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Fixed points of elliptic reversible transformations with integrals

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Abstract. We show that for a certain family of integrable reversible transformations, the curves of periodic points of a general transformation cross the level curves of its integrals. This leads to the divergence of the normal form for a general reversible transformation with integrals. We also study the integrable holomorphic reversible transformations coming from real analytic surfaces in \( \mathbb{C}^2 \) with non-degenerate complex tangents. We show the existence of real analytic surfaces with hyperbolic complex tangents, which are contained in a real hyperplane, but cannot be transformed into the Moser–Webster normal form through any holomorphic transformation.

1. Introduction and results
A transformation \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) is reversible if it is conjugate to its inverse by an involution, or, equivalently if \( \varphi \) is the composition of a pair of involutions. Reversible transformations appeared often in Birkhoff’s work on dynamical systems due mainly to the existence of periodic points of a general elliptic reversible transformation (see [3, 5]). The periodic points of reversible mappings and systems were further discussed by Devaney [7]. It has been observed that reversible mappings and reversible systems have many properties similar to area-preserving transformations or Hamiltonian systems ([10, 1]). For instance, using the curve intersection property, Moser [9] showed that area-preserving mappings have invariant curves surrounding elliptic fixed points. For reversible transformations, the existence of invariant curves was first announced by Arnol’d in [1]. A complete proof was given by Sevryuk in [13] where examples of reversible transformations which do not satisfy the curve intersection property were also constructed.

The purpose of this paper is to bring up a different aspect of reversible transformations concerning the convergence of normal forms. In [4], Birkhoff proved that an area-preserving transformation \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) can always be transformed into its normal form through convergent transformations whenever \( \psi \) has an integral, i.e. a non-constant function invariant under \( \psi \). However, we shall prove that an analytic
A reversible transformation may not be transformed into its normal form by any convergent transformation even if it has integrals.

Let \( z = x + iy \) be the complex coordinate for \( \mathbb{R}^2 \), and denote by \( \tau \) the complex conjugation \( z \mapsto \bar{z} \). To formulate our result, we fix a positive integer \( s \) and let

\[ \Sigma = \{(a_{i,j}); i + j > 2s, a_{i,j} = \bar{a}_{i,j}, |a_{i,j}| < 1\}. \]

We shall also regard \( a_{i,j} \) as a variable and denote by \( a \) the infinitely many variables \( a_{i,j}(i + j > 2s) \). Thus, \( h(a) \) will stand for a power series in all variables \( a_{i,j} \). For each \( a \in \Sigma \), we put

\[ \tilde{a}(z, \bar{z}) = \sum_{i+j>2s} a_{i,j} z^i \bar{z}^j. \]

We define a mapping \( \varphi_a : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[ z \mapsto \varphi_a T \tau \varphi_a^{-1}(z), \tag{1.1} \]

where \( \varphi_a(z) = ze^{i\tilde{a}(z, \bar{z})} \), and \( T \) is the twist mapping

\[ z \mapsto ze^{i(\alpha + z \bar{z})}, \quad \alpha/(2\pi) \in \mathbb{R}\backslash\mathbb{Q}. \tag{1.2} \]

We shall see that \( T \) is the formal normal form of \( \varphi_a \). Notice that \( \varphi_a \) is reversible with respect to \( \tau \), i.e. \( \varphi_a^{-1} = \tau \varphi_a \tau^{-1} \). Moreover, the function \( \kappa(z, \bar{z}) = z\bar{z} \) is an integral for all \( \varphi_a \).

We shall prove the following.

**Theorem 1.1.** For each \( \alpha/(2\pi) \notin \mathbb{Q} \), there exists a sequence of functionally independent power series \( H_k \) in \( a \), of which each \( H_k \) converges for \( a \in \Sigma \), such that for a fixed \( a \in \Sigma \), \( \varphi_a \) cannot be transformed into \( T \) through any convergent transformation of \( \mathbb{R}^2 \), if there are infinitely many \( H_k(a) \neq 0 \).

Theorem 1.1 is motivated by a result of Siegel [14] about Hamiltonian systems as well as a result of Rüssmann [12] about area-preserving transformations. We shall prove Theorem 1.1 by investigating the periodic points of \( \varphi_a \). Our observation is that as a perturbation of the twist mapping, \( \varphi_a \) has Birkhoff curves surrounding the origin. In fact, these curves are the curves of periodic points since \( \varphi_a \) preserves all the circles centered at the origin. However, we shall prove that, as a rule, the curves of periodic points of \( \varphi_a \) are not contained in any circle centered at the origin. More precisely, we have the following result.

**Theorem 1.2.** For each \( \alpha/(2\pi) \notin \mathbb{Q} \), there exists a sequence of functionally independent convergent power series \( H_k(a) \), \( a \in \Sigma \), which satisfies the following property. If \( a \in \Sigma \) is fixed and \( H_k(a) \neq 0 \) for some \( k \), then there is a closed interval \( I_k \) of positive length such that for each \( r \in I_k \), \( \varphi_a \) has a periodic point on \( |z| = r \). Furthermore, \( I_k \) intersects any given interval \( (0, \epsilon) \) where \( k \) is large.

Notice that there are only countably many circles centered at the origin on which the restriction of \( T \) has a periodic point. Thus, Theorem 1.1 follows from Theorem 1.2.
To state our next result, we consider a real analytic surface in \( \mathbb{C}^2 \) with a non-degenerate complex tangent at the origin given by

\[
M : z_2 = z_1 \bar{z}_1 + \gamma z_1^2 + \gamma \bar{z}_1^2 + q(z_1, \bar{z}_1), \quad \gamma \in \mathbb{R}, \tag{1.3}
\]

in which \( q \) is a convergent power series starting with terms of order three. When \( 0 < \gamma < 1/2 \), or \( 1/2 < \gamma < \infty \) with countable exceptional values, Moser and Webster [11] showed that through formal transformations, \( M \) can be transformed into a surface defined by

\[
x_2 = z_1 \bar{z}_1 + (\gamma + \epsilon x_2^s)(z_1^2 + \bar{z}_1^2), \quad y_2 = 0, \tag{1.4}
\]

where either \( \epsilon = 0 \), or \( \epsilon = \pm 1 \) with \( s \) a positive integer. The normal form (1.4) was constructed through a pair of holomorphic involutions which characterizes the real analytic surfaces in \( \mathbb{C}^2 \) with a non-degenerate complex tangent. The existence of an integral for such a pair of involutions corresponds to the property that the surface can be transformed into a real hyperplane in \( \mathbb{C}^2 \) through biholomorphic transformations. Examples of real analytic surfaces of hyperbolic complex tangents which cannot be transformed into any real hyperplane were constructed in [11], and also in [2] by Bedford. Analogous to Theorem 1.1, we have the following.

**THEOREM 1.3.** For each non-exception \( \gamma \in (1/2, \infty) \), there exists a real analytic surface (1.3) which can be transformed biholomorphically into a real hyperplane in \( \mathbb{C}^2 \), but not into the normal form (1.4).

We mention that for a set of \( \gamma \) with Lebesgue measure zero, the above result was proved in [8].

The paper is organized as follows. In §2, we discuss the normal form for reversible transformations. Theorem 1.2 is proved in §4, following the preparation in §3 where the existence of periodic points of holomorphic transformations with integrals is discussed. §6 is devoted to the study of holomorphic reversible transformations in \( \mathbb{C}^2 \) which come from real surfaces with complex tangents. The proof of Theorem 1.3 is given in §6.

**2. Normal form for reversible transformations**

Let \( \varphi, \tau \) be two real analytic transformations of \( \mathbb{R}^2 \) defined near the origin. Assume that \( \varphi(0) = \tau(0) = 0 \), and \( \tau^2 = \text{Id} \). Suppose that \( \varphi \) is reversible with respect to \( \tau \). With a suitable assumption on the eigenvalues of \( d\varphi(0) \), we shall see that after a change of analytic coordinates, \( \tau \) is precisely the complex conjugation \( z \mapsto \bar{z} \). Then we shall derive a normal form for \( \varphi \) under the formal transformations commuting with \( \tau \). This normal form is essentially due to Moser and Webster [11].

For convenience, we shall complexify real analytic or formal transformations of \( \mathbb{R}^2 \). Let \( \Phi \) be a real analytic transformation of \( \mathbb{R}^2 \). Then \( \Phi \) is a power series in \( z \) and \( \bar{z} \). The complexification of \( \Phi \), defined by

\[
(\xi, \eta) \mapsto (\Phi(\xi, \eta), \bar{\Phi}(\eta, \xi)),
\]

is a holomorphic transformation of \( \mathbb{C}^2 \), which will still be denoted by \( \Phi \). Clearly, \( \Phi \) satisfies the reality condition

\[
\rho \Phi = \Phi \rho, \tag{2.1}
\]
in which $\rho$ is the complexification of the complex conjugation, i.e.

$$\rho : (\xi, \eta) \mapsto (\bar{\eta}, \bar{\xi}). \quad (2.2)$$

The complexification of a formal transformation is defined in a similar way. From now on, we shall identify the real analytic transformations of $\mathbb{R}^2$ with the holomorphic transformations of $\mathbb{C}^2$ satisfying the reality condition (2.1).

To discuss the normal form of $\varphi$, we first assume that the eigenvalues of $d\varphi(0)$ are not roots of unity. Let $\lambda$ be an eigenvalue of $d\varphi(0)$ with an eigenvector $e_1$. We have

$$d\varphi(0)(d\tau(0)e_1) = d\tau(0)(d\varphi^{-1}(0)e_1) = \lambda^{-1} d\tau(0)e_1. \quad (2.3)$$

This implies that $\lambda^{-1}$ is also an eigenvalue of $d\varphi(0)$. Since $\lambda^{-1} \neq \lambda$, we see that $d\varphi(0)$ is diagonalizable with distinct eigenvalues $\lambda$ and $\lambda^{-1}$.

We now further assume that $\varphi$ is elliptic, i.e. $|\lambda| = 1$ and $\lambda \neq \pm 1$. Put $e_2 = \rho(e_1)$. Then the reality condition $\rho \varphi = \varphi \rho$ gives

$$d\varphi(0)e_2 = \rho d\varphi(0)e_2 = \tilde{\lambda} e_2. \quad (2.4)$$

Hence, $\tilde{\lambda} = \lambda^{-1}$. Under new coordinates $(\xi, \eta)$ for $\xi e_1 + \eta e_2 \in \mathbb{C}^2$, the anti-holomorphic involution $\rho$ still takes the form (2.2), while the reversible transformation is

$$\varphi : \begin{align*}
\xi' &= \lambda \xi + p(\xi, \eta), \\
\eta' &= \lambda \eta + q(\xi, \eta),
\end{align*} \quad (2.5)$$

in which $p$ and $q$ are convergent power series starting with the second-order terms. Notice that $\lambda$ is replaced by $\tilde{\lambda}$ if one applies a change of coordinates by $(\xi, \eta) \mapsto (\eta, \xi)$. Therefore, one may assume that

$$\text{Im} \lambda > 0. \quad (2.6)$$

We now want to normalize $\tau$. From (2.3), we see that $d\tau(0)e_1$ is an eigenvector. Hence, $d\tau(0)e_1 = \lambda_0 e_2$. On the other hand, $\tau$ satisfies the reality condition. Then

$$d\tau(0)e_1 = \rho d\tau(0)e_1 = \rho d\tau(0)e_2 = \tilde{\lambda}_0^{-1} e_2. \quad (2.7)$$

This implies that $|\lambda_0| = 1$. Thus, we can write

$$\tau(\xi, \eta) = (\lambda_0 \eta, \tilde{\lambda}_0 \xi) + O(2).$$

Consider a change of coordinates defined by

$$\begin{align*}
\xi' &= \lambda_0^{-1/2} (\xi + \lambda_0 \eta \circ \tau(\xi, \eta))/2, \\
\eta' &= \lambda_0^{1/2} (\eta + \tilde{\lambda}_0 \xi \circ \tau(\xi, \eta))/2.
\end{align*}$$

From the reality condition $\rho \tau = \tau \rho$, one can see that under new coordinates $(\xi', \eta')$, $\rho$ is still of the form (2.2). Now $\tau$ is the linear involution

$$\varphi : (\xi, \eta) \mapsto (\eta, \xi). \quad (2.8)$$

**Theorem 2.1.** Let $\rho$ and $\tau$ be given by (2.2) and (2.6) respectively. Suppose that $\varphi$ defined by (2.4) is reversible with respect to $\tau$, and it satisfies the reality condition
$\rho \varphi = \varphi \rho$. If $\lambda$ is not a root of unity, then there exists a formal transformation $\Phi$ such that $\rho \Phi = \Phi \rho$, $\tau \Phi = \Phi \tau$ and

$$\varphi^* = \Phi \varphi \Phi^{-1} : \begin{cases} \xi' = \lambda \xi e^{i\epsilon \xi \eta}, \\ \eta' = \tilde{\lambda} \eta e^{-i\epsilon \xi \eta}, \end{cases} \quad (2.7)$$

in which $\epsilon = \pm 1$ with $s$ a positive integer, or $\epsilon = 0$ with $s = \infty$. Furthermore, $\{\lambda, \epsilon, s\}$ is the full set of invariants of $\varphi$ under real formal transformations.

The normal form (2.7) was derived in [11], except that here a different reality condition is involved. For completeness, we shall verify the reality condition. A transformation

$$\tilde{\varphi} : (\xi, \eta) \mapsto (M \xi \eta \xi, M^{-1} \xi \eta \eta), \quad (2.10)$$

is said to be normalized if $u_{i+1,i} = v_{i,i+1} = 0$ for all $i$. We need the following.

**Lemma 2.2.** ([11]) Let $\tau_j (j = 1, 2)$ be a pair of involutions

$$(\xi, \eta) \mapsto (\lambda_j \eta + f_j(\xi, \eta), \lambda_j^{-1} \xi + g_j(\xi, \eta)). \quad (2.9)$$

Put $\varphi = \tau_1 \tau_2$. If $\lambda_1 \lambda_2^{-1}$ is not a root of unity, then there exists a unique normalized formal transformation $\Phi$ such that $\Phi \varphi \Phi^{-1}$ is of the form

$$\tilde{\varphi} : (\xi, \eta) \mapsto (M(\xi \eta) \xi, M^{-1}(\xi \eta) \eta), \quad (2.10)$$

with $M(t) = \lambda_1 \lambda_2^{-1} + O(1)$. Furthermore, $\tau_j$ is transformed into

$$\tilde{\tau}_j : (\xi, \eta) \mapsto (\Lambda_j(\xi \eta) \eta, \Lambda_j^{-1}(\xi \eta) \xi), \quad (2.11)$$

with $\Lambda = \lambda_j + O(1)$.

**Proof of Theorem 2.1.** We put $\tau_1 = \tau$ and $\tau_2 = \tau \varphi$. Then for the pair of involutions $\{\tau_1, \tau_2\}$, it follows from Lemma 2.2 that there is a unique normalized transformation $\Phi_0$ which transforms $\varphi$ into (2.10). Since $\rho \varphi = \varphi \rho$, we get

$$(\rho \Phi_0 \rho) \varphi(\rho \Phi_0 \rho)^{-1}(\xi, \eta) = (M^{-1}(\xi \eta) \xi, M(\xi \eta) \eta).$$

Clearly, $\rho \varphi \rho$ is still in the form (2.8) except $\tilde{u}, \tilde{v}$ are replaced by $u$ and $v$, respectively. By the uniqueness of the transformation $\Phi_0$, we have

$$\rho \Phi_0 \rho = \Phi_0, \quad M^{-1}(\xi \eta) = M(\xi \eta). \quad (2.12)$$

Hence, $\rho \tilde{\tau}_j = \tilde{\tau}_j \rho$. This implies that $\tilde{\Lambda}_j^{-1}(t) = \Lambda_j(t)$. Put $a(t) = \Lambda_j^{1/2}(t)$ and

$$\Phi_1(\xi, \eta) = (a(\xi \eta) \xi, a^{-1}(\xi \eta) \eta).$$

Then $\Phi_1^{-1}(\xi, \eta) = (a^{-1}(\xi \eta) \xi, a(\xi \eta) \eta)$. It is easy to see that

$$\Phi_1 \tilde{\tau}_j \Phi_1^{-1} = \tau, \quad \Phi_1 \rho = \rho \Phi_1, \quad \Phi_1 \tilde{\varphi} \Phi_1^{-1} = \tilde{\varphi}.$$

We now write

$$M(t) = e^{\Gamma(t)}.$$
Then (2.12) gives $\Gamma ( \xi \eta ) = \tilde{\Gamma} ( \xi \eta )$. If $\Gamma(t)$ is not constant, we can find a real power series $r(t)$ such that

$$\Gamma(t) = \epsilon t^s r^2(t),$$

where $\epsilon$, $s$ are the sign and the order of the first non-vanishing coefficient of $\Gamma(t)$, respectively. Let $\Phi_2(\xi, \eta) = (\xi r(\xi \eta), \eta r(\xi \eta))$. Then one can verify that

$$\Phi_2 \tau = \tau \Phi_2, \quad \Phi_2 \rho = \rho \Phi_2.$$ 

Now $\Phi_2$ transforms $\tilde{\varphi}$ into (2.7). Therefore, $\Phi = \Phi_2 \Phi_1 \Phi_0$ preserves $\tau$ and $\rho$, and it transforms $\varphi$ into (2.7).

To show that $\epsilon$ and $s$ are invariants, we assume that there is another formal transformation $^\dagger$ such that

$$V(\varphi^*) = \frac{x}{V},$$

(2.13)

where $^\dagger$ is a transformation in the form (2.7), where $\epsilon$ and $s$ are replaced by $\epsilon'$ and $s'$. We may assume that $s \leq s'$. From (2.5), we first notice that $\lambda$ is an invariant and $d\Psi(0) = (a_0 \xi, b_0 \eta)$. Notice that $\lambda$ is not a root of unity. By comparing both sides of (2.13) up to terms of order $2s$, one gets

$$\Psi(\xi, \eta) = (\xi a(\xi \eta), \eta b(\xi \eta)) + O(2s + 2).$$

Now the reality condition (2.1) implies that $b = \bar{a}$. By comparing terms in (2.13) of order $2s + 1$, one can get $s = s'$ and $\epsilon = \epsilon'$. This proves Theorem 2.1. □

Let $T$ be the twist mapping (1.2). For simplicity, we call $p \in \mathbb{R}^2$ a periodic point of $T$ of period $n$ if it is the fixed point of the iterate $T^n$. For each positive integer $n$, we put

$$n \alpha = 2g \pi + \beta, \quad -\pi < \beta < \pi, \quad g \in \mathbb{Z}.$$  

(2.14)

Let $C_{n,j}$ be the circle centered at the origin with radius $r_j$ determined by

$$nr_j^2 = -\beta + 2\pi j, \quad j = 1, 2, \ldots.$$  

It is easy to see that for $0 < \beta < \pi$, the set of periodic points of $T$ with period $n$ is the disjoint union of $C_{n,j}(j \geq 1)$. For $-\pi < \beta < 0$, the set of periodic points contains an extra circle

$$C_{n,0} = \{z | n(z\bar{z}) = -\beta\},$$

which is the smallest circle of periodic points of $T$ with period $n$.

A small perturbation of twist mapping may destroy the circle $C_{n,j}$ of periodic points. However, these circles are only deformed slightly into Birkhoff curves, i.e. curves translated radially by the $n$th iterate of the perturbation. We notice that for a perturbation of the twist mapping, the Birhoff curve can be constructed near each periodic circle $C_{n,j}$ which is close to the origin (see [15], in particular pages 176–177). For our purpose, we shall focus on the periodic points of a perturbed mapping near the circle $C_{n,0}$.

Consider the complexification of the mapping (1.1), or more generally, a holomorphic transformation of $\mathbb{C}^2$ defined by

$$\xi' = e^{i\omega(\xi \eta)} \xi (1 + p(\xi, \eta)), $$

$$\eta' = e^{-i\omega(\xi \eta)} \eta (1 + p(\xi, \eta))^{-1},$$

(2.15)
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where \( \omega(\xi \eta) = \alpha + (\xi \eta)^5 \), and \( p = O(2s + 1) \) is a power series converging on

\[
\Delta_R = \{(\xi, \eta); |\xi| < R, |\eta| < R\}, \quad 0 < R < 1.
\]

Set

\[
\|p\|_R = \max_{|\xi|, |\eta| \leq R} |p(\xi, \eta)|.
\]

For a positive number \( \epsilon \), we put

\[
\Omega_{n,\epsilon} = \{(\xi, \eta) \in \mathbb{C}^2; 0 < |\xi|, |\eta| < \epsilon n^{-1/(2s)}, 1/4 < |\xi/\eta| < 4\}.
\]

We have the following.

**Theorem 2.3.** Let \( \varphi \) be given by (2.15) with \( \|p\|_R \leq m_0 \). Then there exist positive constants \( \epsilon \) and \( \delta \), depending only on \( R, m_0 \) and \( s \), such that for any positive integer \( n \), the fixed points of \( \varphi^n \) in the domain \( \Omega_{n,\epsilon} \) are the union of \( s \) holomorphic curves if \( \beta \) in (2.14) satisfies \( -\delta < \beta < 0 \). Furthermore, if \( \varphi \) satisfies the reality condition \( \rho \varphi = \varphi \rho \), then the periodic points in the totally real space \( \mathbb{R}^2 : \eta = \bar{\xi} \) form a closed real analytic curve.

We shall see that, as a rule, the above holomorphic curves cross the level curve \( \xi \eta = c \). This is in contrast to the theory of area-preserving transformations, of which the periodic points are always contained in the level curves of its integrals.

3. Estimates for iterates

Let \( \varphi \) be defined by (2.15). Consider the \( k \)th iterate

\[
\varphi^k : \xi_k = \xi e^{ik\omega(\xi \eta)}(1 + p_k(\xi, \eta)), \quad \eta_k = \eta e^{-ik\omega(\xi \eta)}(1 + p_k(\xi, \eta))^{-1},
\]

with \( p_1(\xi, \eta) = p(\xi, \eta) \). By setting \( p_0(\xi, \eta) \equiv 0 \), we have

\[
p_{k+1} = p_k + p \circ \varphi^k + p \circ \varphi^k \cdot p_k, \quad k \geq 0.
\]

We introduce some notation. For a power series \( f(\xi, \eta) \) with coefficients \( f_{i,j} \), let us denote

\[
\hat{f}(\xi, \eta) = \sum_{i,j \geq 0} |f_{i,j}| \xi^i \eta^j.
\]

We say that \( g \) is a majorant of \( f \), symbolically \( f \prec g \), if \( |f_{i,j}| \leq g_{i,j} \) for all \( i, j \geq 0 \). Assume that \( f \) converges on \( \Delta_R \). By the Cauchy inequality, one gets

\[
|f_{i,j}| \leq \frac{\|f\|_R}{R^{i+j}}.
\]

Assume further that \( f \) starts with terms of order \( k \). Then one has

\[
f(\xi, \eta) \prec \sum_{i+j \geq k} \|f\|_R \frac{\xi^i \eta^j}{R^{i+j}} \prec \|f\|_R \frac{(\xi + \eta)^k}{1 - R^{-1} \xi - R^{-1} \eta}.
\]

From (3.2), we have

\[
\hat{p}_{k+1} \prec \hat{p}_k + \hat{p} \circ \hat{\varphi}_k + \hat{p}_k \cdot \hat{p} \circ \hat{\varphi}_k,
\]
where

\[
\hat{\varphi}_k(\xi, \eta) = \frac{e^{k(\xi \eta)'}}{1 - \hat{p}_k}(\xi, \eta).
\]

From (3.3), we can recursively define a majorant \( f_k \) for \( p_k \) by setting \( f_0 \equiv 0 \) and

\[
f_{k+1}(\xi, \eta) = f_k(\xi, \eta) + \|P\|_{R^2} \cdot \frac{(\xi + \eta)^{2x+1} e^{k(2x+1)\eta'}}{1 - R^{-1}(\xi + \eta)e^{k(\xi \eta)'}(1 - f_k(\xi, \eta))^{-1}}.
\]

for \( k \geq 0 \).

Put

\[
d_0 = \min \left\{ \frac{R^{2x+1}}{2^{6x+6}m_0}, \left( \frac{1}{2n} \right)^{1/2}, \frac{R}{16} \right\}.
\]

Then there exists a positive constant \( c_1 \) depending only on \( s, m_0 \) and \( R \) such that

\[
c_1 < nd_0^{2x} < 1/2.
\]

We want to prove the following lemma.

**Lemma 3.1.** Let \( \varphi \) be as in Theorem 2.3. Then the \( k \)-th iterate of \( \varphi \) satisfies

\[
\varphi^k : \Delta_{d_0} \rightarrow \Delta_{4d_0}, \quad 1 \leq k \leq n.
\]

**Proof.** From (3.6) we have

\[
|e^{k(\xi \eta)'}| \leq e^{nd_0^{2x}} < 2, \quad (\xi, \eta) \in \Delta_{d_0}.
\]

Hence

\[
|\xi_k| \leq 2|\xi|(1 + |p_k(\xi, \eta)|), \quad |\eta_k| \leq 2|\eta||(1 + p_k(\xi, \eta))^{-1}.
\]

Since \( p_k < f_k \), it suffices to show that \( f_k \) converges in a neighborhood of \( \Delta_{d_0} \), and

\[
\|f_k\| \leq \frac{k}{4n}.
\]

We apply the induction on \( k \). For \( k = 0 \), the assertion is trivial since \( f_0 = 0 \). Assume that the assertion holds for some \( k < n \). This implies that \((1 - f_k)^{-1}\) converges on \( \Delta_{d_0} \). From (3.5) one has

\[
d_0 \leq R/16.
\]

Hence, for \((\xi, \eta) \in \Delta_{d_0},

\[
|R^{-1}(\xi + \eta)e^{k(\xi \eta)'}(1 - f_k(\xi, \eta))^{-1}| < 1/2.
\]

Thus the right-hand side of (3.4) converges on \( \Delta_{d_0} \), which gives the convergence of \( f_{k+1} \) in the same domain. Now, one can get

\[
\|P\|_{R^2} \cdot \frac{(\xi + \eta)^{2x+1} e^{k(2x+1)\eta'}}{1 - R^{-1}(\xi + \eta)e^{k(\xi \eta)'}(1 - f_k(\xi, \eta))^{-1}} \leq \frac{m_0 2^{6x+5} d_0^{2x+1}}{R^{2x+1}}.
\]

From (3.6), one can replace \( d_0^{2x} \) by \( 1/2n \) and rewrite the right-hand side of the above inequality as

\[
\frac{m_0}{R^{2x+1}} \cdot \frac{2^{6x+4} d_0}{n},
\]

which, by (3.5), does not exceed \( 1/4n \). This proves (3.8).
Under assumptions as in Lemma 3.1, we now give some estimates for $p_k$ and $f_k$. Notice that $p_k$ and $f_k$ start with terms of order $2s + 1$. By (3.8), the Schwarz lemma gives

$$|p_k(\xi, \eta)|_r, \quad |f_k(\xi, \eta)|_s \leq \frac{1}{4} \left( \frac{r}{d_0} \right)^{2s+1}, \quad r \leq d_0.$$

(3.10)

We are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let $\varphi, m_0, d_k$ and $n$ be as in Lemma 3.1. Set

$$\xi = \zeta w, \quad \eta = \zeta w^{-1}.$$

We require that

$$0 < |\xi| < d_0/2, \quad 1/2 < |w| < 2.$$  

(3.11)

Consider the equation $\xi_n = \xi$ or, equivalently,

$$e^{i(\beta + n\xi^{2s})[1 + p_n(\xi, \zeta w^{-1})]} = 1,$$

(3.12)

in which $\beta$ is determined by (2.14) for a given $n$.

From (3.10) and (3.11), it follows that

$$|p_n(\xi, \zeta w^{-1})| < 1/4.$$  

(3.13)

We also have $n|\xi|^{2s} < 1/2$ and $|\beta| < \pi$. Thus (3.12) is reduced to

$$\beta + n\xi^{2s} - i \log(1 + p_n(\xi, \zeta w^{-1})) = 0,$$

in which the logarithm assumes principal values. We further rewrite the equation as

$$\xi^{2s}(1 + h(\xi, w)) = -\frac{\beta}{n},$$

(3.14)

with

$$h(\xi, w) = \frac{1}{in\xi^{2s}} \log(1 + p_n(\xi, \zeta w^{-1})).$$

(3.15)

Using (3.10), we have

$$|h(\xi, w)| \leq \frac{1}{n|\xi|^{2s}} \left( \frac{2|\xi|}{d_0} \right)^{2s+1}.$$

From (3.6) it follows that

$$|h(\xi, w)| \leq \frac{n^{1/2s}}{c_2} |\xi|,$$

(3.16)

in which $c_2 < c_1$ is a constant depending only on $m_0, R$ and $s$.

We now take

$$\epsilon_0 = c_2, \quad \delta = \left( \frac{c_2}{4} \right)^{2s}.$$  

(3.17)

Put

$$r_0 = \frac{1}{2}\epsilon_0 n^{-1/2s} < \frac{d_0}{2}.$$  

(3.18)

It follows from (3.16) and (3.17) that

$$|h(\xi, w)| \leq 1/4, \quad |\xi| < r_0.$$
Now (3.14) is reduced to
\[ \zeta(1 + h(\zeta, w))^{1/2s} = e^{i(j\pi/s)}\left(\frac{-\beta}{n}\right)^{1/2s}, \quad j = 1, 2, \ldots, 2s. \] (3.19)

Furthermore, for \(|\zeta| = r_0\), we have
\[ |(1 + h)^{1/2s} - 1| = \left| \frac{h}{2s} \int_0^1 (1 + th)^{(1/2s) - 1} \, dt \right| < 1/6. \] (3.20)

We now assume that \(|\beta| < \delta\). Then (3.17) gives
\[ \left| \left(\frac{-\beta}{n}\right)^{1/2s} \right| < \frac{c_2}{4} n^{-1/2s} = r_0/2. \] (3.21)

Thus, the Rouche theorem implies that for each \(j\), (3.19) has a unique solution \(\zeta = \zeta_j(w)\) in the disk \(|\zeta| < r_0\). Clearly, the solution is holomorphic for \(1/2 < |w| < 2\).

Notice that the transformation \((\zeta, w) \mapsto (\xi w, \zeta w^{-1})\) is two-to-one. Also, equation (3.12) is invariant under the transformation \((\zeta, w) \mapsto (-\zeta, -w)\). Hence, the \(2s\) holomorphic curves \(\zeta_j(w)(1 \leq j \leq 2s)\) in \((\xi, w)\)-coordinates give us \(s\) holomorphic curves \((\xi, \eta) = (\zeta_j w, \zeta_j w^{-1})(1 \leq j \leq s)\). Since \(\kappa(\xi, \eta) = \xi \eta\) is invariant under \(\varphi\), the solutions \((\xi, \eta)\) to \(\xi_n = \xi\) give the fixed points of \(\varphi^n\).

We now assume that \(\varphi\) satisfies the reality condition, and we want to show that \(\zeta(w) = \zeta_{2s}(w)\) is a real valued function for \(|w| = 1\). Since \(\varphi \circ \varphi = \varphi \circ \varphi\), we have \(\varphi^n \circ \varphi = \varphi \circ \varphi^n\). Hence
\[ 1 + \tilde{p}_n(\eta, \xi) = (1 + p_n(\xi, \eta))^{-1}. \]

It is easy to see that for \(|w| = 1\),
\[ \tilde{h}(\tilde{\xi}, \tilde{w}) = h(\zeta, w). \]

Conjugating (3.19), we get
\[ \tilde{\zeta}(1 + h(\tilde{\zeta}, w))^{1/2s} = e^{-i(j\pi/s)}\left(\frac{-\beta}{n}\right)^{1/2s}, \]
which is precisely equation (3.19) for \(j = s\). From the uniqueness of the solution, we obtain that \(\zeta(w) = \overline{\zeta(w)}\) for \(|w| = 1\). The proof of Theorem 2.3 is complete. \(\square\)

4. Analytic dependence on coefficients
We shall first discuss the analytic dependence of holomorphic curves of periodic points of \(\varphi\) defined by (2.15) on the coefficients of \(p\). We need power series in infinitely many variables. The convergence of such a power series will always mean that it converges absolutely.

For each positive \(m_0\) and \(R\), we define
\[ D^\infty(m_0, R) = \left\{ p = (p_{i,j}); |p_{i,j}| \leq \frac{m_0}{4(2R)^{i+j}}, i + j > 2s \right\}. \]

For each \(p \in D^\infty(m_0, R)\), we put
\[ \tilde{p}(\xi, \eta) = \sum_{i+j>2s} p_{i,j} \xi^i \eta^j. \]
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Then \( \tilde{\rho}(\xi, \eta) \) converges on \( \Delta_R \) and

\[
\| \tilde{\rho} \|_R \leq \sum_{i+j=2s} |p_{i,j}| R^{i+j} \leq \frac{m_0}{4} \sum_{k>2s} (k+1) \frac{1}{2^k} < m_0.
\]

Consider the transformation \( \varphi \) defined by (2.15), with \( p(\xi, \eta) = \tilde{\rho}(\xi, \eta) \). Put the iterate \( \varphi^k \) in the form (3.1) with \( p_k(\xi, \eta, p) \). We also denote the solution \( \zeta \) to (3.19) by \( \zeta_j(w, p) \). We have seen that for a fixed \( p \in D^\infty(m_0, R) \), \( p_k(\xi, \eta, p) \) converges in \( \xi \) and \( \eta \), and \( \zeta(w, p) \) converges as a Laurent series in \( w \). Next, we want to show that they actually converge as series expansions with respect to all variables. To this end, we put \( p_0^*(\xi, \eta, p) \equiv 0 \) and define recursively

\[
p_{k+1}^* = p_k^* + \tilde{\rho} \circ \varphi_k^*(1 + p_k^*), \quad k = 0, 1, \ldots, n - 1,
\]

in which

\[
\varphi_k^*(\xi, \eta) = (e^{k(\xi \eta)^\gamma} (1 - p_k^*)^{-1}, \eta e^{k(\xi \eta)^\gamma} (1 - p_k^*)^{-1}).
\]

From (3.2), it follows that

\[
p_k(\xi, \eta, p) < p_k^*(\xi, \eta, p).
\]

Put

\[
p^0(\xi, \eta) = \sum_{i+j=2s} \frac{1}{4} m_0 (2R)^{-i-j} \xi^i \eta^j.
\]

Clearly, for a fixed \( p \in D^\infty(m_0, R) \), one has

\[
p_k^*(\xi, \eta, p) < p_k^*(\xi, \eta, p^0) = f_k^*(\xi, \eta),
\]

in which \( f_k^*(\xi, \eta) \) are defined through the recursive formulas (3.4) by setting \( p = p^0 \) and \( f_0^* \equiv 0 \). From (3.8), it follows that \( p_k^*(\xi, \eta, p^0) \) converges on \( \Delta_{d_0} \) with \( \| p_k^*(\cdot, \cdot, p^0) \|_{d_0} \leq 1/4 \). Hence, \( p_k^*(\xi, \eta, p) \) converges on \( \Delta_{d_0} \times D^\infty(m_0, R) \) with

\[
\| p_k^*(\cdot, \cdot, p) \|_{d_0} < 1/4, \quad \text{for } p \in D^\infty(m_0, R).
\]

(4.1)

To show the convergence of \( \zeta_j(w, p) \), we use the residue formula

\[
\zeta_j(w, p) = \frac{1}{2\pi i} \int_{|\zeta|=r_0} \frac{\partial F_j(\zeta, w, p)}{F_j(\zeta, w, p)} d\zeta,
\]

where

\[
F_j(\zeta, w, p) = \zeta(1 + h(\zeta, w, p))^{1/2s} - e^{i(j(\zeta \eta)^s)} \left( \frac{\zeta \eta}{\alpha} \right)^{1/2s},
\]

and \( h(\zeta, w, p) \) is given by (3.15). As series expansion in \( \zeta, w \) and \( p \), we have

\[
h \prec \frac{p_n^*}{n \xi^{2s}(1 - p_n^*)} \equiv \tilde{h}.
\]

Analogous to the estimate (3.18), one can use (4.1) to get

\[
|\tilde{h}(r_0, w, p)| < 1/4, \quad \text{for } 1/2 < |w| < 2, |\zeta| \leq r_0, p \in D^\infty(m_0, R).
\]

(4.2)
From (3.20) and (3.21), it follows that for each \( p \in D^\infty(m_0, R) \), the function \( 1/F_j \) has the expansion
\[
\frac{1}{\xi} \sum_{k=0}^{\infty} \left( (1 + h(\xi, w, p))^{1/2x} - 1 + \frac{e^{i(\pi/2)}}{\xi} \left( -\frac{\beta}{n} \right)^{1/2x} \right)^k, \quad |\xi| = r_0, 1/2 < |w| < 2.
\]
Since \( (1 + x)^{1/2x} < 1/(1 - x) \), then
\[
(1 + h)^{1/2x} < \frac{1}{1 - h}.
\]
Hence, we see that as a series expansion in \( \xi, w, p, 1/F_j \) is majorized by
\[
\frac{1}{\xi} \sum_{k=0}^{\infty} \left( \frac{\tilde{h}(\xi, w, p)}{1 - \tilde{h}(\xi, w, p)} + \frac{1}{\xi} \left( \frac{|\beta|}{n} \right)^{1/2x} \right)^k.
\]
From (3.21) and (4.2), it follows that \( 1/F_j \) converges for \( 1/2 < |w| < 2, p \in D^\infty(m_0, R) \) and \( |\xi| = r_0 \). By the residue formula, we see that \( \zeta_j(w, p) \) converges for \( 1/2 < |w| < 2 \) and \( p \in D^\infty(m_0, R) \). We have proved the following.

**Lemma 4.1.** The solution \( \zeta_j(w, p) \) to (3.19) converges for \( 1/2 < |w| < 2 \) and \( p \in D^\infty(m_0, R) \).

We now want to apply Lemma 4.1 to the family of reversible transformations (1.1). Let us denote
\[
D^\infty = \{a = (a_{i,j})|i + j > 2s, |a_{i,j}| \leq 1\}.
\]
Fixing \( a \in D^\infty \), we consider the holomorphic mapping
\[
\phi_a : \xi' = e^{i\tilde{a}(\xi, \eta)} \xi, \quad \eta' = e^{-i\tilde{a}(\xi, \eta)} \eta,
\]
with \( \tilde{a}(\xi, \eta) = \sum_{i+j>2s} a_{i,j} \xi^i \eta^j \). Let us denote
\[
\phi_a^* : \xi' = e^{\tilde{a}(\xi, \eta)} \xi, \quad \eta' = e^{\tilde{a}(\xi, \eta)} \eta.
\]
We can put
\[
\phi_a^{-1} : \xi' = e^{-i\tilde{a}(\xi, \eta)} \xi, \quad \eta' = e^{i\tilde{a}(\xi, \eta)} \eta,
\]
for some formal power series \( b(\xi, \eta) \) without the constant term. This leads to the identity
\[
b(\xi, \eta) = \tilde{a}(e^{-i\tilde{b}(\xi, \eta)} \xi, e^{i\tilde{b}(\xi, \eta)} \eta).
\]
We consider \( b(\xi, \eta) \) as a power series in \( \xi, \eta \) and \( a \). Then \( b(\xi, \eta) \) is majorized by \( b^*(\xi, \eta, a) \) determined by the equation
\[
b^*(\xi, \eta, a) = \tilde{a}(e^{b^*(\xi, \eta, a)} \xi, e^{b^*(\xi, \eta, a)} \eta).
\]
Put
\[
\psi_a : \xi' = e^{b^*(\xi, \eta, a)} \xi, \quad \eta' = e^{b^*(\xi, \eta, a)} \eta.
\]
We have the following.
LEMMA 4.2. Let \( \psi_a(\xi, \eta) \) be given as above. Then there exists a constant \( R_1, \) \( 0 < R_1 < 1, \) independent of \( a \in D^{\infty}, \) such that \( \psi_a \) is given by power series which are convergent for \((\xi, \eta, a) \in \Delta_{R_1} \times D^{\infty}. \) Moreover, for each \( a \in D^{\infty}, \)
\[
\psi_a : \Delta_{R_1} \to \Delta_{1/3}. 
\] (4.8)

Proof. Put
\[
e(\xi, \eta) = \sum_{i+j \geq 1} \xi^i \eta^j.
\]
By the implicit function theorem, there exists a positive number \( R_1 < 1/6 \) such that \( b^*(\xi, \eta, e) \) converges on \( \Delta_{R_1}, \) and \( |b^*(\xi, \eta, e)| < 1/2. \) Clearly, we have (4.8). Now for each \( a \in D^{\infty}, \) we have \( a(\xi, \eta) < e(\xi, \eta). \) Since \( b^*(\xi, \eta, a) \) has positive coefficients, this shows that \( b^*(\xi, \eta, a) \) converges on \( \Delta_{R_1} \times D^{\infty}. \) The proof of the lemma is complete. □

We are ready to prove the following.

PROPOSITION 4.3. Let \( \varphi_a \) be given by (1.1). Then there exist positive constants \( \epsilon_0 \) and \( \delta \) which are independent of \( n \) and \( a \) such that in the polar coordinates \((r, \theta), \) the fixed points of \( \varphi^a_n \) in \( |z| < \epsilon_0 n^{-1/(2\delta)} \) form a closed analytic curve \( r = \zeta(w, a), \) \( |w| = 1, \) provided \( \beta \) in (2.14) satisfies \(-\delta < \beta < 0. \) Furthermore, \( \zeta(w, a) \) converges for \( 1/2 < |w| < 2 \) and \( a \in \Sigma. \)

Proof. For each \( a \in \Sigma, \) let \( h(\xi, \eta) \) be given by (4.6). We have \( h(z, \tilde{z}) = \tilde{b}(\tilde{z}, z). \) Hence,
\[
\varphi_a = \varphi_a \circ T \circ \tilde{\varphi}^{-1} = \varphi_a \circ T \circ \phi_b.
\]
Put
\[
\varphi_a^*(\xi, \eta) = (\varphi^a \varphi^{-1}(\xi, \eta), \eta e^{i\theta(\xi, \eta)}), \quad T^*(\xi, \eta) = (\varphi^a(\xi, \eta), \eta e^{i\theta(\xi, \eta)}).
\]
It is easy to see that as power series in \( \xi = z, \eta = \tilde{z} \) and \( a, \) one has
\[
\varphi_a(\xi, \eta) < \varphi_a^* \circ T^* \circ \varphi_a(\xi, \eta) = \varphi_a^*(\xi, \eta).
\]
For a fixed \( a \in \Sigma, \) \( \varphi_a^* \) is majorized by \( \varphi_a^* \). Since \( \varphi_a^* \) is convergent in the disk \( \Delta_{R_1}, \) then \( \varphi_a(z, \tilde{z}) \) converges on \( \Delta_{R_1} \times D^{\infty}. \) Put \( \varphi_a \) in the form (2.15), with \( p(\xi, \eta, a) \) in place of \( p(\xi, \eta). \) Then \( p(\xi, \eta, a) \) converges in \( \Delta_{R_1} \times D^{\infty}. \) Therefore, there exists a constant \( m_0, \) say \( m_0 = 4 \tilde{p}(R_1, R_1, e), \) such that for each \( a \in \Sigma \)
\[
p(\xi, \eta, a) \in D^{\infty}(R_1/2, m_0).
\]
Now Theorem 2.3 and Lemma 4.1 give us the required convergence of \( \zeta_j(w, a). \) The proof of Proposition 4.3 is complete. □

5. Proof of Theorem 1.2
From (3.2), one has
\[
p_a(\xi, \eta, p) = \sum_{k=0}^{n-1} p(\xi e^{ik\omega(\eta)}, \eta e^{-ik\omega(\xi, \eta)}) + O(p^2), \quad (5.1)
\]
in which $O(p^2)$ stands for terms of order at least two in variables $p_{i,j}$. Now (3.15) gives

$$h(\zeta, w, p) = \frac{1}{\ln \zeta^{2s}} \sum_{k=0}^{n-1} p(\zeta w e^{ikw(t^2)}, \zeta w^{-1} e^{-ikw(t^2)}) + O(p^2). \quad (5.2)$$

Let $\xi_n(w, p)$ be the solution to equation (3.19) with $j = 2s$. Denote by $K\xi(w, p)$ the constant term of $\xi(w, p)$ with respect to variables $p_{i,j}$, and by $L\xi(w, p)$ the linear part of $\xi(w, p)$. Then from (3.19), we get

$$K\xi(w) = \xi_0, \quad \xi_0 = \left( -\frac{\beta}{n} \right)^{1/2s}. \quad (5.3)$$

Now (5.2) gives

$$L\xi(w, p) = \frac{i}{2sn\zeta^{2s}} \sum_{k=0}^{n-1} p(\zeta_0 w u^k, \zeta_0 w^{-1} u^k), \quad (5.4)$$

with

$$u = e^{i\omega(t^2)}. \quad (5.5)$$

We can now complete the proof of Theorem 1.2. For each $a \in \Sigma$, write

$$\varphi_a = \phi_a \circ T \circ \bar{\phi}_a^{-1} : \begin{cases} \xi' = \xi e^{i\omega(\xi, \eta)}(1 + p(\xi, \eta, a)), \\ \eta' = \eta e^{-i\omega(\xi, \eta)}(1 + p(\xi, \eta, a))^{-1}. \end{cases}$$

From (4.5) and (4.6), we get

$$\bar{\phi}_a^{-1}(\xi, \eta) = (\xi - i\alpha(\xi, \eta), \eta + i\eta(\xi, \eta)) + O(a^2), \quad (5.6)$$

in which $O(a^2)$ stands for terms of order at least two in variables $a_{i,j}$. From $\bar{a}(\xi, \eta) = \tilde{a}(\eta, \xi)$, we have

$$\bar{\phi}_a^{-1}(\xi, \eta) = (\xi + i\alpha(\eta, \xi), \eta - i\eta(\eta, \xi)) + O(a^2). \quad (5.7)$$

Now one gets

$$p(\eta, \xi, \eta) = i\alpha(\xi e^{i\omega(\eta, \xi)}, \eta e^{-i\omega(\xi, \eta)}) + i\tilde{a}(\eta, \xi) + O(a^2). \quad (5.8)$$

Let $\epsilon_0, \delta$ be as in Proposition 4.3. Choose a sequence of positive integers $n_k \to \infty$ such that for some $g_k \in \mathbb{Z}$,

$$n_k \alpha = 2g_k \pi + \beta_k, \quad -\delta < \beta_k < 0.$$ 

Then for each $a \in \Sigma$, the periodic points of $\varphi_a$ in $|\zeta| < \epsilon_0 n_k^{-1/2s}$ form an analytic curve $r = \xi_{n_k}(w, a), |w| = 1$, given by

$$\xi_{n_k}(1 + h(\zeta_{n_k}, w, a))^{1/2s} = \left( -\frac{\beta_k}{n_k} \right)^{1/2s} = \xi_0.$$

By Proposition 4.3, $\xi_{n_k}(w, a)$ converges for $1/2 < |w| < 2$ and $a \in \Sigma$. From (5.4) and (5.5), one sees that with respect to the variables $a_{i,j}$, the linear part of $\xi_{n_k}$ is

$$L\xi_{n_k}(w, a) = -\frac{1}{2snk\zeta_0^{2s}} \sum_{j=0}^{n-1} \tilde{a}(\zeta_0 w u^j, \zeta_0 w^{-1} u^{j+1}) + \tilde{a}(\zeta_0 w^{-1} u^j, \zeta_0 w u^j).$$
Let $H_k(a)$ be the coefficients of $w^{n_k}$ in the Laurent expansion of $\zeta_{n_k}(w, a)$ with respect to $w$. Notice that $u^{n_k} = 1$. Then we have

$$H_k(a) = -\xi_0^{n_k-2s}(a_{n_k,0} + a_{0,n_k})/(2s) + h_k(a),$$

in which $h_k(a)$ contains no linear terms in $a_{j,0}$ and $a_{0,j}$ for all $j$. It is clear that $H_k(a)(k = 1, 2, \ldots)$ are functionally independent.

Fix $a \in \Sigma$. Assume that $H_k(a) \neq 0$. Since $H_k(a)$ is the coefficient of $w^{n_k}$ in the expansion of $\zeta_{n_k}(w, a)$, then $\zeta_{n_k}(w, a)$ is not constant with respect to the variable $w$. Let $I_k = \{\zeta_{n_k}(w, a); |w| = 1\}$. Then $I_k$ is a closed interval of positive length. For each $r \in I_k$, $\varphi_a^{n_k}$ has a fixed point on $|z| = r$. Obviously, $I_k$ intersects any given interval $(0, \varepsilon)$ for large $k$. The proof of Theorem 1.2 is complete. 

6. Complex tangents of real surfaces

It is natural to replace the reality condition of a reversible transformation of $\mathbb{C}^2$ by the reversibility of the transformation with respect to an anti-holomorphic involution. This means that one may ask for the classification of biholomorphic mappings which are reversible through a holomorphic involution as well as an anti-holomorphic involution. Such reversible holomorphic mappings arise from real analytic surfaces with non-degenerate complex tangents, which is the main subject to be discussed in this section.

A real analytic surface $M$ has a complex tangent at $p$ if $T_pM$ is a complex line in $\mathbb{C}^2$. Introduce local coordinates such that $p = 0$, and $M : z_2 = az_1^2 + bz_1\bar{z}_1 + cz_2^2 + O(|z|^3)$. The complex tangent is said to be non-degenerate if $b \neq 0$. Then for a further change of local holomorphic coordinates near 0, one may assume that $M$ is given by (1.3). The $\gamma$ in (1.3) is the Bishop invariant [6]. The complex tangent is said to be elliptic, parabolic or hyperbolic if $0 \leq \gamma < 1/2$, $\gamma = 1/2$ or $\gamma > 1/2$, respectively.

To describe a pair of intrinsic involutions introduced in [11], we consider the complexified surface in $\mathbb{C}^2 \times \mathbb{C}^2$:

$$M^c : z_2 = z_1w_1 + \gamma z_1^2 + \gamma w_1^2 + q(z_1, w_1),$$

$$w_2 = z_1w_1 + \gamma z_1^2 + \gamma w_1^2 + \bar{q}(z_1, w_1).$$

Let $\pi_1(z, w) = w$ and $\pi_2(z, w) = z$ with $z = (z_1, z_2)$, $w = (w_1, w_2)$. Then for $\gamma \neq 0$, $\pi_j|_{M^c}$ is a branched double-covering. The covering transformation $\pi_j|_{M^c}$ gives an involution $\tau_j$. Two involutions $\tau_1$ and $\tau_2$ are conjugate to each other by the anti-holomorphic involution $(z, w) \mapsto (\bar{w}, \bar{z})$ restricted to $M^c$. After a change of local holomorphic coordinates of $M^c \equiv \mathbb{C}^2$, we may assume that $\tau_j(j = 1, 2)$ is given by (2.9).

Let us restrict ourselves to the hyperbolic case. Then we have $\lambda_1 = \lambda = \lambda_2^{-1}$. The Bishop invariant $\gamma$ and $\lambda$ are related by

$$\gamma \lambda^2 - \lambda + \gamma = 0.$$ 

From $\gamma > 1/2$, it follows that $|\lambda| = 1$. We say that $\gamma$ is exceptional if $\lambda$ is a root of unity. Furthermore, the anti-holomorphic involution $(z, w) \mapsto (\bar{w}, \bar{z})$ restricted to $M^c$ is given by

$$\rho(\xi, \eta) = (\tilde{\xi}, \tilde{\eta}).$$

(6.1)
Thus, the reality condition on \( \{\tau_1, \tau_2\} \) is given by

\[
\rho \tau_1 = \tau_2 \rho. \tag{6.2}
\]

The importance of the triple \( \{\tau_1, \tau_2, \rho\} \), described above, is that it characterizes a real analytic surface as follows. A triple \( \{\tau_1, \tau_2, \rho\} \), given by (2.9), (6.1) and (6.2), always generates a real analytic surface (1.3). Moreover, two real analytic surfaces are biholomorphically equivalent if and only if their corresponding pairs of involutions are equivalent through a biholomorphic transformation \( \Phi \) satisfying the reality condition

\[
\Phi \rho = \rho \Phi.
\]

We refer to [11] for details.

For \( a \in D^\infty \), we define an involution

\[
\tau_1 = \phi_a \circ T_1 \circ \phi_a^{-1}, \tag{6.3}
\]

in which \( \phi_a \) is given by (4.4), and

\[
T_1(\xi, \eta) = (e^{i\omega(\xi \eta)/2} \eta, e^{-i\omega(\xi \eta)/2} \xi).
\]

Consider

\[
\varphi \equiv \tau_1 \tau_2, \quad \tau_2 = \rho \tau_1 \rho, \quad T_2 = \rho T_1 \rho. \tag{6.4}
\]

Then as power series in \( \xi, \eta, a \) and \( \bar{a} \), one has

\[
\varphi(\xi, \eta) \sim \phi_a^* T_1^* \psi_a^* \bar{T}_2^* \bar{\psi}_a(\xi, \eta) \equiv \psi(\xi, \eta).
\]

By Lemma 4.2, we know that there is \( 0 < R_2 < 1 \) such that \( \varphi(\xi, \eta) \) converges for \( (\xi, \eta, a, \bar{a}) \in \Delta_{R_2} \times D^\infty \times \bar{D}^\infty \). Put \( \varphi \) in the form (2.15). Thus \( p(\xi, \eta) = p(\xi, \eta, a, \bar{a}) \) converges on \( \Delta_{R_2} \times D^\infty \times \bar{D}^\infty \). By Lemma 4.1, there are positive constants \( \epsilon \) and \( \delta_0 \) such that if \( \beta \) in (2.14) satisfies \( -\delta_0 < \beta < 0 \), then the fixed points of \( \varphi^n \) in the domain \( \Omega_{n,\epsilon} \) are the union of \( s \) holomorphic curves:

\[
\gamma_{n,j} : \xi = \zeta_j w, \quad \eta = \zeta_j w^{-1}, \quad \zeta_j = \zeta_j(w, a, \bar{a}), \quad 1 \leq j \leq s. \tag{6.5}
\]

Moreover, \( \zeta = \zeta_j(w, a, \bar{a}) \) converges for \( 1/2 < |w| < 2 \) and \( (a, \bar{a}) \in D^\infty \times \bar{D}^\infty \).

The following result implies that, in general, the periodic points are not contained in the totally real space

\[
\mathbb{R}^2 : \tilde{\xi} = \xi, \quad \tilde{\eta} = \eta.
\]

We have the following.

**Theorem 6.1.** Let \( \epsilon, \delta_0, n \) and \( \Omega_{n,\epsilon} \) be as above. Then there is a power series \( H(a, \bar{a}) \) converging on \( D^\infty \times \bar{D}^\infty \) such that \( \gamma_{n,j} \) intersects the totally real space \( \mathbb{R}^2 \) at isolated points, provided \( H(a, \bar{a}) \neq 0 \).

**Proof.** For the proof of Theorem 6.1, we need the second-order power expansion of \( \zeta_j(w, a, \bar{a}) \) with respect to variables \( a \) and \( \bar{a} \). In the following discussion, we shall denote by \( Lp \) the linear terms of a power series \( p \) in \( a \) and \( \bar{a} \), and denote by \( Qp \) the second-order terms of \( p \).
To simplify the computation, we first assume that
\[ \tilde{a}(\xi, \eta) = a_{n,0} \xi^n. \]

Let \( \phi_a \) be defined by (4.4). Then one has
\[
\begin{align*}
\phi_a : \\
\xi' &= \xi(1 + i a_{n,0} \xi^n - \frac{1}{2} a_{n,0}^2 \xi^{2n} + O(a_{n,0}^3)), \\
\eta' &= \eta(1 - i a_{n,0} \xi^n - \frac{1}{2} a_{n,0}^2 \xi^{2n} + O(a_{n,0}^3)).
\end{align*}
\]
(6.6)

Denote the inverse of \( \phi_a(\xi, \eta) \) by \( \phi^{-1}_a(\xi, \eta) \). Clearly, we have \( b(\xi, \eta) = a_{n,0} \xi^n + O(a_{n,0}^2) \).

From (6.6), it follows that
\[ b(\xi, \eta) = a_{n,0} \xi^n - i a_{n,0}^2 \xi^{2n} + O(a_{n,0}^3). \]

Thus, one has
\[
\begin{align*}
\phi_a^{-1} : \\
\xi' &= \xi \left(1 - i a_{n,0} \xi^n - \frac{2n}{2} a_{n,0}^2 \xi^{2n} + O(a_{n,0}^3)\right), \\
\eta' &= \eta \left(1 + i a_{n,0} \xi^n + \frac{2n}{2} a_{n,0}^2 \xi^{2n} + O(a_{n,0}^3)\right).
\end{align*}
\]
(6.7)

Noticing that \( \kappa(\xi, \eta) = \xi \eta \) is invariant under \( \tau_1 \), we can put
\[
\tau_1 : \\
\xi' &= e^{i\omega/2} \eta \left(1 + q(\xi, \eta)\right), \\
\eta' &= e^{-i\omega/2} \xi \left(1 + q(\xi, \eta)\right)^{-1}.
\]

From (6.3), (6.6) and (6.7), it follows that
\[ q(\xi, \eta) = ia_{n,0}(\xi^n + e^{i\omega/2} \eta^n) - \frac{1}{2} a_{n,0}^2 (\xi^n + e^{i\omega/2} \eta^n)^2 + na_{n,0}^2 \xi^n (\xi^n - e^{i\omega/2} \eta^n) + O(a_{n,0}^3). \]

From (6.4), we get
\[
\begin{align*}
\tau_2 : \\
\xi' &= e^{-i\omega/2} \eta \left(1 + \bar{q}(\xi, \eta)\right), \\
\eta' &= e^{i\omega/2} \xi \left(1 + \bar{q}(\xi, \eta)\right)^{-1}.
\end{align*}
\]

Put \( \varphi = \tau_1 \tau_2 \) in the form (2.15) with
\[ p(\xi, \eta) = (1 + \bar{q}(\xi, \eta))^{-1}(1 + q \circ \tau_2(\xi, \eta)) - 1. \]

Clearly, one has
\[ Lp(\xi, \eta) = i(e^{i\omega} a_{n,0} \xi^n + \bar{a}_{n,0})\xi^n + i e^{-i\omega/2}(a_{n,0} + \bar{a}_{n,0})\eta^n. \]
(6.8)

Notice that
\[ Q(q \circ \tau_2) = Qq \circ T_2 + L\bar{q}(\eta e^{-i\omega/2} \partial_\xi Lq \circ T_2 - \xi e^{i\omega/2} \partial_\eta Lq \circ T_2). \]

Then we get
\[ Qp = Qq \circ T_2 - Lq \circ T_2 \cdot L\bar{q} - Q\bar{q} + (L\bar{q})^2 + L\bar{q}(\eta e^{-i\omega/2} \partial_\xi Lq \circ T_2 - \xi e^{i\omega/2} \partial_\eta Lq \circ T_2). \]
(6.9)

We now assume that \( 4s \in \mathbb{N} \). Let \( \xi_j(0) \) be the right-hand side of (3.19). Then from (2.14), it is easy to see that
\[ e^{i\omega/2(\xi_j(0))^2} = 1. \]
Thus (6.9) gives

\[ Qp(\xi_j(0)w, \xi_j(0)w^{-1}) = na_n^2(\xi_j(0)w)^{2n} + q_1(a_{n,0}, \bar{a}_{n,0})w^{2n} + e_5(w), \]  

(6.10)
in which, and also in the rest of the discussion, we use \( q_j(a_{n,0}, \bar{a}_{n,0}) \) to denote a polynomial which is symmetric in \( a_{n,0} \) and \( \bar{a}_{n,0} \), and also use \( e_j(w) \) to denote terms of order \(< 2n \) in \( w \), unless it is otherwise stated.

Write \( \phi^n \) in the form (3.1) with \( p_n \) in place of \( p \). From (5.1) and (6.8), we see that the linear part of \( p_n \) with respect to the variables \( a_{n,0} \) and \( \bar{a}_{n,0} \) is given by

\[ Lp_n(\xi, \eta) = i(e^{i\omega a_{n,0} + \bar{a}_{n,0}}\xi^n + e^{i\omega/2}(a_{n,0} + \bar{a}_{n,0})\eta^n \sum_{k=0}^{n-1} e^{-inkw} + e^{i\omega/2}(a_{n,0} + \bar{a}_{n,0})\eta^n \sum_{k=0}^{n-1} -e^{-inkw}. \]  

(6.11)

From (3.2), one gets

\[ Qp_n = \sum_{k=0}^{n-1} (Qp \circ (T_1T_2)^k + Lp \circ (T_1T_2)^k Lp_k) + \sum_{k=0}^{n-1} Lp_k(\xi e^{ikw} \partial_{\xi} Lp \circ (T_1T_2)^k - \eta e^{-ikw} \partial_{\eta} Lp \circ (T_1T_2)^k). \]

From (3.2) and (6.8), it follows that for \( \xi = \xi_j(0)w \) and \( \eta = \xi_j(0)w^{-1} \),

\[ Lp_k(\xi, \eta) = ik(a_{n,0} + \bar{a}_{n,0})(\xi^n + \eta^n). \]  

(6.12)

Hence, we have

\[ Qp_n(\xi_j(0)w, \xi_j(0)w^{-1}) = n^2a_n^2(\xi_j(0)w)^{2n} + q_2(a_{n,0}, \bar{a}_{n,0})w^{2n} + e_7(w). \]

From (6.11), it also follows that

\[ \partial_{\xi} Lp_n(\xi_j(0)w, \xi_j(0)w^{-1}) = -2sn^2\xi_j^{2n+1}(0)a_{n,0}w^n + c_3(a_{n,0} + \bar{a}_{n,0})w^n + e_5(w), \]  

(6.13)
in which \( c_3 \) is a constant and \( e_5(w) \) contains terms of order \(< n \) in \( w \).

By (3.15), one gets

\[ (1 + h(\xi, w))^{1/2} = 1 + \frac{1}{2sn\xi_j^{2m}(0)i} (Qp_n(\xi, \eta) + \partial_{\xi} p_n(\xi, \eta) \cdot L\xi(\xi, \eta)) + c_5 L\xi \cdot Lp_n(\xi, \eta) + c_4 Lp_n^2(\xi, \eta), \]

in which \( c_4 \) is a constant. Then for the solution \( \zeta = \zeta_j(w, a, \bar{a}) \), one has

\[ Q(1 + h(\xi, w))^{1/2} = \frac{1}{2sn\xi_j^{2m}(0)i} (Qp_n(\xi, \eta) + \partial_{\xi} p_n(\xi, \eta) \cdot L\xi(\xi, \eta)) + c_5 L\xi \cdot Lp_n(\xi, \eta) + c_4 Lp_n^2(\xi, \eta), \]

with \( \xi = \xi_j(0)w \) and \( \eta = \xi_j(0)w^{-1} \). Notice that

\[ L\xi = -\bar{\xi}_j(0) \frac{Lp_n(\xi_j(0)w, \xi_j(0)w^{-1})}{2sn\xi_j^{2m}(0)i} = -\bar{\xi}_j(0) \frac{a_{n,0} + \bar{a}_{n,0}}{2s\xi_j^{2m}(0)} \xi_j^n(0)w^n + e_6(w), \]

where \( e_6(w) \) contains terms of order \(< n \) in \( w \). Thus, (6.13) gives

\[ \partial_{\xi} Lp_n \cdot L\xi = n^2a_n^2(\xi_j^{2n}(0)w^{2n} + q_3(a_{n,0}, \bar{a}_{n,0})w^{2n} + e_7(w). \]
Now it is easy to see that the solution $\zeta_j(w, a, \tilde{a})$ to (3.19) can be written as

$$Q\zeta_j = \frac{i n \xi_j^{2n-2n+1}(0)}{s} (a_{n,0}^2 + h_{n,j}(a, \tilde{a})) w^{2n} + e_k(w),$$

(6.14)
in which $h_{n,j}(a, \tilde{a})$ is a power series converging for $(a, \tilde{a}) \in D^\infty \times D^\infty$. Furthermore, $h_{n,j}(a, \tilde{a})$ starts with the quadratic terms which are symmetric in $a_{n,0}$ and $\tilde{a}_{n,0}$, if $\tilde{a}(\xi, \eta) = a_{n,0} \xi^n$.

To complete the proof of Theorem 6.1, we put

$$\zeta_j(w, a, \tilde{a}) = \sum_{k=-\infty}^{\infty} \zeta_{j,k}(a, \tilde{a}) w^k.$$

Assume that $\gamma_{n,j} \cap \mathbb{R}^2$ has an accumulating point in $\Omega$. Then the intersection is a real analytic curve. Notice that $(\xi, \eta) \in \mathbb{R}^2$ implies that $\zeta$ and $w$ are both real or pure imaginary. Hence, we get $\tilde{\xi}_{j,k}(a, \tilde{a}) = \zeta_{j,k}(a, \tilde{a})$, for $w = \tilde{w}$; or $\tilde{\xi}_{j,k}(a, \tilde{a}) = \xi_{j,k}(a, \tilde{a})$, for $w = -\tilde{w}$. In particular, we see that $\xi_{j,2n}(a, \tilde{a})/\xi_{j,0}(a, \tilde{a})$ is real. Since $\xi_{j,2n-2n}(0)$ is real for $s | n$, then (6.14) gives

$$a_{n,0}^2 + \tilde{a}_{n,0}^2 + 2h_{n,j}(a, \tilde{a}) = 0,$$

in which $h_{n,j}$ is the real part of $h_{n,j}(a, \tilde{a})$. Put

$$H_n(a, \tilde{a}) = \prod_{j=0}^{2n-1} (a_{n,0}^2 + \tilde{a}_{n,0}^2 + 2h_{n,j}(a, \tilde{a})).$$

Since $h_{n,j}(a, \tilde{a})$ is symmetric in $a_{n,0}$ and $\tilde{a}_{n,0}$ for $a(\xi, \eta) = a_{n,0} \xi^n$, then one has $H_n(a, \tilde{a}) \neq 0$. Therefore, Theorem 6.1 is proved.

Analogous to Theorem 1.1, we have the following.

**Corollary 6.2.** There is a sequence of power series $H_k(a, \tilde{a})$ such that the transformation $\varphi_k$, defined by (6.3) and (6.4), cannot be transformed into the normal form (2.7) through any convergent transformation satisfying the reality condition, provided $H_k(a, \tilde{a}) \neq 0$ for infinitely many $k$.

For the proof, we keep the notation of Theorem 6.1. Choose a sequence of positive integers $n_k (k = 1, 2, \ldots)$ such that for each $n = n_k$, the number $\beta$ in (2.14) satisfies $-\delta < \beta < 0$. We now let $H_k(a, \tilde{a})$ be the power series $H_n(a, \tilde{a})$ constructed at the end of the proof of Theorem 6.1. From the normal form (2.7), one can see that in any small deleted neighborhood of the origin in $\mathbb{R}^2$, $\varphi$ has a continuum of periodic points with period $n_k$ for $k$ large. Therefore, $\varphi$ cannot be transformed into the normal form (2.7). Finally, without giving details, we mention that the sequence of power series $\{H_k(a, \tilde{a})\}$ constructed above is indeed functionally independent.

Theorem 1.3 is a consequence of Corollary 6.2 and the intrinsic property of pairs of involutions described early in this section.
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