ERRATUM: A COMPLETE CLASSIFICATION FOR PAIRS OF REAL ANALYTIC CURVES IN THE COMPLEX PLANE

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1) Theorem 1.7 in [1] (stated for all $p \geq 3$) is correct if and only if $\delta, p/q$ are even and q/δ is odd (in particular $4|p$). In Erratum we will correct Theorem 1.7 in [1] and strengthen Corollary 1.6 in [1]. Theorem 1.1 below then covers all possible cases. Recall that $\mathbb{R}_p$ is the set of real analytic maps of the form $x \to x + f_{p+1}x^{p+1} + \dots$ with $f_{p+1} \neq 0$. We will keep notations in [1]. Let $d = d_f, q = q_f, \delta = \delta_f$. When $d = 0$, we have $p = q = \delta$. When $d \neq 0$, we have $\hat{\mathcal{C}}(f) =< \tau > \otimes < \omega >$, where $\tau'(0)$ is a primitive q-th root of unity, $\tau^{q/\delta} = \omega f^{1/d}$ and ω generates periodic centralizers of f.

Theorem 1.1. *Suppose that $f, \tilde{f} \in \mathbb{R}_p$, and that there is a biholomorphic g such that $gfg^{-1} = \tilde{f}$. (a) If p is odd, then $f, \tilde{f}$ are real analytically equivalent. (b) Suppose that p is even. b (i) Suppose that $\frac{p}{\delta}$ is odd. Then $f, \tilde{f}$ are real analytically equivalent if they are formally real analytically equivalent; moreover, there exists a biholomorphic u such that ufu^{-1} is real and is not equivalent to f by any real formal map. b (ii) Suppose that $\frac{p}{\delta}$ is even. Then $\tilde{f}, f$ are always formally real analytically equivalent, and they are real analytically equivalent if either δ is odd or $\frac{p}{q}$ is odd or $\frac{q}{\delta}$ is even. If δ and $\frac{p}{q}$ are even and $\frac{q}{\delta}$ is odd, there exists a biholomorphic map h such that hfh^{-1} is real but is not real analytically equivalent to f.*

Proof. Let $\rho(z) = \overline{z}$. Consider $d = 0$ first. We want to show that f is real analytically equivalent to the restriction to the real line of the time-1 map of the vector field $v = \frac{bz^{p+1}}{1+\lambda z^p}\frac{\partial}{\partial z}$, where $\lambda \in \mathbf{R}$ and $b = 1$ when p is odd and $b = \pm 1$ when p is even. The flow of v is $\varphi_t(z) = z + btz^{p+1} + (\frac{(p+1)t^2}{2} - b\lambda t)z^{2p+1} + O(|z|^{2p+2})$. By a real change of coordinates $z \to cz$ we may assume that f_{p+1} is 1 when p is odd and ± 1 when p is even. By a simple computation there exists a unique real change of coordinates $z \to z + \cdots + g_p z^p$ such that f has the form $z \to z + f_{p+1}z^{p+1} + f_{2p+1}z^{2p+1} + O(2p+2)$. Take $b = f_{p+1}$ and λ with $\frac{p+1}{2} - b\lambda = f_{2p+1}$. So f and φ_1 have the same formal holomorphic invariant. Since $d = 0$ by the Ecalle-Voronin theory there is a biholomorphic map g_0 such that $f = g_0\varphi_1 g_0^{-1}$. We must have $g_0'(0)^p = 1$. Let $\nu(z) = g_0'(0)^{-1}z$. Note that ν preserves the complex vector field v, and hence commutes with φ_t. Replacing g_0 by $g_0\nu$, one may assume that $g_0'(0) = 1$. Then $g_0(z) = z + O(p+1)$. Replacing g_0 by $g_0\varphi_t$ for some $t \in \mathbf{C}$, we may assume that $g_0(z) = z + O(p+2)$. Now conjugate $f = g_0\varphi_1 g_0^{-1}$ by ρ. Using $\rho f \rho = f$ and $\rho\varphi_1\rho = \varphi_1$ we get $(g_0^{-1}\rho g_0 \rho)\varphi_1(g_0^{-1}\rho g_0\rho)^{-1} = \varphi_1$. By comparing coefficients or using the fact that φ_t are the only centralizers of φ_1 that are tangent to the identity, we conclude that $g_0^{-1}\rho g_0\rho(z) = z + O(p+2)$ is the identity map. This shows that g_0 is real. We

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have proved that f is real analytically equivalent to φ_1. The same argument also shows that $\tilde f$ is real analytically equivalent to the φ_1 as above when p is odd or p is even and $f_{p+1}\tilde f_{p+1} > 0$. It is obvious that if p is even and $f_{p+1}\tilde f_{p+1} < 0$ then $f, \tilde f$ are not formally real analytically equivalent. Therefore $f, \tilde f$ are real analytically equivalent if and only if p is odd or p is even and $f_{p+1}\tilde f_{p+1} > 0$. Furthermore, when p is even $f_{p+1}\tilde f_{p+1} < 0$ occurs to $\tilde f = u^{-1}g_0^{-1}fg_0u = u^{-1}\varphi_1 u$ by taking $u(z) = e^{\frac{\pi i}{p}}z$. We have obtained the desired conclusions when $d = 0$, which are listed in (a) and b (i).

The above argument is valid if all maps are at the formal level. Hence it shows that $f, \tilde f$ are formally real analytically equivalent, if and only if p is odd or p is even and $f_{p+1}\tilde f_{p+1} > 0$.

Assume now $d \neq 0$. We first assume that $f_{p+1}\tilde f_{p+1} > 0$. By a real linear change of coordinates we may assume that $f_{p+1} = \tilde f_{p+1}$. We want to show that $f, \tilde f$ are real analytically equivalent for the following cases (i), (ii), (iii).

(i) δ is odd. Note that $g^{-1}\mathbf{R}$ is an invariant curve for f. By Theorem 1.4 (ii) in [1], we have $g'(0) = \pm\sqrt{\mu}^{jp/\delta} = \sqrt{\mu}^{pk/\delta}$; moreover, $g^{-1}\mathbf{R} = \mathbf{R}$ if $g'(0) = \pm 1$. It suffices to find a centralizer ν of f such that $\nu'(0) = \pm g'(0)$ (so $\tilde g = g\nu^{-1}$ is real and $\tilde g f \tilde g^{-1} = \tilde f$). If k is even, then $g'(0)$ is the derivative of some periodic centralizer ν of f at the origin. Otherwise, assume that k is odd. Then, since δ is odd $\delta - k = 2l$ is even. Now take a periodic centralizer ν with $\nu'(0) = \mu^{lp/\delta}$ and let $h = g\nu$. We calculate that $h'(0) = \sqrt{\mu}^{p(k+2l)/\delta} = -1$.

(ii) $\frac{p}{q}$ is odd. Since f, gfg^{-1} commute with ρ then $g^{-1}\rho g\rho$ commutes with f. So $g^{-1}\rho g \rho = \tau^a f^{l/d}$. Write $\tau'(0) = \mu^{pq^*/q}$. Since $f_{p+1} = \tilde f_{p+1}$ then $g'(0) = \mu^k$. Thus p/q divides $2k$, i.e. p/q divides k. This shows that there is a centralizer ν satisfying $\nu'(0) = g'(0)$.

(iii) $2\delta | q$. As in (i) we have $g'(0) = \sqrt{\mu}^{pk/\delta} = \mu^{\frac{p}{q}\frac{kq}{2\delta}}$, which is a q-th root of unity. Thus there is $\nu \in \hat{\mathcal{C}}(f)$ such that $g'(0) = \nu'(0)$.

Obviously we can get $\tilde f_{p+1} = f_{p+1}$ by a real change of variable when p is odd. Thus we have obtained (a) from (i) and the first part of b (i) from (ii). Consider the second part of b (i). From the proof of Theorem 1.4 (ii) in [1] (p. 28) we know that there exists u satisfying $u^2 = \omega$ and $\rho u \rho = u^{-1}$. Hence $\rho u f u^{-1} \rho = u^{-1} f u = u f u^{-1}$, i.e. ufu^{-1} is real. Since $\omega'(0)$ is a primitive δ-th root of unity and δ is even, then $u'(0) = \pm\sqrt{\mu}^{p(2k+1)/\delta}$. Hence, $u'(0)^p = (\pm\sqrt{\mu}^{p(2k+1)/\delta})^p = (\sqrt{\mu}^p)^{(2k+1)p/\delta} = (-1)^{(2k+1)p/\delta} = -1$ and $ufu^{-1}(z) = z - f_{p+1}z^{p+1} + O(p+2)$. This completes b (i).

(iv) We want to show that $f_{p+1}\tilde f_{p+1} > 0$ when $\frac{p}{\delta}$ is even. Otherwise we may assume that $\tilde f_{p+1} = -f_{p+1}$. Write $g'(0) = c$. We get $c^p = -1$, i.e. $c = \sqrt{\mu}^{2j+1}$. Since $g^{-1}\mathbf{R}$ is invariant under f, we get $c = \sqrt{\mu}^{kp/\delta}$ as in (i). Thus $(2j+1) - \frac{kp}{\delta} = 2lp$. But p/δ is even, a contradiction. Now using (iv) and then (i)-(iv) we get the first two assertions of b (ii).

We now prove the last assertion of b (ii). So p/q, δ are even and 2δ does not divide q. Since $\rho f = f\rho$, from the proof of Proposition 7.9 [1] (p. 55) we know that $q = \delta$ and $\rho\tau\rho = \tau^{-1}f^{2/d}$. So $\tau = \omega f^{1/d}$ and ω generates periodic centralizers of f. By the proof of Theorem 1.4 (ii) in [1] there exists h satisfying $h^2 = \omega$ and $\rho h \rho = h^{-1}$. We have $h'(0) = \pm\mu^{pj/(2\delta)}$. Since $q = \delta$ and $2q|p$ then $h'(0)^p = (\sqrt{\mu}^{pj/\delta})^p = 1$, i.e. $f, \tilde f = hfh^{-1}$ have the same coefficient for z^{p+1}. Now $\tilde f$ is real because $\rho h f h^{-1} \rho = h^{-1} f h = h f h^{-1}$. If $ghfh^{-1}g^{-1} = f$ for some real analytic g, then $gh = \tau^a\omega^b$. Using $\rho\tau\rho = \tau^{-1}f^{2/d} = \tau\omega^{-2}$, we get $gh^{-1} = (\tau\omega^{-2})^a\omega^{-b}$. Thus $h^2 = \omega^{2b+2a}$. Hence $1-2b-2a = j\delta$ is even, a contradiction. The proof of the theorem is complete. □

The above theorem covers all possible cases of a given $f \in \mathbf{R}_p$. We want to remark that by using the moduli space $\mathcal{N}_{p,\sigma_1}$ (p. 43, [1]) all positive integers d, p, q, δ can be realized by some $f \in \mathbf{R}_p$ when $q = \delta$ or 2δ and $q|p$. The latter conditions are also necessary by the proof of Proposition 7.9 in [1].

We also need the following corrections.

2) Theorem 1.4 (ii): Replace $\omega^{(j-1)/2}$ by $(\omega^{1/2})^{j-1}$, and insert "and $\omega^{1/2}$ is a square root of ω that is reversible by σ_{γ_1}" after $\hat{\mathcal{C}}(f)$. The last sentence of (iii) should read: g is real-valued on $\mathbf{R}$, if and only if $g'(0)$ is real; in particular, g is real if $p=1$ or if $p=2$ and the coefficients of z^3 of f and gfg^{-1} have the same sign.

3) Proposition 7.6: Replace "mod (d,s)" by "$+j_0 \mod (d,s)$ with $j_0 = \frac{s_0'(d,s)-\alpha\delta}{s}$". Line -15 (i.e. line -15 from bottom): Replace "mod (d,s)" by "$+j_0 \mod (d,s)$".

4) Proposition 7.7: In (i), replace if "$q \neq \delta, 2\delta$" by "$q = 1,2$ or $q \neq \delta, 2\delta$". In (ii) replace "$q \neq \delta, 2\delta$" by "$q \geq 3$ and $q = \delta$ or 2δ". Replace (iii) by the following: if $q \geq 3$, and $q = \delta$ or 2δ then $F_{s,j_+,\tau_+}(f) = F_{s,j_-,\tau_-}$ for $j_- = -j_+ - \frac{2s_0'(d,s)}{s} \mod (d,s)$. Replace the last line of the proof of Proposition 7.7 by "which implies $j_- = -j_+ - \frac{2s_0'(d,s)}{s} \mod (d,s)$". In the second last line of the proof, replace $\mathcal{F}_{s,-j_+,\tau_-}$ by $\mathcal{F}_{s,j_-,\tau_-}$.

5) Corollary 7.8: Replace "$\tau_{f^{-1}*} = \tau_{f*}$ for" by "$\tau_{f^{-1}*} = (\tau_{f*})^{-1}$ for $q = 1,2$ or". Replace "$q = \delta$, or 2δ" by "$q \geq 3$ and $q = \delta$ or 2δ".

6) Lemma 9.1 (c): Replace "the square root $\omega_h^{1/2} = \sqrt{\mu}^{p/\delta}z + O(2)$" by "there exists a unique square root $\omega_h^{1/2} = \sqrt{\mu}^{p/\delta}z + O(2)$ that". (The proof for the uniqueness is in the proof of Proposition 9.3.)

7) Proposition 9.3: After the third sentence, insert: Assume that $d_h \neq 0$, $\omega_h^{1/2}h\omega_h^{-1/2} = h^{-1}$, $(\omega_h^{1/2})^{m\delta_h/p} = \omega = (\omega_{\tilde{h}}^{1/2})^{m\delta_{\tilde{h}}/p}$, and $\omega_{\tilde{h}}^{1/2}\tilde{h}\omega_{\tilde{h}}^{-1/2} = \tilde{h}^{-1}$.

Page 17, line -9: Replace Ω_{2k} by Ω_{2k+1} and Ω_{2l+1} by Ω_{2l}.

Page 24, line 16: There should be $\omega_1 = \tau_1^{q/\delta}f^{-1/d}$. Proposition 3.5, line 2: Replace "positive integer" by "non-negative integer with $(q,q_0) = 1$".

Page 26, line 1: Replace (a,d) by (a,q). Theorem 3.7, line 3: Replace $\mathcal{A}_p$ by $\hat{\mathcal{C}}(f)$.

Page 27. Replace line -7 by "$f^l = H^{2d} = Hf^lH^{-1} = f^{-l}$". In lines -2 and -3, replace all σ by σ_1.

Page 28, line 8: Replace "square roots" by "some square roots". Line 11: Replace $(D\omega)^{-1} \circ \omega$ by $(D\omega)^{-k} \circ \omega^k$.

Page 29, line 11: Replace $(-1)^p$ by $(\operatorname{sign} c)^p$.

Page 39, line 16: Replace $A_j(\Omega_j)$ by $A_{2k+2-j}(\Omega_{2k+2-j})$. Line 19: Replace $A_{2j}f$ by $A_{2j}f^{-1}$. Line 24: Replace f^{-1} by $\tilde{f}^{-1}$.

Page 42, line 1 above Lemma 6.1: Replace σ by σ_1. In Lemma 6.1: Replace σA_{2j}^{-1} by A_{2j}^{-1}.

Page 43, line 6: Replace "$j \neq m, m+p \mod (2p)$" by "$m < j < m+p$".

Page 45, line -13: Replace $-\overline{c}_{2m-j}$ by $\overline{c}_{2m-j}$. Line -5: Replace μ^{m+k} by $\sqrt{\mu}^{2m+2k-1}$. Lines -1 and -2: Replace $\overline{c}_j$ by $-\overline{c}_j$.

Page 46: Replace the proof of Theorem 1.7 by the proof of Theorem 1.1 in this Erratum.

Page 48, line -19: Replace "$F \cap \mathcal{A}_0$ by "$F \cap \mathcal{A}_0$ is in the flow or".

Page 49, line 12: Replace d_0 by d'.

Page 51, line 15: Replace c_j by $-c_j$. In line 17, delete "$\tilde{\Phi}, \Phi$ are both in $\mathcal{M}'_{p,q,\delta,d}$". In line 18, delete "$\tilde{\Phi}, \Phi$ are both in $\mathcal{M}''_{p,q,\delta,d}$".

Page 52, line 10: Replace both $\sqrt{\mu}$ by $\sqrt{\mu}\lambda$. Line 12: Replace the second $\tilde{h}$ by $\tilde{h}^{-1}$. Line 15: Replace the last f_* by f_*^{-1}.

Page 54, line 14: Replace $2s$ by 2δ. Line -9: Replace $\tilde{f} = gfg^{-1}$ by $\tilde{F} = gFg^{-1}$.

Page 55: Add $-s \leq a < 0$ to the end of formula (7.3).

Page 56, line 10. Replace Proposition 7.9 by "The proof of Proposition 7.9". Line -4: Replace $\tilde{F}$ by $\tilde{F} \cap \mathcal{A}_0$.

Page 57, line 14: Replace A_p^* by $A_{p,\sigma}^*$.

Page 58, line 6: Replace $F \cap A_0$ by F.

Page 64, line 8: Replace $\lambda > 0$ by $0 \leq \arg \lambda < 2\pi/p$.

Page 67, line 14: Replace ξ/d by ξd. Line -4: Replace $\kappa h \kappa$ by $\kappa h \kappa^{-1}$.

Page 68, line 14: Replace m by p/δ. Line -4: Replace A_j by $\tilde{A}_j$. Line -1: Replace the second h by $\tilde{h}$.

Page 69, lines 7 and 18: Replace $\mathcal{M}$ by $\mathcal{M}^r_{p,\delta,d}$. Line 19: Replace "choose" by "choose g_1 with", and replace g by g_1. Line 21: Replace κ_1 by κ.

References

[1] P. Ahern and X. Gong, *A complete classification for pairs of real analytic curves in the complex plane*, J. Dynam. Control Systems 11(2005), no. 1, 1-71.

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