NORTH-HOLLAND

Perron-Frobenius Theory Over Real Closed Fields and Fractional Power Series Expansions*

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ABSTRACT

Some of the main results of the Perron-Frobenius theory of square nonnegative matrices over the reals are extended to matrices with elements in a real closed field. We use the results to prove the existence of a fractional power series expansion for the Perron-Frobenius eigenvalue and normalized eigenvector of real, square, nonnegative, irreducible matrices which are obtained by perturbing a (possibly reducible) nonnegative matrix. Further, we identify a system of equations and inequalities whose solution

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yields the coefficients of these expansions. For irreducible matrices, our analysis assures that any solution of this system yields a fractional power series with a positive radius of convergence.

1. INTRODUCTION

In this paper we obtain convergent fractional power series expansions for the Perron-Frobenius eigenvalue and normalized eigenvector for real, square, nonnegative, irreducible matrices which are obtained by perturbing (not necessarily irreducible) nonnegative matrices. Further, we present a nonlinear system of equations and inequalities whose solution yields the coefficients of these expansions (under linear perturbations), and show that any solution of this system yields a fractional power series with a positive radius of convergence. The main tool in our analysis is the extension of the Perron-Frobenius theory to matrices with elements in the field of convergent Puiseux series which is a real closed field. In turn, these results are derived from an extension of the Perron-Frobenius theory to nonnegative matrices with elements in an arbitrary real closed field.

Let P and B be two $n \times n$ real matrices such that for all sufficiently small positive ε, the matrix $P + \varepsilon B$ is nonnegative and irreducible. The Perron-Frobenius theory assures that for such ε, the spectral radius of $P + \varepsilon B$, which we denote $\rho(P + \varepsilon B)$, is a simple eigenvalue of $P + \varepsilon B$ having a corresponding positive eigenvector. Results of Kato (1966) show the existence of fractional power series expansions of eigenvalues of $P + \varepsilon B$, suggesting that there is a fractional power series expansion for $\rho(P + \varepsilon B)$ of the form $\sum_{k=0}^{\infty} \rho_k \varepsilon^{k/p}$. In particular, once a normalization condition is determined, there is a unique eigenvector $u(\varepsilon)$ of $P + \varepsilon B$ corresponding to $\rho(P + \varepsilon B)$, and the results of Kato suggest that this eigenvector has a fractional power series expansion in fractional powers of ε, say $\sum_{k=0}^{\infty} u_k \varepsilon^{k/p}$. Here, we establish these expansions by using the Perron-Frobenius theory over the real closed field of real convergent Puiseux fractional power series with finite radius of convergence. We also identify a nonlinear system of equations and inequalities over the reals which characterize the coefficients of such expansions. In particular, we prove that every solution of this set of equations defines a fractional power series with a positive radius of convergence.

The Perron-Frobenius eigenvalue and normalized eigenvector yield important characteristics of dynamic systems that are governed by nonnegative transition matrices; see the many examples in Berman and Plemmons (1979). In particular, expansions of these characteristics for perturbed transition matrices are useful for sensitivity analysis. For example, perturbed stochastic matrices were studied by Schweitzer (1986) and Meyer and Stewart (1988), and perturbations of general (not necessarily stochastic) nonnegative matrices were explored by Cohen (1978), Deutsch and Neumann (1984), and Haviv, Ritov, and Rothblum (1992) among others. All previous schemes for computing explicit expansions of the Perron-Frobenius eigenvalue and corresponding normalized eigenvector with which we are familiar are confined to regular power series, rather than fractional power series; see, for example, Schweitzer (1986) and Haviv, Ritov, and Rothblum (1992).

It has long been observed that the Perron-Frobenius theory does not extend to arbitrary ordered fields and that ordered fields over which corresponding results hold must satisfy extra structure; see Barker and Schneider (1975). The main tool for proving our results about fractional power series expansions is the extension of the Perron-Frobenius theory to nonnegative square matrices over real closed fields, in particular, over the Puiseux field of fractional power series. This is done by observing that results of the Perron-Frobenius theory can be expressed with closed formulae in the language of ordered fields and by applying the Tarski-Seidenberg principle, which asserts that the validity of a closed formula over all real closed fields follows from its validity over the reals.

In Section 2 we summarize preliminaries about ordered fields, the corresponding predicate language, and the Tarski-Seidenberg principle. In Section 3, we extend some of the main results of the classical Perron-Frobenius theory to nonnegative matrices over real closed fields. The results are applied in Section 4 to Puiseux fields of formal fractional power series over real closed fields. In Section 5 we consider the real convergent Puiseux field, which is the real closed subfield of the formal Puiseux field over the reals consisting of power series with positive radius of convergence. The earlier results are then used to obtain expansions of the Perron-Frobenius eigenvalue and normalized eigenvector of a nonnegative, irreducible perturbation of real matrices. In Section 6, we relate results of Barker and Schneider (1975) concerning the existence of Perron-Frobenius eigenvalues over ordered fields to the results of Section 3. Finally, we discuss our results and some possible extensions in Section 7.

2. ORDERED FIELDS AND THE TARSKI-SEIDENBERG PRINCIPLE

Essential to our development are ordered fields and closed formulae which are used to express certain facts about ordered fields. These concepts are standard in mathematics, and we describe them here briefly; for a more complete development see MacLane and Birkhoff (1967) or Enderton (1970), for example.

As usual, a *field* is a set F containing three distinct elements -1 (*negative one*), 0 (*zero*), and 1 (*one*) and two binary operations $+$ (*addition*) and $\cdot$ (*multiplication*) defined over pairs of elements of F, where the following conditions hold: 0 is the addition identity, 1 is the multiplication identity, -1 is the additive inverse of 1, addition and multiplication are both associative and commutative, multiplication is distributive over addition, all elements in F have an additive inverse, and all elements in $F \setminus \{0\}$ have a multiplicative inverse. An *ordered field* is a field F with a binary relation $>$ (*bigger than*) which is a total order and is preserved under addition and under multiplication by positive elements; that is, for a, b, and c in F with $a > b$ one has that $a + c > b + c$ and if $c > 0$ then $ac > bc$. Throughout we use standard notation for fields and ordered fields; for example, the multiplicative inverse of an element $c \neq 0$ is denoted c^{-1}. We identify fields and ordered fields with their underlying sets.

An *atomic formula* is an expression of the form $p = q$, $p \geqslant q$, $p \neq q$, or $p > q$, where p and q are polynomials with integer coefficients. *Formulae* are expressions constructed by a finite number of steps from atomic formulae via the application of the connectives $\rceil$ (*negation*), $\wedge$ (*and*), $\vee$ (*or*), and the quantifiers $\exists$ (*there exists*) and $\forall$ (*for every*) as follows: all atomic formulae are formulae, and if φ and ψ are formulae, then so are $\rceil\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists x\ \varphi$, and $\forall x\ \varphi$, where x is any variable. A variable x is *free* in a formula φ if nowhere in the formula is it modified by a quantifier $\exists$ or $\forall$. The free variables x of a formula φ are sometimes displayed, and we write $\varphi(x)$ interchangeably with φ. A *closed formula* is a formula with no free variables. For example, φ defined by

$$\varphi\colon \big((y < 0) \vee \big(\exists x\ \big((x^2 = y) \wedge (x \geqslant 0)\big)\big)\big) \tag{2.1}$$

is a formula with free variable y, and we sometimes write $\varphi(y)$. Also,

$$\psi\colon \forall y\ \big((y < 0) \vee \big(\exists x\ \big((x^2 = y) \wedge (x \geqslant 0)\big)\big)\big) \tag{2.2}$$

is a closed formula. A formula is called *quantifier-free* if is has no quantifiers.

To enhance readability, we use matrix and vector notation whenever convenient; for example, we refer to the formula $\exists x\ (Ax = b)$, where A is a $m \times n$ array of variables and x and y are $n \times 1$ and $m \times 1$ arrays of variables, respectively. The number of elements in a vector y whose coordinates are variables or elements in a given field is denoted $\#y$. Also, we use the connective $\rightarrow$ (*implies*).

An *extension* of a field F is a field G which contains F where addition and multiplication in F are, precisely, the restriction of those operators in G. In this case we also say that F is a *subfield* of G. Similarly, we use extensions of ordered fields and refer to *ordered subfields* and *ordered field extensions*. We note that every ordered field is an ordered field extension of an isomorphic copy of the *rational ordered field* Q.

A closed formula φ determines a mathematical statement over each ordered field F, with the interpretation that the variables take values in F. We say that φ is *satisfied* or is *not satisfied over* an ordered field F if the corresponding mathematical statement is, respectively, true or false over F. A closed formula can be satisfied over one ordered field but false over another. For example, the closed formula ψ defined by (2.2) is true over the reals but false over the rationals, and vice versa for $\rceil\psi$. Similarly, given a formula $\varphi(y)$ and an assignment of the variables to elements of an ordered field F, we obtain a mathematical statement which has an interpretation over any given ordered field extension G of F. If under such an assignment the vector $y = (y_1, \ldots, y_n)$ of free variables of φ is assigned to the vector $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \in F^n$, we say that $\varphi(\bar{y})$ is *satisfied* or is *not satisfied* over G if the corresponding mathematical statement over G is, respectively, true or false. We say that two formulae are *equivalent* over an ordered field F if they have the same set of free variables and if one is true under a corresponding assignment of variables to the elements of F if and only if the other is also.

A field F is called *algebraically closed* if every polynomial with coefficients in F has a root. A *real closed field* is an ordered field F satisfying the following two conditions:

(1) every positive element in F has a square root, and
(2) any polynomial of odd degree with coefficients in F has a root in F;

see Jacobson (1964, pp. 273–277). For example, the *real field* R is a real closed field.

Every field is contained in an algebraically closed field, and the smallest such field is called its *algebraic closure*; see Jacobson (1964, pp. 142–145). Also, every ordered field has a real closed ordered field extension, and the smallest such extension is called its *real closure*; see Jacobson (1964, pp. 284–286). A necessary and sufficient condition for an ordered field F to be real closed is that its extension by the square root of -1, denoted ι, results in an algebraically closed field, that is, if and only if $F + \iota F$ is algebraically closed; see Jacobson (1964, Theorem 5 and 6, pp. 275–276).

Tarski (1951) constructed an algorithm for deciding in a finite number of steps whether a closed formula is satisfied over the reals, that is, whether a given closed formula is true when the variables appearing in it are thought of as real numbers. Tarski's algorithm eliminates from a formula all quantifiers $\exists$ and $\forall$ and variables they quantify, while introducing no new variables. The elimination method leads to an equivalent quantifier-free formula. Tarski originally applied his decision method to the reals. Seidenberg (1954) observed that the properties of the reals used in the decision method are properties of every real closed field. The following result follows from Tarski's algorithm; see Tarski (1951) and Seidenberg (1954) for details.

THEOREM 2.1. *Let φ be a formula. Then there exists a quantifier-free formula ψ which is equivalent to φ over every real closed field, and ψ has the same free variables as φ.*

The first two corollaries of Theorem 2.1 are commonly referred to as the Tarski-Seidenberg principle; see Seidenberg (1954) or Jacobson (1964, pp. 307, 313, 314).

COROLLARY 2.2. *Let F be a real closed field, $\varphi(x)$ be a formula, and $\bar{x} \in F^{\#x}$. If $\varphi(\bar{x})$ is satisfied over one real closed extension of F, then $\varphi(\bar{x})$ is satisfied over all real closed extensions of F.*

Proof. By Theorem 2.1, φ can be replaced by a quantifier-free formula ψ, and the conclusion of the corollary for such a formula is trivial, as the testing of whether or not $\psi(\bar{x})$ is satisfied over any ordered field extension of F requires a finite number of tests of statements of the form $a = b$ or $a > b$, where a and b are elements of F. ■

COROLLARY 2.3. *Let φ be a closed formula. Then φ is true over one real closed field if and only if it is true over every real closed field.*

Corollary 2.2 shows that if $\varphi(x)$ is a formula and $\bar{x} \in F^{\#x}$, then $\varphi(\bar{x})$ is true over F if and only if $\varphi(\bar{x})$ is true over every real closed extension of F; in this case we say that $\varphi(\bar{x})$ is *satisfied*. Similarly, we say that a closed formula is satisfied if it is satisfied over the real closure of the rationals (the smallest real closed field). Given a formula $\varphi(x, y)$, a field F and a vector $\bar{y} \in F^{\#y}$, we denote the set of vectors $\bar{x} \in F^{\#x}$ for which $\varphi(\bar{x}, \bar{y})$ is satisfied by $\bar{X}(F, \varphi, \bar{y})$.

COROLLARY 2.4 (INVARIANCE OF FINITE SOLUTION SETS). *Let F be a real closed field, $\varphi(x, y)$ be a formula, and $\bar{y} \in F^{\#y}$. If $\bar{X}(G, \varphi, \bar{y})$ is finite for some real closed extension of F, then $\bar{X}(G, \varphi, \bar{y})$ is invariant over all real closed extensions of F.*

Proof. Let G and H be two real closed extensions of F. For each nonnegative integer k, the assertion that there are exactly k distinct vectors $\bar{x}$ for which $\varphi(\bar{x}, \bar{y})$ is true can be expressed by the formula $\psi_k(y)$ given by

$$\exists x^1, x^2, \dots, x^k \left(\left(\bigwedge_{j=1}^{k} \varphi(x^j, y) \right) \wedge \left(\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{k} (x^i \neq x^j) \right) \right.$$

$$\left. \wedge \left(\forall x \left(\varphi(x, y) \rightarrow \bigvee_{j=1}^{k} (x = x^j) \right) \right) \right).$$

By Corollary 2.3, satisfiability of each $\psi_k(\bar{y})$ over F, G, and H is the same, implying that the cardinalities of $\bar{X}(F, \varphi, \bar{y})$, $\bar{X}(G, \varphi, \bar{y})$, and $\bar{X}(H, \varphi, \bar{y})$ coincide. By Corollary 2.2, we have that $\bar{X}(G, \varphi, \bar{y}) \supseteq \bar{X}(F, \varphi, \bar{y})$ and $\bar{X}(H, \varphi, \bar{y}) \supseteq \bar{X}(F, \varphi, \bar{y})$. Combining these facts, we conclude that $\bar{X}(F, \varphi, \bar{y}) = \bar{X}(G, \varphi, \bar{y}) = \bar{X}(H, \varphi, \bar{y})$. ■

COROLLARY 2.5. *Let F be a real closed field, $\varphi(x, y)$ be a quantifier-free formula, and $\bar{y} \in F^{\#y}$. If $\bar{X}(H, \varphi, \bar{y})$ is finite for some real closed extension H of F, then $\bar{X}(G, \varphi, \bar{y})$ is a subset of $\bar{X}(F, \varphi, \bar{y})$ for every ordered field extension G or F.*

Proof. Suppose $\bar{X}(H, \varphi, \bar{y})$ is finite for some real closed extension H of F, and let G be an ordered field extension of F. By Jacobson (1964, pp. 142–145), G has a real closed extension, say H', and as φ is quantifier-free $\bar{X}(H', \varphi, \bar{y}) \supseteq (G, \varphi, \bar{y})$. By Corollary 2.4, $\bar{X}(F, \varphi, \bar{y}) = \bar{X}(H', \varphi, \bar{y})$; hence, $\bar{X}(F, \varphi, \bar{y}) \supseteq \bar{X}(G, \varphi, \bar{y})$. ■

COROLLARY 2.6. *Let F be a real closed field, G an ordered field extension of F, $p(x, y)$ a polynomial with integer coefficients where $\#x = 1$, and $\bar{y}$ a vector in $F^{\#y}$. If $\bar{x}$ in G satisfies $p(\bar{x}, \bar{y}) = 0$, then $\bar{x}$ is in F.*

Proof. Euclid's algorithm implies that there are only finitely many elements $\bar{x}$ in F for which $p(\bar{x}, \bar{y}) = 0$. The corollary now follows directly from Corollary 2.5. ■

3. PERRON-FROBENIUS THEORY OVER REAL CLOSED FIELDS

In this section we extend some of the main results of the Perron-Frobenius theory to nonnegative matrices over real closed fields. Let F be a given ordered field. A vector a with elements in F is called *nonnegative* or *positive*, written $a \geqslant 0$ or $a \gg 0$, if all of its coordinates are nonnegative or positive, respectively. A vector a is called *semipositive*, written $a > 0$, if $a \geqslant 0$ and $a \neq 0$. Corresponding definitions apply to matrices. An $n \times n$ nonnegative matrix P is called *irreducible* if $\sum_{i=0}^{n-1} P^i \gg 0$. We note that the 1×1 zero matrix is nonnegative and irreducible; when we wish to exclude this case we refer to *square, semipositive*, irreducible matrices.

Let F be a given real closed field, and let $A \in F^{n \times n}$. We say that a pair of elements (α, β) is a *eigenvalue* of A if there exist vectors $v, w \in F^n$ such that

$$Av = \alpha v - \beta w \quad \text{and} \quad Aw = \alpha w + \beta v, \tag{3.1}$$

and in this case we call the pair (v, w) an *eigenvector of A corresponding to* the eigenvalue (α, β), or briefly an *eigenvector* of A. We observe that (3.1) is equivalent to the usual assertion that $\alpha + \iota\beta$ and $u + \iota v$ are standard eigenvalue and corresponding eigenvector of A in the algebraic closure of F, namely, $A(u + \iota v) = (\alpha + \iota\beta)(u + \iota v)$. If $\alpha \in F$ and $(\alpha, 0)$ is an eigenvalue of A, we simply say that α is a eigenvalue of A. In this case there is an eigenvector of A corresponding to $(\alpha, 0)$ having the form $(v, 0)$ where $v \in F^n$, and we refer to v as an eigenvector of A corresponding to α.

THEOREM 3.1. *Let F be a real closed field, let f in F^n be a semipositive vector, and let P in $F^{n \times n}$ be a semipositive, irreducible matrix. Then there exists a positive element ρ in F and a positive vector u in F^n such that ρ is an eigenvalue of P, and u is a corresponding eigenvector satisfying $f^T u = 1$. Further*:

(a) *ρ is the unique eigenvalue of P having a semipositive eigenvector*,

(b) *u is the unique eigenvector of P corresponding to ρ that satisfies $f^T u = 1$, and*

(c) *for every eigenvalue (α, β) of P, $\alpha^2 + \beta^2 \leqslant \rho^2$.*

Proof. When F is the real field, the conclusions of our theorem are part of the classic Perron-Frobenius theory; see Berman and Plemmons (1979). Now, for a fixed positive integer n, the conclusion of our theorem is expressible by the validity of the following closed formula:

$$
\begin{aligned}
&\forall P\ \forall f\ \big((P > 0) \wedge (f > 0) \wedge \big(\textstyle\sum_{i=0}^{n-1} P^i \gg 0\big)\big) \\
&\quad \to \big(\exists \rho\ \exists u\ ((\rho > 0) \wedge (u \gg 0) \wedge (Pu = \rho u) \wedge (f^T u = 1)\big) \\
&\quad \wedge (\forall \alpha\ \forall \beta\ (\exists v\ \exists w((Pv = \alpha v - \beta w) \wedge (Pw = \alpha w + \beta v) \\
&\qquad\qquad\qquad \wedge (v > 0) \wedge (w = 0))) \\
&\quad \to ((\alpha = \rho) \wedge (\beta = 0))) \\
&\quad \wedge \big(\forall u'\ (((Pu' = \rho u') \wedge (f^T u' = 1)) \to (u = u'))\big) \\
&\quad \wedge (\forall \alpha\ \forall \beta\ (\exists v\ \exists w\ ((Pv = \alpha v - \beta w) \wedge (Pw = \alpha w + \beta v))) \\
&\quad \to (\alpha^2 + \beta^2 \leqslant \rho^2))\big).
\end{aligned}
$$

In particular, we conclude that this closed formula is true over the reals. By the Tarski-Seidenberg principle (Corollary 2.2), the validity of this closed formula over the reals implies its validity over all real closed fields, establishing the conclusion of our theorem when F is an arbitrary real closed field. ∎

Given a real closed field F and a square, nonnegative, irreducible matrix P and a semipositive vector f over F, conditions (a) and (b) of Theorem 3.1 assert the existence and uniqueness of the scalar ρ and vector u. We shall refer to this scalar and this vector as the *PF-eigenvalue* and *PF-f-eigenvector* of P and denote them by $\rho(P)$ and $u(P, f)$, respectively. We sometime omit the reference to f and simply refer to a *normalized PF-eigenvector* of P. The next result shows invariance of these entities over real closed extensions of a given real closed field.

THEOREM 3.2. *Let F and G be real closed fields where G is an extension of F, let f in F^n be a semipositive vector, and let P in $F^{n\times n}$ be a semipositive, irreducible matrix. Then the PF-eigenvalue and PF-f-eigenvector of P over F and over G coincide; in particular, the PF-eigenvalue of P and coordinates of the PF-f-eigenvector over G are in F.*

Proof. The proof of Theorem 3.1 shows that the assertion that ρ and u are the PF-eigenvalue and PF-f-eigenvector of P over F is expressible by a formula $\varphi(x, y)$ where x represents ρ and u, and y represents f and P. For a particular realization of P and f, Theorem 3.1 shows that ρ and u are determined uniquely and are invariant over all real closed extensions of F. Hence, Corollary 2.4 implies that over each such extension, ρ and the coordinates of u are in F. ∎

The next two results adapt Theorems 3.1 and 3.2 to the case where the underlying matrix need not be irreducible.

THEOREM 3.3. *Let F be a real closed field, let f in F^n be a positive vector, and let P in $F^{n \times n}$ be a nonnegative matrix. Then there exists a nonnegative element ρ in F and a semipositive vector u in F^n such that ρ is an eigenvalue of P, and u is a corresponding eigenvector satisfying $f^T u = 1$. Further, for every eigenvalue (α, β) of P, $\alpha^2 + \beta^2 \leqslant \rho^2$.*

Proof. The proof follows from the Tarski-Seidenberg principle by the arguments used to establish Theorem 3.1. Of course, a different closed formula is required. ■

Given a real closed field F and a square, nonnegative matrix P and positive vector f over F, Theorem 3.3 uniquely determines the asserted element ρ. We shall refer to this unique element as the *PF-eigenvalue* of P and denote it by $\rho(P)$. But the vector u need not be unique. The next result extends the invariance of the *PF*-eigenvalue over real closed extensions of an underlying ordered field to the cases where P is not irreducible.

THEOREM 3.4. *Let F and G be real closed fields where G is an extension of F, and let P in $F^{n \times n}$ be a nonnegative matrix. Then the PF-eigenvalue of P over F and over G coincide; in particular, the PF-eigenvalue of P over G is in F.*

Proof. The proof follows the arguments in the proof of Theorem 3.2. ■

We observe that the Tarski-Seidenberg principle can be used to extend, to real closed fields, any result of the Perron-Frobenius theory that is expressible by formulae.

4. PERRON-FROBENIUS THEORY OVER FORMAL PUISEUX FIELDS

In this section we consider formal fractional power series in an infinitesimal over ordered fields. When the underlying ordered field is real closed, these series form a real closed field. We characterize the *PF*-eigenvalue and normalized *PF*-eigenvector of square, nonnegative, irreducible matrices over this ordered field through a set of equations and inequalities in the original field.

Throughout, let F be a given ordered field, and let ω be an indeterminate symbol. Also, let Z be the ring of integers, Q the ordered field of rationals, and Z_+ and Q_+ the positive elements in Z and Q, respectively. Define $F_*\lfloor\omega\rfloor$ to be the collection of all triplets (r, p, a) where $r \in Q_+$, $p \in Z$, and $a : Z \to F$ where $a_i = 0$ for all $i \leqslant p$. For $(r, p, a) \in F_*\lfloor\omega\rfloor$, we define r to be the *exfactor* and p to be the base; also, with $j \equiv \min\{i \in Z: a_i \neq 0\}$, we define j to be the *order* and a_j to be the *order coefficient*. If $a \equiv 0$, the order and order coefficient are defined to be $+\infty$ and 0, respectively. For convenience, we denote a triple $(r, p, a) \in F_*\lfloor\omega\rfloor$ by the formal sum $\sum_{i=p}^{\infty} a_i \omega^{ir}$, or more briefly $\sum_p a_i \omega^{ir}$. As ω is an indeterminate symbol, we can recover (r, p, a) from the representation by a formal sum. If $r = 1$, we omit r in the formal sum.

We next introduce an equivalence relation $\approx$ over $F_*\lfloor\omega\rfloor$. We say that elements $\sum_p a_i \omega^{ir} \in F_*\lfloor\omega\rfloor$ and $\sum_q b_j \omega^{js} \in F_*\lfloor\omega\rfloor$ satisfy the relation, written $\sum_p a_i \omega^{ir} \approx \sum_q b_j \omega^{js}$, if

(1) $a_i \neq 0$ implies that $j \equiv ir/s$ is an integer and $b_j = a_i$, and
(2) $b_j \neq 0$ implies that $i \equiv js/r$ is an integer and $a_i = b_j$.

It is easily seen that $\approx$ is reflexive, symmetric, and transitive. We denote the collection of corresponding equivalence classes which partition $F_*\lfloor\omega\rfloor$ by $F\lfloor\omega\rfloor$. As usual, we routinely use elements of $F_*\lfloor\omega\rfloor$ to represent elements of $F\lfloor\omega\rfloor$.

The underlying ordered field F is embedded in $F_*\lfloor\omega\rfloor$, where an element $u \in F$ is identified with the element $(1, 0, a) \in F_*\lfloor\omega\rfloor$ with $a_i = 0$ for all $i \neq 0$ and $a_0 = u$. We shall identify the elements of F with the corresponding elements of $F_*\lfloor\omega\rfloor$, that is, we consider F to be a subset of $F_*\lfloor\omega\rfloor$. The element $(r, 1, a) \in F_*\lfloor\omega\rfloor$ with $a_i = 0$ for all $i \neq 1$ and $a_1 = 1$ is denoted ω^r. If $r = 1$, we write ω for ω^1. Also, if $r = 1/q$ for some $q \in Z_+$, we write $\omega^{k/q}$ for $\omega^{(1/q)k}$ for each $k \in Z$.

The order and order coefficients are invariant over the elements of a common equivalence class in $F_*\lfloor\omega\rfloor$; hence they are well defined for all elements in $F\lfloor\omega\rfloor$. But the exfactor and the base are not invariant over an equivalence class of $F_*\lfloor\omega\rfloor$. In fact, we have the following.

LEMMA 4.1. *Let $\sum_p a_i \omega^{ir}$ be in $F_*\lfloor\omega\rfloor$. Then:*

(a) *for every integer $q \leqslant p$, $\sum_p a_i \omega^{ir} \approx \sum_q a_i \omega^{ir}$, and*
(b) *if $s \in Q_+$ is obtained by dividing r by a positive integer k, then $\sum_p a_i \omega^{ir} \approx \sum_q b_j \omega^{js}$, where $q = kp$ and*

$$b_j = \begin{cases} a_{j/k} & \text{if } j \in \{kp, k(p+1), k(p+2), \dots\}, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 4.1 shows that the base of a representation of an element in $F\lfloor\omega\rfloor$ can be arbitrarily reduced and the exfactor can be divided by an arbitrary positive integer. It follows that every pair of element in $F\lfloor\omega\rfloor$ have representations with common base and exfactors. By using such representations, we define addition and multiplication of the elements in $F\lfloor\omega\rfloor$; namely, for elements in $F\lfloor\omega\rfloor$ with representation $\Sigma_p a_i \omega^{ir}$ and $\Sigma_p b_i \omega^{ir}$, we have

$$\sum_p a_i \omega^{ir} + \sum_p b_i \omega^{ir} = \sum_p (a_i + b_i)\omega^{ir}$$

and

$$\left(\sum_p a_i \omega^{ir}\right)\left(\sum_p b_i \omega^{ir}\right) = \sum_{2p}\left(\sum_{j=p}^{i-p} a_j b_{i-j}\right)\omega^{ir}.$$

It is easy to see that the outcome of addition and multiplication is independent of the selected representations; hence, these operations are well defined on $F\lfloor\omega\rfloor$. We note that when $\Sigma_p a_i \omega^{ir} \in F\lfloor\omega\rfloor$ and the set I of indices i with $a_i \neq 0$ is finite, the formal sum $\Sigma_p a_i \omega^{ir}$ equals the finite power series $\Sigma_{i\in I} a_i(\omega^r)^i$, where superscripts i denote powers.

Addition and multiplication in $F\lfloor\omega\rfloor$ are both commutative and associative, multiplication is distributive over addition, and the additive and multiplicative identities of F are respectively, additive and multiplicative identities in $F\lfloor\omega\rfloor$. Also, for an element α in $F\lfloor\omega\rfloor$ with representation $\Sigma_p a_i \omega^{ir} \in F_*\lfloor\omega\rfloor$, $\Sigma_p(-a_i)\omega^{ir}$ is the additive inverse of α, and if α is nonzero and $a_p \neq 0$, then $\Sigma_{-p} c_i \omega^{ir}$ with

$$c_i = \begin{cases} 0 & \text{if } i < -p, \\ 1/a_p & \text{if } i = -p, \\ -\Sigma_{j=p+1}^{i} a_j c_{i-j+p}/a_p & \text{if } i > -p \end{cases}$$

is the multiplicative inverse of α. Hence, $F\lfloor\omega\rfloor$ with the above definitions of addition and multiplication is a field, to which we refer as the formal Puiseux field.

An order on $F\lfloor\omega\rfloor$ is obtained by defining a nonzero element $\Sigma_p a_i \omega^{ir} \in F\lfloor\omega\rfloor$ to be *positive*, written $\Sigma_p a_i \omega^{ir} > 0$, if its order coefficient is positive. As the order coefficient is invariant over equivalence classes in $F\lfloor\omega\rfloor$, the order in $F\lfloor\omega\rfloor$ is well defined. Given two elements α and β in $F\lfloor\omega\rfloor$, we say that α is *greater than* β, written $\alpha > \beta$, if $\alpha - \beta > 0$. Trivially, the relation "greater than" is preserved under addition and under multiplication by positive elements. Hence, with the above definition of addition, multiplication, and order, $F\lfloor\omega\rfloor$ is an ordered field. Further, we have the following fact.

THEOREM 4.2. *F is a real closed field if and only if $F\lfloor\omega\rfloor$ is a real closed field.*

Proof. We first establish a modified version of the theorem asserting that F is algebraically closed if and only if is $F\lfloor\omega\rfloor$ also. That fact that $F\lfloor\omega\rfloor$ is algebraically closed whenever F is algebraically closed is proved in Walker (1950, Theorem 3.1, p. 98). To establish the reverse implication, suppose $F\lfloor\omega\rfloor$ is algebraically closed, and let $p(x) = \sum_{h=0}^{M} c_h x^h$ be a nonzero polynomial whose coefficients $c_0, \dots, c_M$ are in F. Without loss of generality we assume that $c_M \neq 0$. As $F\lfloor\omega\rfloor$ is algebraically closed, it contains a root of $p(\cdot)$, say $a = \sum_p a_i \omega^{ir}$. We observe that the order j of a is not negative, for otherwise the order coefficient of $\sum_{h=0}^{M} c_h a^h$ would be $c_M(a_j)^M \neq 0$, in contradiction to the assertion that $\sum_{h=0}^{M} c_h a^h = 0$. Now, the coefficient of $0 = \sum_{h=0}^{M} c_h a^h = \sum_{h=0}^{M} c_h(\sum_p a_i \omega^{ir})^h$ corresponding to ω^0 is $\sum_{h=0}^{M} c_h(a_0)^h$. As this coefficient (like all others) equals zero, we see that $a_0 \in F$ is a root of $p(\cdot)$, showing that $p(\cdot)$ has a root in F.

We next recall that an ordered field G is real closed if and only if $G + \iota G$ is algebraically closed, where ι is the square root of -1. Also, we observe that $(G + \lambda G)\lfloor\omega\rfloor$ is isomorphic to $G\lfloor\omega\rfloor + \iota G\lfloor\omega\rfloor$. These observations combine with the established modified version of our theorem to show that for each ordered field F the following assertions are equivalent:

(1) F is real closed.
(2) $F + \iota F$ is algebraically closed.
(3) $(F + \iota F)\lfloor\omega\rfloor$ is algebraically closed.
(4) $F\lfloor\omega\rfloor + \iota F\lfloor\omega\rfloor$ is algebraically closed.
(5) $F\lfloor\omega\rfloor$ is real closed. ■

We say that a statement depending on a parameter ε holds *for all sufficiently small positive* ε if for some $\gamma > 0$ the statement holds for all $0 < \varepsilon < \gamma$. We say that such a statement holds *for arbitrarily small positive* ε if for every $\delta > 0$ the statement holds for some $0 < \varepsilon < \delta$.

The following lemma characterizes nonnegativity and irreducibility over $F\lfloor\omega\rfloor$ via these properties over F. The proof is standard; see Eaves and Rothblum (1989, Lemma 5.5, p. 518).

LEMMA 4.3. *Suppose P and B are matrices in $F^{n\times n}$. Then the following are equivalent:*

(a) *$P + \varepsilon B$ is nonnegative and irreducible in F for sufficiently small positive ε,*

(b) *$P + \varepsilon B$ is nonnegative and irreducible in F for arbitrarily small positive ε, and*

(c) *$P + \omega B$ is nonnegative and irreducible in $F\lfloor\omega\rfloor$.*

We next explore the PF-eigenvalue and normalized PF-eigenvector over $F\lfloor \omega \rfloor$ of a nonnegative irreducible matrix of the form $P + \omega B$ where P and B are square matrices over a given ordered field F. The key issue is the characterization of the coefficients of the fractional power series expansions by equations that are defined over F.

THEOREM 4.4. *Let f in F^n be a semipositive vector, and let P and B be matrices in $F^{n\times n}$ where $P + \omega B$ is semipositive and irreducible over $F\lfloor \omega \rfloor$. Then the PF-eigenvalue and PF-f-eigenvector of $P + \omega B$ have representations*

$$\rho(P + \omega B) = \sum_0 \rho_k \omega^{k/q} \tag{4.1}$$

and

$$u(P + \omega B, f) = \omega^{-t/q}\Big(\sum_0 u_k \omega^{k/q}\Big), \tag{4.2}$$

where t is a nonnegative integer and q is a positive integer.

Proof. By applying Theorem 3.1 to $P + \omega B$ over the real closed field $F\lfloor \omega \rfloor$ we have that it has a PF-eigenvalue $\rho(P + \omega B)$ and PF-f-eigenvector $u(P + \omega B, f)$. By part (b) of Lemma 4.1, we may assume that the exfactor of the representation of $\rho(P + \omega B)$ and $u(P + \omega B, f)$ are reciprocals of an integer q; thus, $\rho(P + \omega B)$ and $u(P + \omega B, f)$ have representations $\rho(P + \omega B) = \sum_p \rho_k \omega^{k/q}$ and $u(P + \omega B, f) = \omega^{-t/q}(\sum_0 u_k \omega^{k/q})$ where $p, t \in Z$, $q \in Z_+$, and ρ_p is a positive element in F. To obtain the representations (4.1) and (4.2), it remains to show that we may select $p = 0$ and $t \geqslant 0$.

To show that we can select $p = 0$ it suffices to show that $\rho_k = 0$ for all $k < 0$. Let $L \equiv \min\{k \geqslant p\colon\ \rho_k \neq 0\}$, and let $H \equiv \min\{k \geqslant 0\colon\ u_k \neq 0\}$; as $\rho(P + \omega B) \neq 0$ and $u(P + \omega B, f) \neq 0$, these minima are finite. As

$$(P + \omega B)u(P + \omega B, f) = \rho(P + \omega B)u(P + \omega B, f),$$

we have that

$$(P = \omega B)\,\omega^{-t/q}\Big(\sum_0 u_k \omega^{k/q}\Big) = \Big(\sum_p \rho_k \omega^{k/q}\Big)\omega^{-t/q}\Big(\sum_0 u_k \omega^{k/q}\Big). \tag{4.3}$$

By the multiplication rule in $F\lfloor\omega\rfloor$, the coefficient of $\omega^{(L+H-t)/q}$ on the right hand side of (4.3) is $\rho_L u_H \neq 0$, and for each $k < H$, the coefficient of $\omega^{(k-t)/q}$ on the left hand side of (4.3) is zero. So $L + H - t \geqslant H - t$, implying that $L \geqslant 0$, that is, $\rho_k = 0$ for all $k < 0$.

We next show that $t \geqslant 0$. As

$$1 = f^T u(P + \omega B, f),$$

we have that

$$1 = \sum_{0} f^T u_k \omega^{(k-t)/q} = \sum_{-t} f^T u_{k+t} \omega^{k/q}. \tag{4.4}$$

The coefficient of $\omega^{0/q}$ on the left hand side of (4.4) is 1, and the coefficient on $\omega^{k/q}$ on the right hand side of (4.4) is zero for each $k < -t$. Hence $0 \geqslant -t$, that is, $t \geqslant 0$. ∎

We next obtain a slight addendum to Theorem 4.4.

ADDENDUM TO THEOREM 4.4. *If $f \gg 0$, then $t = 0$.*

Proof. As $f \gg 0$, $f^T a > 0$ for every semipositive vector a. Let $H \equiv \min\{k \geqslant 0 \colon u_k \neq 0\}$. Then the positivity of $u(P + \omega B, f)$ implies that $u_H > 0$ and therefore $f^T u_H > 0$. The normalization condition

$$1 = f^T\left(\omega^{-t/q} \sum_{0} u_k \omega^{k/q}\right) = \sum_{-t} f^T u_{k+t} \omega^{k/q}$$

then implies that $H + t = 0$, that is, $t = -H \leqslant 0$. As we have seen in the original version of Theorem 4.4 that $t \geqslant 0$, we conclude that, indeed, $t = 0$. ∎

COROLLARY 4.5. *Let f be a semipositive vector in F^n, let P and B be matrices in $F^{n\times n}$ where $P + \omega B$ is semipositive and irreducible over $F\lfloor\omega\rfloor$, and suppose $\rho(P + \omega B)$ and $u(P + \omega B, f)$ have representations given by* (4.1) *and* (4.2). *Then*

$$(P + \omega^q B)\left(\sum_{0} u_k \omega^k\right) = \left(\sum_{0} \rho_k \omega^k\right)\left(\sum_{0} u_k \omega^k\right) \tag{4.5}$$

and

$$\omega^{t} = \sum_{0} f^{T} u_k \omega^k. \tag{4.6}$$

Proof. Substituting (4.1) and (4.2) into the equalities $(P + \omega B)u(P + \omega B, f) = \rho(P + \omega B)u(P + \omega B, f)$ and $1 = f^T u(P + \omega B, f)$, we have that

$$(P + \omega B)\Big(\sum_{0} u_k \omega^{k/q}\Big) = \Big(\sum_{0} \rho_k \omega^{k/q}\Big)\Big(\sum_{0} u_k \omega^{k/q}\Big)$$

and

$$1 = \omega^{-t/q}\Big(\sum_{0} f^T u_k \omega^{k/q}\Big),$$

and (4.5) and (4.6) follow from the fact that the rules of addition and multiplication in $F\lfloor \omega \rfloor$ allow one to replace in each occurrence of ω in any given equality with ω^q. ∎

Our next result provides a characterization of the coefficients of the expansion of the *PF*-eigenvalue and normalized *PF*-eigenvector given in (4.1)–(4.2) via a system of equations/inequalities in F.

THEOREM 4.6 (CHARACTERIZATION OF *PF*-EIGENVALUE AND NORMALIZED *PF*-f-EIGENVECTOR OVER $F\lfloor \omega \rfloor$). *Let $f \in F^n$ be a semipositive vector, and let P and B be matrices in $F^{n \times n}$ where $P + \omega B$ is semipositive and irreducible over $F\lfloor \omega \rfloor$. Further, let $t \in Z_+ \cup \{0\}$, $q \in Z_+$, $\rho_0, \rho_1, \ldots \in F$, and $u_0, u_1, \ldots \in F$. Then the representations* (4.1) *and* (4.2) *of $\rho(P + \omega B)$ and $u(P + \omega B, f)$ hold with the given $t, q, \rho_0, \rho_1, \ldots$ and $u_0, u_1, \ldots$ if and only if they satisfy the following system of equations and inequalities*:

$$(P - \rho_0 I)u_k = \begin{cases} \rho_k u_0 + \rho_{k-1}u_1 + \rho_{k-2}u_2 + \cdots + \rho_2 u_{k-2} + \rho_1 u_{k-1} \\ \qquad \text{for} \quad 0 \leqslant k < q, \\ \rho_k u_0 + \rho_{k-1}u_1 + \rho_{k-2}u_2 + \cdots + \rho_2 u_{k-2} + \rho_1 u_{k-1} - Bu_{k-t} \\ \qquad \text{for} \quad q \leqslant k, \end{cases} \tag{4.7}$$

$$f^T u_k = \begin{cases} 0 & \text{if} \quad k \neq t, \\ 1 & \text{if} \quad k = t, \end{cases} \tag{4.8}$$

and

$$\sum_{0} u_k \omega^{k/q} > 0 \qquad \text{in } F\lfloor \omega \rfloor. \tag{4.9}$$

Proof. First assume that $\rho(P + \omega B)$ and $u(P + \omega B, f)$ have representations given by (4.1) and (4.2). By Corollary 4.5, t, q, and the coefficients of the representations of $\rho(P + \omega B)$ and $u(P + \omega B, f)$ satisfy (4.5) and (4.6), from which (4.7) and (4.8) follow immediately by the rules of addition and multiplication in $F\lfloor \omega \rfloor$. Also, (4.9) is part of the definition of the *PF-f*-eigenvector; see Theorem 3.1.

Next assume that $t, q, \rho_0, \rho_1, \ldots$ and $u_0, u_1, \ldots$ satisfy (4.7)–(4.9). Let $\rho \equiv \sum_0 \rho_k \omega^{k/q}$ and $u \equiv \omega^{p/q} \sum_0 u_k \omega^{k/q}$. Then (4.9) implies that $u > 0$; further, by the multiplication and addition rules in $F\lfloor \omega \rfloor$, (4.7) and (4.8) imply, respectively, that $(P + \omega B)u = \rho u$ and $f^T u = 1$. So ρ is an eigenvalue with a semipositive eigenvector. By Theorem 3.1 it follows that ρ and u are the *PF*-eigenvalue and *PF-f*-eigenvector of $P + \omega B$. ■

For $k = 0, 1, \ldots$, we define (4.7.k) as the constraint of (4.7) determining $(P - \rho_0 I)u_k$, similarly, we define (4.8.k) as the constraint of (4.8) specifying the value of $f^T u_k$. Also, we emphasize that for given q and t the system consisting of (4.7)–(4.8) is not linear.

We next consider semipositive perturbations which are not necessarily irreducible.

THEOREM 4.7. *Let P and B be matrices in $F^{n \times n}$ where $P + \omega B$ is semipositive over $F\lfloor \omega \rfloor$. Then the PF-eigenvalue of $P + \omega B$ has a representation of the form given in* (4.1). *Further, suppose f is a positive vector in F^n. Then* (4.7)–(4.9) *is satisfied by $q, \rho_0, \rho_1, \ldots,$ some $t \in Z_+ \cup \{0\}$, and some elements $u_0, u_1, \ldots$ in F.*

Proof. The conclusion of the theorem follows from the arguments used to prove Theorem 4.4 and 4.6, except that Theorem 3.3 is used rather than Theorem 3.1 and the eigenvector corresponding to the *PF*-eigenvalue of $P + \omega B$ is not unique. ■

In Lemma 4.3, Theorems 4.4., 4.6, and 4.7, and Corollary 4.5 we explored linear, nonnegative, irreducible perturbations over $F\lfloor \omega \rfloor$ of a matrix $P \in F^{n \times n}$. We observe, without providing formal details, that the analysis we developed holds unchanged for polynomial perturbations of the form $P + \sum_{i=1}^{m} B_i \omega^i$, but the analogs of (4.7)–(4.8) become much more complicated when these more general perturbations are considered. Further, we observe that Theorems 4.4, 4.6, and 4.7 and Corollary 4.5 extend to even more general perturbations of the form $P + \Sigma_p B_i \omega^{ir} \in F[\omega]^{n\times n}$. But in this case there is no analog of Lemma 4.3, and the obtained expansions cannot necessarily be used to determine the PF-eigenvalue and normalized PF-eigenvector of perturbations of matrices over F with respect to a sufficiently small value of a parameter. The next section allows for such perturbations by considering convergence.

5. PERRON-FROBENIUS THEORY OVER REAL CONVERGENT PUISEUX FIELDS

In this section we consider a real closed subfield of the formal Puiseux field over the reals consisting of the fractional power series with positive radius of convergence. We then use Theorem 3.2 to argue that the PF-eigenvalue and normalized PF-eigenvector of the matrices considered in Section 4 are always in the smaller field, and conclude that PF-eigenvalue and normalized PF-eigenvector of perturbed matrices over the original field have power series expansions with positive radius of convergence. The main case of interest is where the underlying field is R, and we restrict attention to this case. Obstacles to the consideration of Puiseux series with positive radius of convergence over real closed fields that strictly contain the reals are discussed in Eaves and Rothblum (1985, revised 1987).

Consider the subset $R_+[\omega]$ of $R[\omega]$ consisting of those elements $\Sigma_p a_k \omega^k \in R[\omega]$ where the power series $\Sigma a_k \varepsilon^k$ has a positive radius of convergence, that is, for all sufficiently small positive ε the series $\Sigma_{k=p}^{\infty} a_k \varepsilon^k$ converges absolutely, or equivalently, for all sufficiently small positive ε the series $\Sigma_{k=p}^{\infty} a_k \varepsilon^{k/q}$ converges absolutely. Standard results show that $R_+[\omega]$ is closed under $R[\omega]$-addition, under $R[\omega]$-multiplication, under $R[\omega]$-additive inversion, and under $R[\omega]$-multiplicative inversion. So $R_+[\omega]$ is an ordered subfield of $R[\omega]$, and we refer to $R_+[\omega]$ as the *convergent Puiseux field*.

We have the following fact about $R_+[\omega]$; see Picard (1905) or Bewley and Kohlberg (1976, Section 10).

THEOREM 5.1. *The ordered field $R_+[\omega]$ is real closed.*

The following lemma characterizes the relations $>$, $\geqslant$, $<$, $\leqslant$, and $\neq$ in $R_+[\omega]$.

LEMMA 5.2. *Let a in $R[\omega]$ have the representation $\Sigma_p a_k \omega^{k/q}$. Then the following are equivalent*:

(a) *a is in $R_+[\omega]$ and $a > 0$ in $R[\omega]$*,

(b) *for all sufficiently small positive* ε, *the series* $\sum_{k=p}^{\infty} a_k \varepsilon^{k/q}$ *converges absolutely and* $\sum_{k=p}^{\infty} a_k \varepsilon^{k/q} > 0$, *and*

(c) *for arbitrarily small positive* ε, *the series* $\sum_{k=p}^{\infty} a_k \varepsilon^{k/q}$ *converges absolutely and* $\sum_{k=p}^{\infty} a_k \varepsilon^{k/q} > 0$.

Moreover, the above equivalences hold if $>$ *is replaced by* $\geqslant$, $<$, $\leqslant$, *or* $\neq$.

Proof. The conclusions of the lemma follow from standard results in analysis; see Picard (1905) or Bewley and Kohlberg (1976). ■

We are now ready to present the expansions of the *PF*-eigenvalue and normalized *PF*-eigenvector of nonnegative, irreducible (linear) perturbations of a real matrix from the solution of (4.7)–(4.9).

THEOREM 5.3. *Let* f *be a semipositive vector in* F^n, *and let* P *and* B *be matrices in* $R^{n\times n}$ *where* $P + \varepsilon B$ *is semipositive and irreducible for all sufficiently small positive* ε. *Suppose* $t \in Z_+ \cup \{0\}$, $q \in Z_+$, $\rho_0, \rho_1, \ldots \in F$, *and* $u_0, u_1, \ldots \in F$ *satisfy* (4.7) *and not all* u_i*'s are zero. Then for all sufficiently small positive* ε, *the series* $\sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q}$ *converges absolutely. Further, if* (4.8) *and* (4.9) *are satisfied, then for all sufficiently small positive* ε *the series* $\sum_{k=0}^{\infty} u_k \varepsilon^{k/q}$ *converges absolutely, and* $\sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q}$ *and* $\varepsilon^{-t/q}(\sum_{k=0}^{\infty} u_k \varepsilon^{k/q})$ *are the PF-eigenvalue and PF-f-eigenvector of* $P + \varepsilon B$.

Proof. Suppose $t \in Z_+ \cup \{0\}$, $q \in Z_+$, $\rho_0, \rho_1, \ldots \in F$, and $u_0, u_1, \ldots \in F^n$ satisfy (4.7). Then $\rho \equiv \sum_0 \rho_k \omega^{k/q} \in R\lfloor \omega \rfloor$ and $u \equiv \omega^{-t/q}(\sum_0 u_k \omega^{k/q}) \in F\lfloor \omega \rfloor$ satisfy $(P + \omega B)u = \rho u$ and $u \neq 0$; thus, ρ is an eigenvalue of $P + \omega B$. View the characteristic polynomial as a polynomial $p(x, y)$ with integer coefficients, a scalar variable x, and n^2 variables represented by y that correspond to the elements of an $n \times n$ matrix. Standard arguments assure that $p(\rho, P + \omega B) = 0$. As $P + \omega B \in R_+\lfloor \omega \rfloor^{n\times n}$, $R_+\lfloor \omega \rfloor$ is real closed, and $R\lfloor \omega \rfloor$ is a real closed extension of $R_+\lfloor \omega \rfloor$, we conclude from Corollary 2.6 that $\rho \in R_+\lfloor \omega \rfloor$, that is, for all sufficiently small positive ε, the series $\sum_{k=p}^{\infty} \rho_k \varepsilon^{k/q}$ converges absolutely.

Next assume that, in addition, (4.8) and (4.9) are satisfied. Then (4.9) implies that $u > 0$, and, by Lemma 4.3, $P + \omega B$ is semipositive and irreducible in $R\lfloor \omega \rfloor^{n\times n}$. Further, by (4.8), $f^T u = 1$. So, by Theorem 4.6, ρ and u are the *PF*-eigenvalue and *PF-f*-eigenvector of $P + \omega B$ over $R\lfloor \omega \rfloor$. Obviously, $P + \omega B \in R_+\lfloor \omega \rfloor^{n\times n}$; hence, Theorem 3.2 implies that the coordinates of u are in the ordered subfield $R_+\lfloor \omega \rfloor$ of $R\lfloor \omega \rfloor$, that is, for all sufficiently small positive ε, the series $\sum_{k=0}^{\infty} u_k \varepsilon^{k/q}$ converges absolutely. For relevant ε, denote the converging sums of the series $\sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q}$ and $\sum_{k=0}^{\infty} u_k \varepsilon^{k/q}$ by $\rho(\varepsilon)$ and $u(\varepsilon)$, respectively. As $(P + \omega B)u = \rho u$ and $f^T u = 1$, it follows from the definition of addition and multiplication in $R\lfloor \omega \rfloor$

and standard rules for multiplying and adding converging real power series that for all sufficiently small positive ε, $(P + \varepsilon B)u(\varepsilon) = \rho(\varepsilon)u(\varepsilon)$ and $f^T u(\varepsilon) = 1$. Further, as $u > 0$, Lemma 5.2 implies that for all sufficiently small positive ε, $u(\varepsilon) > 0$. So, for all sufficiently small positive ε, we have $\rho(\varepsilon) \in R$, $u(\varepsilon) \in R^n$, $(P + \varepsilon B)u(\varepsilon) = \rho(\varepsilon)u(\varepsilon)$, $f^T u(\varepsilon) = 1$, and $u(\varepsilon) > 0$. By Theorem 3.1, for such ε, the scalar and vector $\rho(\varepsilon)$ and $u(\varepsilon)$ are the *PF*-eigenvalue and *PF-f*-eigenvector of $P + \varepsilon B$. ■

We obtain the following corollary of Theorem 5.3, which asserts that the *PF*-eigenvalue and normalized *PF*-eigenvector of nonnegative, irreducible (linear) perturbations of a square matrix have fractional power series expansions.

COROLLARY 5.4. *Let f be a semipositive vector in R^n. Suppose P and B are matrices in $R^{n\times n}$ where for all sufficiently small positive ε, the matrix $P + \varepsilon B$ is semipositive and irreducible. Then for all sufficiently small positive ε, the PF-eigenvalue and PF-f-eigenvector of $P + \varepsilon B$ have representations through the converging series*

$$\rho(P + \varepsilon B) = \sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q} \tag{5.1}$$

and

$$u(P + \varepsilon B, f) = \varepsilon^{-t/q}\left(\sum_{k=0}^{\infty} u_k \varepsilon^{k/q}\right), \tag{5.2}$$

where $t \in Z_+ \cup \{0\}$ and $q \in Z_+$. In particular, t, q, and the coefficients of these representations satisfy (4.7)–(4.9).

Proof. By Lemma 4.3, $P + \omega B$ is semipositive and irreducible in $R\lfloor \omega \rfloor^{n\times n}$. Thus, Theorem 4.4 implies that the *PF*-eigenvalue $\rho(P + \omega B)$ and *PF-f*-eigenvector $u(P + \omega B, f)$ over $R\lfloor \omega \rfloor$ have representations $\rho \equiv \sum_0 \rho_k \omega^{k/q}$ and $u \equiv \omega^{-t/q}(\sum_0 u_k \omega^{k/q})$, where $t \in Z_+ \cup \{0\}$, $q \in Z_+$, $\rho_0, \rho_1, \ldots \in R$, and $u_0, u_1, \ldots \in R^n$ satisfy (4.7)–(4.9). The representations (5.1) and (5.2) for $\rho(P + \varepsilon B, f)$ and $u(P + \varepsilon B, f)$ now follow directly from Theorem 5.3. ■

We observe that if the requirement $f > 0$ in Corollary 5.4 is tightened to $f \gg 0$, then we can require $t = 0$; see the Addendum to Theorem 4.4.

Theorem 5.3 and Corollary 5.4 suggest that the solution of (4.7)–(4.9) is important. We emphasize that t and q are unknown in this system and their determination is part of the task of solving the system. Further, even after t and q are determined, the system remains nonlinear and complicated. The difficulty in solving (4.7)–(4.8) is demonstrated in Rothblum and Schneider (1995), where the system is solved under restrictive assumptions.

We next consider semipositive perturbations which are not necessarily irreducible.

THEOREM 5.5. *Let P and B be matrices in $R^{n\times n}$ where $P + \varepsilon B$ is semipositive for all sufficiently small positive ε. Suppose $t \in Z_+ \cup \{0\}$, $q \in Z_+$, $\rho_0, \rho_1, \ldots \in F$, and $u_0, u_1, \ldots \in F^m$ satisfy (4.7) and not all u_i's are zero. Then for all sufficiently small positive ε, the series $\sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q}$ converges absolutely.*

Proof. The conclusion of the theorem follows directly from the argument used to prove the first part of Theorem 5.3. ■

COROLLARY 5.6. *Suppose P and B are matrices in $R^{n\times n}$ where for all sufficiently small positive ε, the matrix $P + \varepsilon B$ is semipositive. Then for all sufficiently small positive ε, the PF-eigenvalue of $P + \varepsilon B$ has a representation through a converging series*

$$\rho(P + \varepsilon B) = \sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q} \tag{5.3}$$

where $q \in Z_+$. Further, for each positive vector f in F^n, there exist $t \in Z_+$ and $u_0, u_1, \ldots \in F$ such that $t, q, \rho_0, \rho_1, \ldots$ and $u_0, u_1, \ldots$ satisfy (4.7)–(4.9).

Proof. As in Lemma 4.3, $P + \omega B$ is semipositive in $R\lfloor\omega\rfloor^{n\times n}$. Thus, by Theorem 4.7, the PF-eigenvalue $\rho(P + \omega B)$ over $R\lfloor\omega\rfloor$ has a representation $\rho \equiv \sum_0 \rho_k \omega^{k/q}$ where $q \in Z_+$ and $\rho_0, \rho_1, \ldots \in R$ satisfy (4.7)–(4.9). The representation (5.1) for $\rho(P + \varepsilon B, f)$ now follows directly from Theorem 5.5. ■

In Lemma 5.2, Theorem 5.3, Corollary 5.4, Theorem 5.5, and Corollary 5.6 we considered linear perturbations. Without providing formal details, we extend the observation made at the end of Section 4 and note that the analysis we developed holds unchanged for arbitrary absolutely convergent perturbations of the form $P + \sum_{i=0}^{\infty} B_i \varepsilon^i$, but the analogs of (4.7)–(4.8) become even more complicated.

6. REAL CLOSURE AND THE BARKER-SCHNEIDER CONDITION

Barker and Schneider (1975) identified a condition which implies the existence of a PF-eigenvalue and normalized PF-eigenvector for square, nonnegative, irreducible matrices with elements in ordered fields that are not necessarily real closed. Here, we relate their condition to the assumption that the underlying ordered field is real closed.

Suppose F is an ordered field. For a subset S of F, the *infimum of S over F*, denote $\inf_F(S)$, is defined as an element σ in F having the property that S contains no element $\sigma' < \sigma$ and for every element $\sigma' > \sigma$ there exists an element $\sigma'' \in S$ where $\sigma'' < \sigma'$. Of course, not every subset of F has an infimum. If S has an infimum over F and the infimum is in S, we call it the *minimum of S over F* and denote it $\min_F(S)$.

Let F be an ordered field, and let P in $F^{n\times n}$ be semipositive and irreducible. Barker and Schneider show that P has a PF-eigenvalue and a normalized PF-eigenvector under the assumption that the set

$$T \equiv \{\bar{x} \in F\colon Pz \leqslant \bar{x}z \text{ for some vector } z \in F^n \text{ satisfying } z > 0\} \quad (6.1)$$

has a minimum over F. It is shown in the Appendix that if T has an infimum over F, then the infimum is in T, that is, T then has a minimum over F.

Let F be an ordered field. A subset S of F is called *algebraic* if there is a formula $\varphi(x, y)$ with $\#x = 1$, such that for some $\bar{y}$ in $F^{\#y}$, $S = \{\bar{x} \in F\colon \varphi(\bar{x}, \bar{y})$ is satisfied over $F\}$. If $\varphi(x, y)$ is a quantifier-free formula, we call S a *quantifier-free algebraic set*, and if $\varphi(x, y)$ is a formula having the form $\sum_{i=0}^{d} y_i x^i \geqslant 0$, we call S a *simple algebraic set*.

Part, possibly all, of the next result is known.

THEOREM 6.1. *Let F be an ordered field. Then the following conditions are equivalent*:

(a) *F is real closed*,

(b) *every nonempty simple algebraic set that is bounded from below has an infimum*,

(c) *every nonempty quantifier-free algebraic set that is bounded from below has an infimum, and*

(d) *every nonempty algebraic set that is bounded from below has an infimum.*

Proof. (d) ⇒ (c) ⇒ (b): These implications are trivial.

(b) $\Rightarrow$ (a): Assume that (b) holds. A polynomial over F in a single variable x has a representation $p(x, \bar{y}) = \bar{y}_0 + \bar{y}_1 x + \cdots + \bar{y}_{n-1}x^{n-1} + \bar{y}_n x^n$ where $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n$ are elements in F and $\bar{y}_n \neq 0$. Suppose n is odd; we will show that $p(\bar{x}^*, \bar{y}) = 0$ for some $\bar{x}^* \in F$. By possibly dividing the polynomial by -1 we may assume $\bar{y}_n > 0$. The set $S \equiv \{\bar{x} \in F\colon\ p(\bar{x}, \bar{y}) \geqslant 0\}$ is a simple algebraic set. Further, as n is odd, there exists a element K in F such that very $\xi > K$ is in S and no $\xi < -K$ is in S. Thus, S is nonempty and bounded from below, and condition (b) assures that it has an infimum, say $\bar{x}^*$. We next argue that $p(\bar{x}^*, \bar{y}) = 0$. Indeed, we cannot have $p(\bar{x}^*) > 0$, for then $p(\bar{x}^* - \varepsilon) > 0$ for all sufficiently small positive ε, contradicting the assumption that S contains no element $\bar{x}' < \bar{x}^*$. Similarly, we cannot have that $p(\bar{x}^*) < 0$, for then $p(\bar{x}^* + \varepsilon) > 0$ for all sufficiently small positive ε, contradicting the assumption that for every element $\bar{x}' > \bar{x}^*$ there exists an element $\bar{x}'' \in S$ where $\bar{x}^* < \bar{x}'' < \bar{x}'$.

Next, assume that $\bar{y} \in F$ is positive, and consider the polynomial (with coefficients in F) given by $p(x, \bar{y}) = \bar{y} - x^2$. Then $S \equiv \{\bar{x} \in F\colon\ p(\bar{x}, \bar{y}) \geqslant 0\}$ is a simple algebraic set. It is nonempty, as $0 \in S$, and it is obviously bounded from below. Thus, by condition (b), S has an infimum, say $\bar{x}^*$, and the above arguments show that $p(\bar{x}^*, \bar{y}) = 0$. So $\bar{y}$ has a square root.

(a) $\Rightarrow$ (d): Suppose F is real closed and S is a nonempty algebraic set that is bounded from below. Let φ be a formula that defines S. We consider the closed formula ψ given by

$$\psi : \forall y \left(\left((\exists x\ \varphi(x, y)) \wedge (\exists z\ \forall x' (\varphi(x', y) \to (x' \geqslant z))) \right) \right.$$

$$\left. \to (\exists x^* (\varphi(x^*, y)) \wedge (\forall x'' (\varphi(x'', y) \to (x'' \geqslant x^*)))) \right).$$

By the Weierstrass theorem, every nonempty set over the reals that is bounded from below has an infimum, implying that ψ is true over the reals. Hence, by the Tarski-Seidenberg principle (Corollary 2.2) ψ is true over the real closed field F. ∎

Let P be a square, nonnegative, irreducible matrix over an ordered field F, and consider the set T defined by (6.1). The set T is an algebraic set. It is nonempty, as it contains 0, and it is bounded from below by zero. Hence, Theorem 6.1 implies that if F is real closed, then T has an infimum. By Theorem A.1 of the Appendix, it then follows that T has a minimum. Thus, the assumptions of Barker and Schneider (1975) are implied by the assumption that the underlying ordered field is real closed.

7. DISCUSSION AND EXTENSIONS

In Rothblum and Schneider (1995), the problem (4.7)–(4.9) is solved under restrictive assumptions which assert, among other things, that the matrix P has a unique Jordan chain corresponding to its PF-eigenvalue. The obtained solution has $q = \nu$ and $t = \nu - 1$, where ν is the index of P corresponding to $\rho(P)$. Also, in Haviv, Ritov, and Rothblum (1992), the problem (4.7)–(4.8) is solved when the matrix P is in some class of matrices with index 1 that contains the class of irreducible matrices. The obtained solution has $q = 1$ and $t = 0$. The following example demonstrates that (4.7)–(4.9) may have a solution with $q < \nu$. Yet, we still conjecture that there exists a finite algorithm that can determine q and t for which (4.7)–(4.9) has a solution.

EXAMPLE 7.1. Let

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then the index of P corresponding to $\rho(P) = 1$ is 2. Also, $P + \varepsilon B$ is nonnegative and irreducible for all $0 < \varepsilon < 0.5$, its characteristic polynomial is $(1 - \varepsilon - x)[(1 - \varepsilon - x)(0.5 - \varepsilon - x) - 2\varepsilon]$, and its eigenvalues are $1 - \varepsilon$, $1 + z(\varepsilon)_1$, and $1 + z(\varepsilon)_2$, where $z(\varepsilon)_1$ and $z(\varepsilon)_2$ are the roots of the quadratic equation $z^2 + (2\varepsilon + 0.5)z + \varepsilon^2 - 1.5\varepsilon = 0$. In particular,

$$\begin{aligned} \rho(P + \varepsilon B) &= 2 - (2\varepsilon + 0.5) + \left[(2\varepsilon + 0.5)^2 - 4(\varepsilon^2 - 1.5\varepsilon)\right]^{1/2} \\ &= 2 - (2\varepsilon + 0.5) + (8\varepsilon + 0.25)^{1/2}. \end{aligned}$$

As $(8\varepsilon + 0.25)^{1/2}$ has a power series expansion in integer powers of ε, $\rho(P + \varepsilon B)$ does also. It follows from the analysis of Section 5 that $\rho(P + \omega B)$ has a representation with unit exfactor. As the elements in $R\lfloor\omega\rfloor$ with unit exfactor form an ordered subfield of $R\lfloor\omega\rfloor$, standard arguments about solvability of linear systems over ordered fields (e.g., Eaves and Rothblum, (1993)) show that for every semipositive vector $f \in F^n$, the coordinates of $u(P + \omega B, f)$ have representations with unit exfactor. Thus, Theorem 4.4 implies that (4.7)–(4.8) has a solution with $q = 1$ and $\Sigma_0 u_k \omega^{k/q} > 0$.

In the solutions of (4.7)–(4.9) obtained in Rothblum and Schneider (1995) and in Haviv, Ritov, and Rothblum (1992), the coefficients of the fractional power series of the normalized PF-eigenvector turn out to yield a "preferred basis" of the generalized eigenspace corresponding to the PF-eigenvalue of P; see Rothblum (1975), Richman and Schneider (1978), and Schneider (1986) for formal definitions. Though this phenomenon does not occur in general, there seems to be considerable spectral information in the coefficients of the fractional power series expansions of the normalized PF-eigenvector for perturbed nonnegative matrices.

APPENDIX

Let F be an ordered field, and let P in $F^{n\times n}$ be semipositive and irreducible. We observe that the set T defined by (6.1) has an equivalent representation

$$T \equiv \Big\{\bar{x} \in F\colon Pa \leqslant \bar{x}z \text{ for some vector } z \in F^n \\ \text{satisfying } z \geqslant 0 \text{ and } \textstyle\sum_{i=1}^{n} z_i = 1\Big\}. \tag{A.1}$$

The purpose of this appendix is to establish that T defined by (A.1) has an infimum over F then the infimum is in T.

Throughout this appendix let $\| \ \|_\infty$ denote the l_∞ norm in F^n and the corresponding matrix norm in $F^{n\times n}$; that is, for a in F^n, let $\|a\|_\infty = \max\{a_i\colon i = 1,\dots,n\}$, and for a matrix A in $F^{n\times n}$, let $\|A\|_\infty = \max\{\sum_{j=1}^{n} A_{ij}\colon i = 1,\dots,n\}$.

Given that T has an infimum $\bar{x}^*$ over F, we prove that $\bar{x}^*$ is in T. In this case, for arbitrarily small positive ε, $\bar{x}^* + \varepsilon$ is in T, that is, the (linear) system

$$[P - (\bar{x}^* + \varepsilon)I]z \leqslant 0, \qquad e^T z = 1, \qquad z \geqslant 0 \tag{A.2}$$

is feasible where $e^T = (1,\dots,1) \in F^n$. By adding slack variables, (A.2) can be cast in the form

$$(A + \varepsilon E)w = b, \qquad w \geqslant 0, \tag{A.3}$$

where

$$w = \begin{pmatrix} z \\ s \end{pmatrix}, \quad A = \begin{pmatrix} P - \bar{x}^* I & I \\ e & 0 \end{pmatrix}, \quad E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.4}$$

For $\varepsilon \leqslant 1$ and $i = 1, \ldots, n$, (A.2) implies that $|z_i| \leqslant 1$ and $|s_i| \leqslant \|P\|_\infty + \bar{x}^* + 1$; hence, with $K \equiv \max\{1, \|P\|_\infty + \bar{x}^* + 1\}$, every solution y of (A.3) satisfied $\|w\|_\infty \leqslant K$. We also observe that the rows of the matrix A defined through (A.4) are linearly independent. The next result implies that (A.2) is feasible for $\varepsilon = 0$, that is, $\bar{x}^* \in T$.

THEOREM A.1. *Let F be an ordered field, $K \in F$, $b \in F^m$, and $A, E \in F^{m \times n}$. Suppose:*

(I) *the rows of A are linearly independent,*

(II) *for arbitrarily small positive ε,* (A.3) *has a solution in F, and*

(III) *for all sufficiently small positive ε, every solution of* (A.3) *has $\|y\|_\infty \leqslant K$.*

Then (A.3) *has a solution for $\varepsilon = 0$.*

Proof. Standard results about linear inequalities show that if assumption (I) holds and if (A.3) has a solution for a given ε, then it has a solution y^ε having the form

$$(y^\varepsilon)\alpha = [(A + \varepsilon E)_\alpha]^{-1} b \quad \text{and} \quad (y^\varepsilon)_{\alpha^c} = 0, \tag{A.5}$$

where α is a subset of $\{1, \ldots, n\}$, $\alpha^c = \{1, \ldots, n\} \setminus \alpha$, and subscripts are used to denote corresponding submatrices and subvectors. As the number of subsets of $\{1, \ldots, n\}$ is finite, assumptions (I) and (II) imply that for some subset α, y^ε has the representation (A.5) for arbitrarily small positive ε. It now follows from Cramer's rule that there exist polynomials $p_0, p_1, \ldots, p_n$ such that for all sufficiently small positive ε

$$(y^\varepsilon)_j = p_j(\varepsilon) \setminus p_0(\varepsilon) \qquad \text{for} \quad j = 1, \ldots, n, \tag{A.6}$$

and $p_0(\varepsilon) \neq 0$.

Standard results show that zero is a root of a polynomial if and only if the polynomial is divisible by a positive power of ε (Euclid's algorithm is used to prove one direction). For $j = 0, 1, \ldots, n$, let k_j be the maximal power of ε that divides $p_j(\varepsilon)$; in particular, $p_j(\varepsilon) = \varepsilon^{k_j} q_j(\varepsilon)$ for all $\varepsilon \in F$ for some polynomial q_j where $q_j(0) \neq 0$. Now, assumption (III) implies that for sufficiently small positive ε

$$\varepsilon^{k_j} |q_j(\varepsilon)| = |p_j(\varepsilon)| \leqslant K |p_0(\varepsilon)| = K \varepsilon^{k_0} |q_0(\varepsilon)| \qquad \text{for} \quad j = 1, \ldots, n.$$

It is easy to show that these inequalities imply that $k_j \geqslant k_0$ for $j = 1, \ldots, n$. Hence, in the representation of the coordinates of y^{ε} through (A.6), it is possible to cancel the common factor ε^{k_0} in all the polynomials p_j. So, without loss of generality, we may assume that $k_0 = 0$, that is, $p_0(0) \neq 0$. Consider the vector y^0 defined coordinatewise by

$$(y^0)_j \equiv p_j(0)/p_0(0) \qquad \text{for} \quad j = 1, \ldots, n.$$

It follows that y^0 satisfies (A.3) with $\varepsilon = 0$. So (A.3) with $\varepsilon = 0$ is feasible, proving the conclusion of our theorem. ∎

REFERENCES

Barker, G. P. and Schneider, H. 1975. Algebraic Perron-Frobenius theory, *Linear Algebra Appl.* 11:219–233.

Berman, A. and Plemmons, R. J. 1979. *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York.

Bewley, T. and Kohlberg, E. 1976. The asymptotic theory of stochastic games, *Math. Oper. Res.* 1:197–208.

Cohen, J. E. 1978. Derivatives of the spectral radius as a function of non-negative matrix elements, *Math. Proc. Cambridge Philos. Soc.* 83:183–190.

Deutsch, E. and Neumann, M. 1984. Derivatives of the Perron root at an essentially nonnegative matrix and the group inverse of an *M*-matrix, *J. Math. Anal. Appl.* 102:1–29.

Eaves, B. C. and Rothblum, U. G. 1985; rev. 1987. A theory on extending algorithms for parametric problems, Tech. Rep. SOL 85-13, System Optimization Lab., Dept. of Operations Research, Stanford Univ., Stanford, CA 94305.

Eaves, B. C. and Rothblum, U. G. 1989. A theory on extending algorithms for parametric problems, *Math. Oper. Res.* 14:502–533.

Enderton, H. B. 1970. *A Mathematical Introduction to Logic*, Academic, New York.

Haviv, M., Ritov, Y., and Rothblum, U. G. 1992. Taylor expansions of eigenvalues of perturbed matrices with applications to spectral radii of nonnegative matrices, *Linear Algebra Appl.* 168:159–188.

Jacobson, N. 1964. *Lectures in Abstract Algebra III*, Van Nostrand, Princeton, N.J.

Kato, T. 1966. *Perturbation Theory for Linear Operators*, Springer-Verlag, New York.

MacLane, S. and Birkhoff, G. 1967. *Algebra*, Chelsea, New York.

Meyer, C. D. and Steward, G. W. 1988. Derivatives and perturbations of eigenvectors, *SIAM J. Numer. Anal.* 3:679–691.

Picard, É. 1905. *Traité d'Analyse*, Vol. II, 2nd ed., Gauthier-Villars, Paris.

Richman and Schneider, H. 1978. On the singular graph and the Weyr characteristic of an *M*-matrix, *Aequationes Math.* 17:208–234.

Rothblum, U. G. 1975. Algebraic eigenspaces of nonnegative matrices, *Linear Algebra Appl.* 12:281–292.

Rothblum, U. G. and Schneider, H., 1995. A solution of a nonlinear system arising in spectral perturbation theory of nonnegative matrices, in preparation.

Schneider, H. 1986. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey, *Linear Algebra Appl.* 84:161–189.

Schweitzer, P. J. 1986. Perturbation series for nearly completely decomposable Markov chains, in *Teletraffic Analysis and Computer Performance Evaluation* (O. J. Boxima, J. W. Cohen, and H. C. Tijms, Eds.), Elsevier Science (North Holland).

Seidenberg, A. 1954. A new decision method for elementary algebra, *Ann. of Math.* 60:365–374.

Tarski, A. 1951. *A Decision Method for Elementary Algebra and Geometry*, 2nd ed., rev., Univ. of California Press, Berkeley.

Walker, R. J. 1950. *Algebraic Curves*, Princeton U. P., Princeton, N. J.

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