SPLITTINGS of LINEAR OPERATORS: LYAPUNOV and PERRON-FROBENIUS

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Théorème I: Si les équations differentiélles du mouvement troublé sont telles qu'il est possible de trouver une fonction définie V, don't la dérivée V' soit une fonction de signe fixe et contraire á celui de V, ou se réduise identiquement à zéro, le mouvement non troublé est stable.

M. A. Liapunoff (Lyapunov): Problème général de la stabilité du mouvement
Russian 1892, French 1907, 1947,
English ?

GANTMACHER (1952?)

$$\dot{x} = Ax, \ A \in \mathbb{C}^{nn}$$

\$ A a constant matrix

$$x(t) = e^{At} x_0$$

$x(t) \longrightarrow 0 \,\text{as}\, t \longrightarrow \infty \Leftrightarrow \Re(\alpha) < 0, \ \text{all}\, \alpha \in \text{spec}(A)$

\$ V a homogeneous polynomial of degree 2.
\$ = Special Case

$$V(x) = x^* H x, \quad H^* = H$$

$$\dot{V}(x) = \dot{x}^* H x + x^* H \dot{x} = x^*(A^* H + HA)x$$

$$H \succ 0$$

$$-(A^* H + HA) \succ 0$$

$$H \succ 0 := H \text{ positive definite}$$

"Lyapunov's Theorem": Let $A \in \mathbb{C}^{nn}$. Then there exists $H \gtrdot 0$ such that $AH + HA^* \gtrdot 0$ if and only if A is stable (i.e has all eigenvalues in the *right* half plane).

Gantmacher's Theorem: Let $A \in \mathbb{C}^{nn}$ and let $K \gtrdot 0$. Then there exists $H \gtrdot 0$ such that $AH + HA^* = K$ if and only if A is stable (i.e has all eigenvalues in the *right* half plane).

Change:
Lyapunov : Existence
Gantmacher: Solving equations (for all)

$$\forall K \gtrdot 0,\ \exists H \gtrdot 0,\ AH + HA^* = K$$

$$\Longleftrightarrow$$

$$\exists H \gtrdot 0,\ AH + HA^* \gtrdot 0$$

MESSAGE

$$\forall \Longleftrightarrow \exists$$

WHERE BEFORE?

Define

$$P > 0 := p_{ij} > 0, \text{ all } (i,j)$$

$$P \geq 0 := p_{ij} \geq 0, \text{ all } (i,j)$$

$$\rho(A) = \max\{|\lambda| : \lambda \in \text{spec}(A)\}$$

By Perron-Frobenius, $\rho(P)$ is an eigenvalue of P, $P \geq 0$

Theorem 1: Let $A = \sigma I - P$ where $P \geq 0$. Then TFAE:

1. $\sigma > \rho(P)$
2. $\forall y > 0, \ \exists x > 0, \ Ax = y$
   $(A^{-1} > 0)$
3. $\exists y, \ \exists x > 0, \ Ax = y$
   $(\exists x > 0, \ Ax > 0)$

$$\forall \iff \exists$$

Varga+ (Regular splitting)

Theorem 1': Let $A = M - N$ where $M^{-1} \geq 0,\ N \geq 0$. Then TFAE:

1. $\rho(M^{-1}N) < 1$
2. $\forall y > 0,\ \exists x > 0,\ Ax = y$
   $(A^{-1} > 0)$
3. $\exists y,\ \exists x > 0,\ Ax = y$
   $(\exists x > 0,\ Ax > 0)$

$$\forall \iff \exists$$

PROBLEM:
FIND UNIFIED TREATMENT

Positive Operators and an Inertia Theorem
Num. Math. 7: 11–17 (1965)

MESSAGES

A. Perron–Frobenius $\Longrightarrow$ Lyapunov

B. $\exists \Longleftrightarrow \forall$

Definition: A subset $\mathcal{C}$ of a (finite dimensional) space V (over $\Re$) is a pointed, full, closed)) *cone* if

1. $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$,
   viz. $x + y \in \mathcal{C}, \ \forall x, y \in \mathcal{C}$
2. $\Re_+ \mathcal{C} \subseteq \mathcal{C}$,
   viz. $\alpha x \in \mathcal{C}, \ \forall \alpha \geq 0, x \in \mathcal{C}$
3. $\mathcal{C} \cap -\mathcal{C} = \{0\}$
   viz. $x, -x \in \mathcal{C} \Rightarrow x = 0$
4. $\mathcal{C} - \mathcal{C} = V$,
   $(\forall z \in V, \exists x, y \in \mathcal{C}, z = x - y)$
   or $\mathcal{C}^0 \neq \ \phi$
5. $\mathcal{C}$ is closed

Redefine for $x \in V$

$$x \geq 0 \ : \ x \in \mathcal{C}$$

$$x > 0 \ : \ x \in \mathcal{C}^0$$

Redefine for $T \in \text{Hom}(V)$

$$T \geq 0 : T\mathcal{C} \subseteq \mathcal{C}$$

$$T > 0 : T\mathcal{C} \subseteq \mathcal{C}^0$$

Perron–Frobenius (Krein–Rutman) applies:

$$T \geq 0 : \rho(T) \in \text{spec}(T)$$

Example

Orthant $V = \Re^n$, $\mathcal{C} = \Re^n_+$

$x \geq 0 : x_i \geq 0, \ i = 1, \ldots, n$

$A \geq 0 : \ a_{ij} \geq 0, \ i, j = 1, \ldots, n$

$$R^{-1} \geq 0 \Longleftrightarrow R\mathcal{C}^0 \supseteq \mathcal{C}^0$$

Theorem 2: Let $\mathcal{C}$ be a cone. Let $T = R - S \in$ Hom(V):

$T = R - S,\ R\mathcal{C}^0 \supseteq \mathcal{C}^0$ or $R\mathcal{C}^0 \cap \mathcal{C}^0 = \emptyset,\ S \geq 0.$

TFAE:

1. $R^{-1} \geq 0,\ \rho(R^{-1}S) < 1$
2. $T^{-1}\mathcal{C}^0 \subseteq \mathcal{C}^0 \ \ (T^{-1} \geq 0)$
3. $T\mathcal{C}^0 \cap \mathcal{C}^0 \neq \phi$

$$\text{MESSAGE}: \quad \forall \Longleftrightarrow \exists$$

Example:

$$V = \mathcal{H}_n,\ \mathcal{C} = \mathcal{P}_n$$

$\mathcal{H}_n =$ real space of $n \times n$ Hermitians

$\mathcal{P}_n =$ cone of $n \times n$ positive semidefinites

typical $S \geq 0$: $\quad S(H) = \Sigma_k C_k^* H C_k$

$$S = \Sigma_k C_k \times \bar{C}_k$$

$R(H) = AHA^*,\ A$ nonsingular:

$R \geq 0, R^{-1} \geq 0$ Sylvester

$$T(H) = R(H) - S(H) = AHA^* - \Sigma_k C_k^* H C_k$$
$$(R^{-1}S)(H) = \Sigma_k A^{-1} C_k^* H C_k A^{-1*}$$

Theorem 3: Let A, C_k, $k = 1, \ldots, s$ be complex $n \times n$ matrices. Then the following are equivalent:

1. A is nonsingular and
$$\rho((R^{-1}S) < 1$$

2. For all $K \succ 0$, there exists a unique $H \succ 0$ such that
$$T(H) = K.$$

3. There exists an $H \succ 0$ such that
$$T(H) \succ 0.$$

$A_k, C_k \in \mathbb{C}^{nn}, k = 1, \ldots, s$ commute in pairs
$\Longrightarrow$
$\exists Q,\ Q^{-1} C_k Q$ triangular
("simultaneous triangulation")
$\Longleftrightarrow$
$\exists$ ordering of the eigenvalues $\alpha_i, \gamma_i^{(k)}$, such that the eigenvalues of of $p(A, C_1, \ldots, C_s)$ are
$p(\alpha_i, \gamma_i^{(1)}, \ldots, \gamma_i^{(s)}),\ i = 1, \ldots, n$.
"natural corrrespondence"

$$\underline{\gamma_i} := (\gamma_i^{(1)}, \ldots \gamma_i^{(s)}),\ i = 1, \ldots, s$$

Theorem 4: Let $A,\ C_k,\ k=1,\ldots,s$ be complex $n \times n$ matrices which can be simultaneoulsy triangulated. Suppose the eigenvalues of $A,\ C_k$ under a natural correspondence are $\alpha_i,\ \gamma_i^{(k)}$, $i=1,\ldots,\ k=1,\ldots,s$. For Hermitian H, let

$$T(H) = AHA^* - \Sigma_{k=1}^s C_k H C_k^*.$$

Then the following are equivalent:

1. $\epsilon_i := |\alpha_i|^2 - \Sigma_{k=1}^s |\gamma_i^{(k)}|^2 > 0,\ i=1,\ldots,n$
2. For all $K \gtrdot 0$, there exists a unique $H \gtrdot 0$ such that T(H) = K.
3. There exists an $H \gtrdot 0$ such that $T(H) \gtrdot 0$.

Notes:

A. Simultaneous triangulability is needed for Condition 1. only.

B. We apply Cauchy's inequality to bound the spectrum of $\Sigma_{k=1}^s A^{-1} C_k \times \bar{A}^{-1} \bar{C}_k$:

$$\{\Sigma_{k=1}^s \alpha_i^{-1} \gamma_i^{(k)} \bar{\alpha_j} \bar{\gamma}_j^{(k)} : i,j = 1,\ldots,n\}.$$

Special case:

$$T_\ell = (B+I)H(B+I)^* - (BHB^* + IHI^*)$$

$$T_\ell = BH + HB^*$$

$$\epsilon_i = \beta_i + \bar{\beta}_i$$

Gantmacher-Lyapunov Theorem: Let B be a complex $n \times n$ matrix. TFAE:

1. $$\Re(\beta) > 0, \text{ all } \beta \in \operatorname{spec}(B)$$
2. For all $K \succ 0$, there exists a unique $H \succ 0$ such that $BH + HB^* = K$.
3. There exists an $H \succ 0$ such that $BH + HB^* \succ 0$.

Special case:

$$T = IHI^* - CHC^*$$
$$\epsilon_i = 1 - |\gamma_i|^2$$

Stein's Theorem: Let C be a complex $n \times n$ matrix. TFAE:

1. $\rho(C) < 1$
2. For all $K \gtrdot 0$, there exists a unique $H \gtrdot 0$ such that $H - CHC^* = K$.
3. There exists an $H \gtrdot 0$ such that $H - CHC^* \gtrdot 0$.

Recent developments
R resolvent nonnegative:

$$\exists \alpha_0,\ \forall \alpha > \alpha_0, (\alpha I - R)^{-1} \geq 0$$

$$R^{-1} \geq 0 \Longrightarrow R \text{ res nonneg}$$

THM 5: (Elsner 1970, S-Vidyasagar 1970). TFAE:

1. R resolvent nonneg
2. $\exp(tR) \geq 0,\ t \geq 0$
3. $y \in \mathcal{C}^*, x \in \mathcal{C}, y^* x = 0 \Longrightarrow y^* R x \geq 0$
   R cross positive

Lemma (? Tam-S 2006):

$-R$ res nonneg $\Longrightarrow R\mathcal{C}^0 \supseteq \mathcal{C}^0$ or $R\mathcal{C}^0 \cap \mathcal{C}^0 = \emptyset$

Converse false

$$K = \Re^2_+$$

$$R = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$R^{-1} \geq 0, \quad (\alpha I + R)^{-1} \not\geq 0\,, \alpha > 0$$

THM 6: (Damm-Hinrichsen 2003):
$T = R - S, -R$ res nonneg, $S \geq 0$.
TFAE:

1. $R^{-1} \geq 0,\ \rho(R^{-1}S) < 1$
2. $T^{-1}\mathcal{C}^0 \subseteq \mathcal{C}^0 \ \ (T^{-1} \geq 0)$
3. $T\mathcal{C}^0 \cap \mathcal{C}^0 \neq \phi$
4. T is pos stable, viz spec$(T) \subseteq \mathbb{C}_+$
5. R is pos stable and $\rho(R^{-1}S) < 1$

Example

$$K = \Re^2_+$$

$$R = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T = R - S = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$T^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad R^{-1}S = \frac{1}{4}R$$

1. $R^{-1} \geq 0, \quad \rho(R^{-1}S) = \frac{1}{2}$
2. $T^{-1} \geq 0$
3. $u = [1\ 1]^*,\ Tu > 0$
4. $\mathrm{spec}(T) = \{-3, 1\}$ FALSE
5. $\mathrm{spec}(R) = \{-2, 2\}$ FALSE

$\alpha I + R^{-1}$ not nonneg for $\alpha > 0$

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THAT'S IT   THANKS FOR LISTENING