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Schneider, Hans

Positive operators and an inertia theorem.

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The main object of the author is to point out a common source for the following two facts (in generalized form). (1) If P is a non-negative real $n \times n$ matrix with largest characteristic root $<$ 1, then a non-negative n-vector x can be found such that $x - Px$ is non-negative. This follows from the fact that $I - P$ has an inverse which is non-negative. (2) If C is a complex matrix with $C^n \to 0$, then there exists a positive definite matrix H such that $H - CHC^* > 0$ [P. Stein, J. Res. Nat. Bur. Standards **48** (1952), 82–83; MR0047001 (13,813f)]. The link is provided by taking a closed cone $\mathfrak{C}$ (with interior) in a vector space and a positive operator P, i.e., $P\mathfrak{C} \leq \mathfrak{C}$; for (1), the vector space is formed by the set of real n-vectors, the cone by the non-negative vectors; for (2), the vector space is the space of $n \times n$ hermitian matrices, with the reals as scalars, the cone is the set of positive semidefinite hermitian matrices, and the operator P is the transformation CXC^*. The tool for proving the link is the Kreĭn-Rutnam theorem. By using this (an alternative matrix proof was given by Wielandt), the author proves a theorem which generalizes Lyapunov's theorem concerning stable matrices, as well as Stein's theorem. Namely, let A, C_k, $k = 1, 2, \cdots$, be complex matrices which can be triangulated simultaneously, and let α_i be the characteristic roots of A; under a natural correspondence let ${\gamma_i}^{(k)}$ be those of C_k. Then the two conditions "$|\alpha_i|^2 - \sum |{\gamma_i}^{(k)}|^2 > 0$" and "there exists a positive semidefinite H for which $T(H) = AHA^* - \sum C_k H {C_k}^*$ is positive semidefinite" are equivalent. For $s = 2$, $A = B + I$, $C_1 = B$, and $C_2 = I$, Lyapunov's theorem emerges, and for $s = 1$, $A = I$, $C_1 = C$, Stein's theorem. Actually, Lyapunov's and Stein's theorems are equivalent [see the reviewer, J. Algebra **1** (1964), 5–10; MR0161865 (28 #5069)].

It is further shown that a linear transformation on the space of $n \times n$ hermitian matrices, which maps the cone of positive semidefinite matrices onto itself, is either AHA^* or $AH'A^*$. This makes use of a theorem of M. Marcus and B. N. Moyls [Canad. J. Math. **11** (1959) 61–66; MR0099996 (20 #6432)]. It is shown that the extensions of Lyapunov's theorem to general matrices, found independently by A. Ostrowski and H. Schneider [J. Math. Anal. Appl. **4** (1962), 72–84; MR0142555 (26 #124)] and the reviewer [J. Soc. Indust. Appl. Math. **9** (1961), 640–643; MR0133336 (24 #A3170)], are not true for $T(H)$.

Reviewed by *O. Taussky-Todd*