Math 221  
Worksheet: Inverse functions  
September 3, 2015

What is the inverse of a function? Technically, the inverse of a function \( f(x) \) (if it even has one!) is a function \( g(x) \) with the property that, for all \( x \) in the domain of \( f(x) \), we have \( g(f(x)) = x \), and for all \( y \) in the range of \( f \), we have \( g(f(x)) = y \). We denote the inverse of \( f(x) \) by \( f^{-1}(x) \).

Informally, \( f^{-1} \) is the function whose inputs and outputs are the same as those of \( f \), only switched. For example, if \( f \) is a function that has an inverse (such a function is called invertible), then if \( f(3) = 1 \), we know that \( f^{-1}(1) = 3 \). A function is invertible if, for any value \( y \) that is an output of \( f \), there is only one input that gives that output. (Otherwise, what would \( f^{-1}(y) \) be? There can only be one value for this, or else \( f^{-1} \) isn’t a function!)

1. The graph of \( y = f(x) \) is shown. What is \( f^{-1}(2) \)?

2. Find the domain and range of each function, and draw a rough graph. Then find the function’s inverse, find its domain and range, and draw this graph on the same set of axes.
   
   (a) \( f(x) = x^3 + 5 \)
   
   (b) \( g(x) = \sqrt{1 - x} \)
   
   (c) \( h(x) = \sqrt[3]{1 - x^3} \) (What is unusual about this function and its inverse?)

3. Draw a nice graph of \( y = \sin(x) \). Notice that, if we restrict our domain to \([ -\pi/2, \pi/2 ] \), the sine function is one-to-one (explain why). Use this to draw a graph of the inverse of \( y = \sin(x) \), when the domain of \( \sin(x) \) is restricted to \([ -\pi/2, \pi/2 ] \). This inverse is called arcsin, or \( \sin^{-1} \). What are its domain and range?

4. Repeat the previous problem with \( y = \tan(x) \) instead of \( y = \sin(x) \). This time, you choose the domain restriction by looking at the graph of the tangent function.

5. What is...

   (a) \( \arcsin(\sin(\pi/4)) \)
   
   (b) \( \arcsin(\sin(5\pi/4)) \)
   
   (c) \( \sin(\arcsin(\sqrt{2}/2)) \)
   
   (d) \( \sin(\arctan(\sqrt{2}/3)) \)

6. (a) Let \( f(x) = \arcsin(\sin(x)) \). What are the domain and range of \( f \)? Can you simplify the formula for this function?

   (b) Repeat part (a) with \( f(x) = \sin(\arcsin(x)) \).