On visualization scaling, subeigenvectors and Kleene stars in max algebra

Hans Schneider
based on joint work with
Peter Butkovic
and Sergei Sergeev

Haifa 2009

Definition

$G(A) = (N, E)$

$$N = \{1, \ldots, n\}$$

$$i \to j \iff a_{ij} > 0$$

Definition

path π in $G(A)$

$$i(0) \to i(1S) \to \cdots \to i(k)$$

$\pi(i, j)$, $(\pi(i, j; r))$
path from i to j (length r)

$$\pi(A) =$$
$$a_{i(0),i(1)} a_{i(2),i(3)} \cdots a_{i(k-1),i(k)}$$

Definition

cycle γ: path
$i(0), \ldots, i(k-1)$ distinct,
$i(0) = i(k)$

$$\mu(\gamma(A)) = (\gamma(A)^{1/k}$$

$$\mathrm{mcm(A)} = \max_{\gamma(\mathrm{A})} \mu(\gamma(\mathrm{A}))$$

max (mean) cycle,
max cycle mean - $\lambda(A)$

Max Algebra

$$a, b \in \mathbb{R}_+$$

$$a + b, \ ab$$

$$A, B \in \mathbb{R}_+^{m \times n}$$

$$A + B, AB,$$

.

$$a \oplus b = \max(a, b), a \otimes b = ab$$

$$A \oplus B = \max(A, B)$$

$$C = A \otimes B$$

$$(A^r)_{ij} = \max_{\pi(i,j;r)} \pi(i,j;r)(A)$$

$$(I \oplus A \oplus \cdots \oplus A^{n-1})_{ij} =$$
$$\max_{\pi(i.j,r)} \pi(i,j;r): \;\; r = 0, \ldots n-1$$

A definite: $\lambda(A) = 1$

$$(I \oplus A \oplus \cdots \oplus A^{n-1})_{ij} =$$
$$\max_{\pi(i,j)} \pi(i,j): \;\; r = 0, \ldots,$$

Kleene star

A definite

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$$

$$(A^*)^2 = A^*$$

$$A \oplus x \leq x \iff A^* \oplus x = x$$

$$K_{\max}(A^*) := \{A^* \otimes x : x \geq 0\}$$

$$x \in K_{\max}(A^*) \iff A^* \oplus x = x$$

$K_{\max}(A^*)$ is convex cone

$$K_{\mathrm{cnv}}(A^*) = \{A^* x : x \geq 0\}$$

$$K_{\mathrm{cnv}}(A^*) \subseteq K_{\max}(A^*)$$

Cycle γ *critical*: $\mu(\gamma) = \lambda(A)$

Crit graph $C(A) \subseteq G(A)$:
union of critic γ

A visualized:
$a_{ij} \leq \gamma(A)$

A strictly visualized:
$a_{ij} \leq \lambda(A)$
$a_{ij} = \lambda(A) \Longleftrightarrow (i,j) \in \mathrm{crit}\gamma$

Diagonal similarity:

$$A \to X^{-1}AX : \; X = \mathrm{diag}(x)$$

$$X^{-1}AX \mathrm{vis} \iff x \in K_{\mathrm{cnv}}(A^*)$$

$$X^{-1}AX \mathrm{str\ vis} \iff x \in \mathrm{rint}(K_{\mathrm{cnv}}(A^*)), x > 0$$

$$x \in \mathrm{rint}(K_{\mathrm{cnv}}(A^*)) \iff x = A^* u, u > 0$$

$$C = \begin{bmatrix} \frac{1}{2} & 4 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{2} & 2 & \frac{1}{2} \end{bmatrix}$$

max cyc $1 \to 2 \to 1$

max cyc mean $= 1$

$$C^* = \begin{bmatrix} 1 & 4 & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$$

$$u = \begin{bmatrix} \frac{8}{3} & 0 & \frac{8}{3} \end{bmatrix}$$

$$x = (C^* u) = \begin{bmatrix} 4 & 1 & 4 \end{bmatrix}$$

$$CC = X^{-1}CX = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$CC^* = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Theorem

$$\mathrm{maxdim}(K_{\max}(A^*)) = \mathrm{lindim}(K_{\max}(A^*))$$

$=$ *no. cpts of* $C(A^*)$

$(v^1, v^2, \ldots, v^n)$ max indep:
No v^j a max comb of other v^i
$\max \dim(K_{\max}(A^*)) =$
max no of max indep cols in A^*

$$K_{\mathrm{cnv}}(A^*) \subseteq K_{\max}(A^*)$$

$$\mathrm{lindim}(K_{\mathrm{cnv}}(A^*)) \le \mathrm{lindim}(K_{\max}(A^*))$$

Example of strict inequ
C.Johnson & R. Smith
path matrices

$$A = A^* = \frac{1}{11}\begin{bmatrix} 11 & 5 & 5 & 7 & 7 & 7 \\ 5 & 11 & 5 & 7 & 7 & 7 \\ 5 & 5 & 11 & 7 & 7 & 7 \\ 7 & 7 & 7 & 11 & 5 & 5 \\ 7 & 7 & 7 & 5 & 11 & 5 \\ 7 & 7 & 7 & 5 & 5 & 11 \end{bmatrix}$$

$\text{rank}(A) = 5$, $\text{'maxrank'}(A) = 6$

THANK YOU