows from (4.20) and (4.23) that

$$b_1(0) = \lambda\bar{\mu}.$$

Collecting the coefficients of the term t^2 from (4.19), we get

$$\bar{\mu}^2 = c_1(0)u_t(0) + u(0)c_{1,t}(0) = c_{1,t}(0),$$

where the last identity is obtained from (4.23). Thus, we obtain

$$(4.24) \quad d_1(0) = b_1^2(0) - c_{1,t}(0) = (\lambda^2 - 1)\bar{\mu}^2 \neq 0.$$

Next, we want to show that Δ has a square root $\sqrt{\Delta}(z', t)$, which is a convergent power series near $(z', t) = 0$. From (4.21) and (4.24), it follows that $\Delta(z', t)$ has a square root if and only if $d_0(z') \equiv 0 \equiv a(0, 0, z')$ near $(z', t) = 0$. Assume for contradiction that d_0 is not identically zero. Note that

$$r(z, \bar{\gamma}(t))|_{z_1=0} = \bar{\mu}^2 t^2 (1 - i\alpha(t))^2 + \bar{\mu} t (1 - i\alpha(t)) \bar{a}(\bar{\gamma}(t), 0, z')$$

has the expansion

$$t \left[\bar{\mu} \bar{a}(0, 0, z') + (\bar{\mu}^2 + O(|z'|))t + O(t^2) \right] \equiv tf(z', t).$$

Since $\bar{a}(0, 0, z')$ is not identically zero, one can use Rouché's theorem to verify that $f(z', t)$, and hence $r(0, z', \bar{\gamma}(t))$, vanishes at a point z' near 0 whenever t is small. Therefore, $Q_{\gamma(t)}$ intersects with Q_0 for all small t . Let $N_t \subset \mathbf{C}^{n-1}$ be the complex variety defined by $f(z', t) = 0$, and N the union of N_t . Note that $f_t(0) = \bar{\mu}^2 \neq 0$. Hence, for a fixed $z'_0 \in N_{t_0}$ with $|t_0|, |z'_0|$ small, the equation $f(z', t) = 0$ can be solved for $t = t_0 + h(z')$ with h a holomorphic function defined near z'_0 . Since $\bar{a}(0, 0, z') \not\equiv 0$, h is not constant. Thus, N contains a Levi-flat hypersurface of \mathbf{C}^{n-1} defined by $\Im h(z') = 0$. Notice that $Q'_{\gamma(t)}$ can only intersect with Q_0 at singular points of M . Denote by T the set of real numbers t such that $Q_{\gamma(t)}$ is contained in M . Then T has no interior points, since $\dim M_s < 2n - 3 = \dim N$. Now for $t \notin T$, $Q_{\gamma(t)}$ is reducible with one branch in M and another branch not fully in M . For $t \in T$, we choose a sequence of $t_j \notin T$ with $t_j \rightarrow t$. Since $Q_{\gamma(t)}$ is a 2-to-1 branched covering over a fixed domain $D \subset \mathbf{C}^{n-1}$, one can represent $Q'_{\gamma(t_j)}$ as the graph of a holomorphic function $f_j(z')$. By passing to a subsequence, one may assume that the sequence $f_j(z')$ converges to a holomorphic function $f_0(z')$. The graph $z_1 = f_0(z')$ remains inside M and contains $\gamma(t)$, so it is one branch of $Q_{\gamma(t)}$. This shows that either $Q_{\gamma(t)} = Q'_{\gamma(t)}$ as a set, or $Q_{\gamma(t)}$ is reducible with two branches.

Without loss of generality, we may assume that $d_0(z_2, 0) \neq 0$. Using (4.24), it is easy to verify that for small $t \neq 0$, all zeros of $d_1(z_2, 0)t - d_0(z_2, 0)$ near $z_2 = 0$ are simple. Hence, for each fixed small $t \neq 0$, there exists z'_0 such that $\Delta(z'_0, t) = 0$, but $\nabla_{z'} \Delta(z'_0, t) \neq 0$. Clearly, $(-b(z'_0, t), z'_0)$ is a smooth branch point of the branched covering $Q_{\gamma(t)}$ over D . In particular, all $Q_{\gamma(t)}$ for small t are irreducible, which contradicts our earlier conclusion.

We now know that $\sqrt{\Delta}(z', t)$ is a convergent power series. We take the root with $\sqrt{\Delta}(z', t) = -i\bar{\mu}\sqrt{1 - \lambda^2}(t + O(t^2))$. Next, we want to show that the branch of $\mathcal{Q}_{\gamma(t)}$ which contains $\gamma(t)$ is given by

$$(4.25) \quad z_1 + b(z', t) + \sqrt{\Delta}(z', t) = 0.$$

Equivalently, we need to show that $\gamma(t)$ is not on the other branch, i.e.,

$$\gamma_1(t) + b(0, t) - \sqrt{\Delta}(0, t) \neq 0$$

for $t \neq 0$. A simple computation shows that

$$\gamma_1(t) + b(0, t) - \sqrt{\Delta}(0, t) = \bar{\mu}(\mu^2 + \lambda + i\sqrt{1 - \lambda^2})t^2 + O(t^3).$$

From (4.16), we get $\mu^2 + \lambda + i\sqrt{1 - \lambda^2} = 2i\sqrt{1 - \lambda^2} \neq 0$, i.e., (4.25) holds.

Finally, for a small neighborhood D of the origin of $\mathbf{R} \times \mathbf{C}^{n-1}$, we define a real analytic CR mapping $F: D \rightarrow \mathbf{C}^n$ by

$$F(t, z') = (-b(z', t) - \sqrt{\Delta}(z', t), z').$$

From (4.25), it is clear that $M_1 = F(D)$ is contained in M and $F(t, 0) = \gamma(t)$, $F(0, z') = z'$. Therefore, F is a real analytic embedding and M_1 is a smooth Levi-flat real analytic hypersurface. This means that r_1 divides r , where r_1 is a defining function of M_1 with $dr_1(0) \neq 0$. Write $r = r_1 r_2$. Then $r_2 = 0$ defines another smooth Levi-flat real analytic hypersurface M_2 . Obviously, $r_1(z, 0)r_2(z, 0) = z_1^2$; in particular, each M_j contains \mathcal{Q}_0 . Take a point $z_0 \in \mathcal{Q}_0 \setminus M_s$. Since M_j, M are smooth Levi-flat hypersurfaces near z_0 with $M_j \subset M$, then each M_j coincides with M near z_0 ; in particular, $M_1 = M_2$. By Cartan's theorem, $r_j = u_j \Re h$, where h is a holomorphic function with $h(0) = 0$. Thus, $q(z, \bar{z}) = u_j(0)(\Re h_1)^2$, where h_1 is the linear part of h . Obviously, this contradicts the assumption $0 \leq \lambda < 1$. The proof of Theorem 4.3 is complete. \square

Recall that Proposition 3.4 gives all possible quadratic tangent cones to a reducible Levi-flat hypersurface. Let us now summarize what we have found about possible quadratic tangent cones to an irreducible Levi-flat hypersurface. Let M be such a (nonsmooth) Levi-flat hypersurface, and C the quadratic tangent cone of M . Note that by Propositions 2.4, 3.5, and 4.1, one may assume that C is $z_1 \bar{z}_1 = 0$ or one of the quadrics in section 2. Now, one of the following holds:

- (a) $C = \mathcal{Q}_{0,2k}$ ($k \geq 2$) and M_s is a complex variety in \mathbf{C}^n of codimension $\geq k$.
- (b) C and the range of $d = \dim M_s$ are given by Table 4.1.

Note that $x_1^2 + y_1^2 = 0$ is a Levi-flat hypersurface for which the tangent cone is $\mathcal{Q}_{1,1}$ and the singular set is of dimension $2n - 2$. One can construct other examples of Levi-flat hypersurfaces M with $\dim M_s$ described in (a) and (b) by pulling

Table 4.1. Quadratic tangent cones.

C	d even			d odd	
	$\leq 2n - 6$	$2n - 4$	$2n - 2$	$\leq 2n - 5$	$2n - 3$
$\mathcal{Q}_{0,2}$	✓	✓	*	×	**
$\mathcal{Q}_{1,1}$	**	**	✓	**	**
$\mathcal{Q}_{1,2}^\lambda$	×	×	*	×	**
$\mathcal{Q}_{2,4}$	×	✓	×	×	×
$z_1 \bar{z}_1 = 0$	×	✓	*	×	*

✓ = existence × = non-existence *, ** = unknown, \mathcal{Q}_0 a double hypersurface if ** occurs.

back the real cone (1.1) or the complex cone (1.2) through suitable holomorphic mappings. For instance, the Levi-flat hypersurfaces with $C = \mathcal{Q}_{0,2k}$ ($k > 1$) and $\dim M_s = d$ can be constructed by pulling back (1.1) through $(z_1, \dots, z_n) \rightarrow (z_1, \dots, z_k, z_{k+1}^2, \dots, z_{n-d/2}^2, 0)$. The remaining statements in (a) and (b) are the content of Theorems 3.9–10, 4.2–3, and Proposition 4.1. The details are left to the reader.

5. Levi-flat hypersurfaces and meromorphic functions. In this section, we shall discuss the singularity of Levi-flat hypersurfaces arising from meromorphic functions and then give a sufficient condition for Levi-flat hypersurfaces to be equivalent to real algebraic hypersurfaces.

Recall that a germ of meromorphic function k is given by $k = f/g$, where f, g are two germs of holomorphic function at $0 \in \mathbf{C}^n$ which are relatively prime. We shall define $\mathcal{L}_k: \Re k(z) = 0$ by the real analytic set $\Re\{f(z)\bar{g}(\bar{z})\} = 0$. We shall also denote by $C_{f/g}$ the complex variety defined by $fdg - gdf = 0$. Then on each component of $C_{f/g}$, f/g is a constant meromorphic function, i.e., $g \equiv 0$, or $f - cg \equiv 0$ for some constant c . We shall denote by $C'_{f/g}$ the union of the indeterminacy variety $I_k = \{f = g = 0\}$ with all the components of $C_{f/g}$, on which f/g is pure imaginary. We remark that C_f is precisely the critical variety of the holomorphic function f . As germs at 0, one has $C'_f = C_f \subset \{f = f(0)\}$.

PROPOSITION 5.1. *Let $k = f/g$ be a germ of meromorphic function with $f(0) = 0$ or $g(0) = 0$. Then \mathcal{L}_k is a real analytic Levi-flat hypersurface of which the singular set is the complex variety C'_k .*

Proof. We first need to show that \mathcal{L}_k is of dimension $2n - 1$. Let V be the union of the hypersurfaces $f = 0$ and $g = 0$. Put $V = V^* \cup V_s$, where V_s is the singular locus of V . Take any point $p \in V^*$. One may further assume that $p = 0$, $f(0) = 0$, and $g(0) \neq 0$. Choose local coordinates so that

$$(5.1) \quad k(z) = z_1^m.$$

Near $p = 0$, \mathcal{L}_k is given by $z_1^m + \bar{z}_1^m = 0$, which is a real hyperplane when $m = 1$, or the union of m real hyperplanes with $\{z_1 = 0\}$ being contained in $(\mathcal{L}_k)_s$ when $m > 1$. Thus, $\dim \mathcal{L}_k = 2n - 1$. The above arguments also show that $\mathcal{L}_k \setminus C_k$ is smooth and Levi-flat. Since the real dimension of C_k is less than $2n - 1$, \mathcal{L}_k is Levi-flat.

One sees that $p \in (\mathcal{L}_k)_s \setminus I_k$ if and only if either k or $1/k$ is holomorphic near p such that p is a critical point for which the critical value is pure imaginary. Therefore, $(\mathcal{L}_k)_s \setminus I_k$ equals $C'_k \setminus I_k$. Since both $(\mathcal{L}_k)_s$ and C'_k contain I_k , this completes the proof of the proposition. \square

COROLLARY 5.2. *Let M be as in Theorem 3.9. Assume further that $\dim M_s \leq 1$ if $n \geq 3$. M is biholomorphically equivalent to $\Re h(z) = 0$, where h is a polynomial with isolated singularity at 0.*

Proof. From Theorem 3.9, it follows that M is defined by $\Re h = 0$ with h a holomorphic function. We also know that as a germ at 0 the singular locus M_s is precisely the critical variety of h . Since $\dim M_s \leq 1$, then the singular locus of $h = 0$ is isolated. By a theorem of Arnol'd [1] and Tougeron [12], h is holomorphically equivalent to a finite jet of h . \square

We remark that not every irreducible real analytic Levi-flat hypersurface can be defined by the real part of a meromorphic function. To see this, we need the following.

LEMMA 5.3. *Let M be defined by $\Re(f/g) = 0$ with $f(0) = g(0) = 0$. Then for all values of $c \in \mathbf{R}$, the subvariety $f/g = ic \subset M$ passes through each point of the indeterminacy locus of f/g , and for an open set of $z \in M^*$, the irreducible component of Q_z passing through z also passes through 0.*

Proof. Indeed, for the first statement, we can assume without loss of generality that the dimension $n = 2$, and that the indeterminacy locus of f/g is just the origin. We can blow-up \mathbf{C}^2 suitably so that we obtain a proper modification

$$\pi : \hat{\mathbf{C}}^2 \rightarrow \mathbf{C}^2$$

with $\pi^{-1}(0) := E$ a (connected) union of copies of \mathbf{P}^1 such that the function $F := \pi^*(f/g)$ extends holomorphically across E . On one of the irreducible components, say E_1 , of E the map F is a finite ramified covering onto \mathbf{P}^1 . Let $\gamma \subset E_1$ be the real analytic curve $F^{-1}(\{\Re z = 0 \cup \infty\} \subset \mathbf{P}^1)$, and let ϕ be the map F restricted to γ . Note that ϕ is surjective onto $\{\Re z = 0 \cup \infty\} \subset \mathbf{P}^1$, and that for all but a finite number of points z_0 in γ , $\phi(z_0)$ is a regular value of ϕ and hence (locally) of F . In a neighborhood of such a point, the real subvariety $\Re F = 0$ is smooth, and the holomorphic curve $F = F(z_0)$ intersects γ transversally at z_0 . We now conclude, after pushing these curves down to $M \subset \mathbf{C}^2$ using the map π , that every level $f/g = ic$ passes through 0, and that for an open set of $z \in M^*$

the irreducible component of Q_z passing through z also passes through 0. This shows the openness in the statement of the lemma when $n = 2$. The general case follows by a similar argument, but involves the resolution of singularities in higher dimensions, which is much deeper than the elementary result in two dimensions. The case $n = 2$ will suffice for our purposes below. The lemma is proved. \square

PROPOSITION 5.4. *Let C be a germ of real analytic curve at $0 \in \mathbf{C}$. Then $C \times \mathbf{C}^{n-1}$ is the zero locus of the real part of a meromorphic function if and only if C is conformally equivalent to m straight lines $\Re(\mu^j z_1) = 0$, where $\mu = i^{1/m}$ and $j = 0, \dots, m-1$.*

Proof. We may assume that 0 is an isolated singular point of C . The singular locus of $M = C \times \mathbf{C}^{n-1}$ is $0 \times \mathbf{C}^{n-1}$. The branch of Segre variety Q_z containing $z = (c_1, \dots, c_n) \in M^*$ is the hyperplane $z_1 = c_1$, which does not pass through the origin. Lemma 5.3 implies that M cannot be defined by the real part of a meromorphic function. Assume now that M is defined by $\Re f = 0$, where f is a germ of holomorphic function at 0. Proposition 5.1 implies that the critical variety of f is $z_1 = 0$, i.e., $f(z) = u(z)z_1^m$ with $u(0) \neq 0$ and $m > 1$. Now, $C: \Re(u(z_1, 0)z_1^m) = 0$ is conformally equivalent to the germ of the m lines stated in the proposition. \square

Note that the claim in the introduction, that the Levi flat hypersurface $M := \{x_1^2 + y_1^3 = 0\}$ cannot be defined by the vanishing of the real part of a holomorphic or meromorphic function, follows directly from any of the results above.

We conclude the paper by noting that there are obviously several questions left unanswered by what we have done. We point out a few of them here.

(1) Do there exist real analytic Levi-flat hypersurfaces in \mathbf{C}^n for which the singular point is isolated and the tangent cone is $Q_{1,1}$? More generally, can one find a real analytic hypersurface with isolated singularity which is the union of a family of smooth complex hypersurfaces? (Note that $V: x^2 + zy^3 + z^2 = 0$ in \mathbf{R}^3 (resp. \mathbf{C}^3) is a union of *smooth* real (resp. complex) hypersurfaces in \mathbf{R}^3 (resp. \mathbf{C}^3) parameterized by $t = z$, for which $V_s = \{0\}$.)

(2) Is the Levi-flat hypersurface $\Re(f/g) = 0$ finitely determined if it has an isolated singularity at 0, where f/g is a germ of meromorphic function at 0? Are there topological invariants of a Levi-flat hypersurface near an isolated singular point, analogous to the Milnor number of a complex hypersurface with isolated singularity?

(3) What are the singularities and rigidity properties of Levi-flat real analytic varieties of higher codimension or of lower CR dimension?

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