



**NORTH-HOLLAND**

## **A Solution of a Nonlinear System Arising in Spectral Perturbation Theory of Nonnegative Matrices\***

Uriel G. Rothblum

*Faculty of Industrial Engineering and Management  
Technion-Israel Institute of Technology*

*Haifa 32000, Israel*

and

*RUTCOR—Rutgers Center for Operations Research*

*Rutgers University*

*New Brunswick, New Jersey 08903*

and

Hans Schneider

*Department of Mathematics*

*University of Wisconsin*

*Madison, Wisconsin 53706*

Submitted by Richard A. Brualdi

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### ABSTRACT

Let  $P$  and  $E$  be two  $n \times n$  complex matrices such that for sufficiently small positive  $\varepsilon$ ,  $P + \varepsilon E$  is nonnegative and irreducible. It is known that the spectral radius of  $P + \varepsilon E$  and corresponding (normalized) eigenvector have fractional power series expansions. The goal of the paper is to develop an algorithm for computing the coefficients of these expansions under two (restrictive) assumptions, namely that  $P$  has a single Jordan block corresponding to its spectral radius and that the (unique up to scalar multiples) left and right eigenvectors of  $P$  corresponding to its spectral

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radius, say  $v$  and  $w$ , satisfy  $v^T E w \neq 0$ . Our approach is to consider an associated countable system of nonlinear equations and solve this system recursively. At each step, we consider the coefficients of the expansion of the spectral radius of  $P + \varepsilon E$  as parameters and solve a related linear system parametrically. The next coefficient of the expansion of the spectral radius is then determined from feasibility considerations for a linear system. This solution method is novel and seems useful for computing coefficients of corresponding expansions when the two (restrictive) assumptions are relaxed. Also, interestingly, the coefficients we compute yield a preferred basis of the generalized eigenspace corresponding to the spectral radius of the unperturbed matrix  $P$ . © Elsevier Science Inc., 1997

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## 1. INTRODUCTION

A real matrix  $A$  is called *nonnegative*, written  $A \geq 0$ , if all entries are nonnegative;  $A$  is called *positive*, written  $A \gg 0$ , if all entries of  $A$  are positive; and  $A$  is called *semipositive*, written  $A > 0$ , if  $A \geq 0$  and  $A \neq 0$ . Corresponding definitions apply to vectors. An  $n \times n$  nonnegative matrix  $A$  is called *irreducible* if  $\sum_{i=0}^n A^i \gg 0$  (note that this formulation excludes the  $1 \times 1$  zero matrix). The *spectrum* of an  $n \times n$  matrix  $A$  will be denoted  $\sigma(A)$ , and its *spectral radius* will be denoted  $\rho(A)$ , i.e.,  $\sigma(A)$  is the set of eigenvalues of  $A$ , and  $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ .

The Perrone-Frobenius theorem (e.g., Berman and Plemmons, 1979) asserts that if  $A$  is a square, nonnegative, irreducible matrix, the spectral radius  $\rho(A)$  is a simple eigenvalue of  $A$ . Further, if  $f$  is a semipositive vector in  $R^n$ , then  $A$  has a unique right eigenvector  $u$  and a unique left eigenvector  $v$  corresponding to  $\rho(A)$  that satisfy  $f^T u = f^T v = 1$ . We denote these eigenvectors by  $u(A, f)$  and  $v(A, f)$ , respectively; in particular,  $u(A, f)$  and  $v(A, f)$  are positive and span the sets of left and right eigenvectors of  $A$  corresponding to  $\rho(A)$ , respectively.

The spectral radius and corresponding normalized eigenvectors of matrices govern the evolution of dynamic systems, and hence they are important characteristics of such systems; see numerous examples in Berman and Plemmons (1979). In particular, expansions of these characteristics for perturbed transition matrices are useful for sensitivity analysis of such systems. For example, perturbed stochastic matrices were studied by Schweitzer (1986) and Meyer and Stewart (1988), and perturbations of general (not necessarily stochastic) nonnegative matrices were explored in Cohen (1978), Deutsch and Neumann (1984), and Haviv, Ritov, and Rothblum (1992), among others. Of particular interest are explicit expansions of the above characteristics for dynamic systems under small scalar linear perturbation of their transition matrices.

Throughout the remainder of this paper, we assume that  $P$  and  $E$  are two given matrices in  $R^{n \times n}$  having the property that for sufficiently small positive  $\varepsilon$ ,  $P + \varepsilon E$  is nonnegative and irreducible. For  $\varepsilon > 0$ , we then let  $\rho(\varepsilon) \equiv \rho(P + \varepsilon E)$ , and for each semipositive vector  $f$ ,  $u(\varepsilon, f) \equiv u(P + \varepsilon E, f)$  and  $v(\varepsilon, f) \equiv v(P + \varepsilon E, f)$ .

Using algebraic methods, Eaves, Rothblum, and Schneider (1995) showed that  $\rho(\varepsilon)$  has a fractional power series expansion in  $\varepsilon$  of the form  $\sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q}$  for some positive integer  $q$ ; see also Kato [1966], where classic methods relying on the theory of functions of complex variables are used to establish the existence of such expansions for arbitrary eigenvalues. The expansion of  $\rho(\varepsilon)$  combines with standard arguments about solvability of linear systems over ordered fields to show that there is an expansion  $\varepsilon^{-p/q} \sum_{k=0}^{\infty} u_k \varepsilon^{k/q}$  of  $u(\varepsilon, f)$ . Eaves, Rothblum, and Schneider (1995) obtained an explicit system of (nonlinear) equations that characterizes the coefficients of these series. The purpose of the current paper is to describe a method for solving this system under two restrictive assumptions (described formally in Section 2); thus, under these assumptions, we get a method for computing the coefficients of the fractional power series expansions of  $\rho(\varepsilon)$  and  $u(\varepsilon, f)$ . All previously obtained explicit expansions of the Perron-Frobenius eigenvalue and corresponding normalized eigenvector with which we are familiar considered cases where the expansions are in the form of regular power series, rather than fractional power series; see, for example, Schweitzer (1986) and Haviv, Ritov, and Rothblum (1992). For computational simplicity, we consider only the case where  $f$  is a left eigenvector of the matrix  $P$ , but the case with general vector  $f$  can be derived from the particular case we consider by scaling.

We next outline the method we use for computing the coefficients of the expansions. Consider the equations defining  $\rho(\varepsilon)$  and  $u(\varepsilon, f)$ ,

$$(P + \varepsilon E)u(\varepsilon, f) = \rho(\varepsilon)u(\varepsilon, f) \quad \text{and} \quad f^T u(\varepsilon, f) = 1. \quad (1.1)$$

By substituting formal fractional power series for  $\rho(\varepsilon) = \sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q}$  and  $u(\varepsilon, f) = \varepsilon^{-p/q} \sum_{k=0}^{\infty} u_k \varepsilon^{k/q}$  and using Cauchy's formula for multiplying power series, one obtains a countable set of *nonlinear* equations for which the variables are  $p, q$  and the corresponding coefficients of the asserted expansions of  $\rho(\varepsilon)$  and  $u(\varepsilon, f)$ ; see Eaves, Rothblum, and Schneider (1995). We develop an iterative method for computing the general solution of the resulting system with the particular selection of  $q$  as the index of  $P$  for its spectral radius and with the selection of  $p$  as  $q - 1$ . We then show that one of the solutions yields the desired expansions of  $\rho(\varepsilon)$  and  $u(\varepsilon, f)$ ; in particular, we show that the corresponding fractional power series defined by this solution have positive radius of convergence.

The calculated coefficients of the expansion of the normalized Perron-Frobenius eigenvector  $u(\varepsilon, f)$  of the perturbed matrix  $P + \varepsilon E$  corresponding to nonpositive powers turn out to form a basis of the generalized eigenspace of the unperturbed matrix  $P$ ; in fact, this basis has nonnegativity properties that qualify it to be a “preferred basis” as constructed in Rothblum (1975) and Richman and Schneider (1978) [see also the survey in Schneider (1986)]. The derivation of preferred basis in these references filled a gap in the theory of nonnegative matrices that had lasted for over half a century. Specifically, the classical results of Perron and Frobenius for irreducible matrices show the existence of a positive, unique (up to scalar multiples) eigenvector corresponding to the spectral radius. Perturbing each zero entry of a nonnegative matrix by replacing it by a positive element  $\varepsilon$ , one concludes the existence of a positive eigenvector of the perturbed matrix for each such  $\varepsilon$ . Normalizing these eigenvectors and using a continuity argument then yields the existence of a semipositive eigenvector for the unperturbed matrix that corresponds to its spectral radius. But the restriction of this conclusion to irreducible matrices does not yield the existence of a positive eigenvector, suggesting that more can be said than the mere existence of a semipositive eigenvector. Indeed, the derivation of the preferred basis filled this gap, as a preferred basis for irreducible matrices consists of the single (normalized) positive eigenvector of the classical Perron-Frobenius theory for irreducible matrices. Our current results show that, under our two restricted assumptions, the preferred basis can be obtained from the Perron-Frobenius eigenvector of the perturbed matrix by truncating the positive powers of its power series expansion. In contrast, normalization and letting  $\varepsilon \rightarrow 0$  yields the first term of the expansion, which is a single element of the preferred basis. Thus, our analysis points out useful information, namely the preferred basis, that is available in the Perron-Frobenius eigenvector of the perturbed matrix. The extra information is obtained from the coefficients of the nonpositive powers of the fractional power series expansion of the eigenvector; it is not available just from the first term of the expansion obtained by normalizing and taking corresponding limits. We are currently exploring the derivation of preferred basis from the expansion of the eigenvector of the perturbed matrix without the restricted assumptions imposed in the current paper.

The two (restrictive) assumptions that we impose are described in Section 2. In Section 3 we describe the target system (over the complex field) of countably many (nonlinear) equations. We then describe an algorithm that is shown to produce, with all (finitely many) inputs, all solutions of the target system (under our two assumptions). In Section 4, we briefly describe a construction of fields of formal fractional power series in an indeterminate symbol over an arbitrary field  $F$ . In Section 5, we use a representation of the given matrix perturbation  $P + \varepsilon E$  as a nonnegative irreducible matrix over

the field  $R[\omega]$  of formal fractional power series over the reals. We then use the Perron-Frobenius theorem over  $R[\omega]$  to show that one of the solutions of the target system defines formal power series that yield its Perron-Frobenius eigenvalue and corresponding normalized eigenvector over  $R[\omega]$ . Results developed elsewhere are used to show that the resulting formal fractional power series have a positive radius of convergence, and an elaborate argument is used to show that expansions of the spectral radius and corresponding normalized eigenvector of the perturbed matrix are attainable from the algorithm. Finally, in Section 6, we discuss our results and the extension of the methods we introduce to more general cases (where the imposed assumptions are relaxed).

## 2. THE ASSUMPTIONS

Recall that  $P$  and  $E$  are two given  $n \times n$  real matrices where for sufficiently small positive  $\varepsilon$ ,  $P + \varepsilon E$  is nonnegative and irreducible. In the current section we state the (restrictive) assumptions we impose on the matrices  $P$  and  $E$ . Under the first assumption, we identify parameters which depend on  $P$ . These parameters are used to state the second assumption, and are further used in the forthcoming analysis. For convenience (and for consistency with notation used in the following sections) we use underlining in denoting the parameters we identify.

Let  $\underline{\rho} \equiv \rho(P)$ , and let  $\nu$  be the *index* of  $P$  corresponding to  $\underline{\rho}$ , i.e.,  $\nu \equiv \min\{k = 0, 1, \dots : \text{null}[(P - \underline{\rho}I)^{k+1}] = \text{null}[(P - \underline{\rho}I)^k]\}$ . Recall that a *generalized eigenvector of  $P$  corresponding to  $\underline{\rho}$*  is a solution of the system  $(P - \underline{\rho}I)^j x = 0$  for some positive integer  $j$ .

The first assumption we impose concerns the matrix  $P$ . It is introduced below:

**ASSUMPTION I.** The matrix  $P$  has a unique Jordan chain corresponding to its Perron-Frobenius eigenvalue  $\underline{\rho}$ .

Results of Schneider (1956) yield necessary and sufficient conditions for Assumption I in terms of the class structure of the matrix  $P$ ; see also Rothblum (1975) and Richman and Schneider (1978). We do not state these conditions here explicitly, because they are not used in our development.

Haviv, Ritov, and Rothblum (1992) obtained power series expansions of the spectral radius and corresponding normalized eigenvector under the assumption that the matrix  $P$  is irreducible. Irreducibility is known to imply that the spectral radius of  $P$  is a simple eigenvalue, i.e.,  $P$  has a single Jordan

chain of size 1, and that  $P$  has a positive eigenvector corresponding to  $\underline{\rho}$ . As the results of Haviv, Ritov, and Rothblum (1992) easily extend to the general case where  $P$  has a unique Jordan chain of size 1, our case of interest under Assumption I is when the size of the unique Jordan chain of  $P$  is 2 or more.

Assumption I, combined with results of Schneider (1956) [see also Rothblum (1975) and Richman and Schneider (1978)], implies the existence of vectors  $\underline{w}_0 \neq 0, \underline{w}_1, \dots, \underline{w}_{\nu-1}$  in  $R^n$  satisfying

$$(P - \underline{\rho}I)\underline{w}_j = \underline{w}_{j-1} \quad \text{for } j = 0, \dots, \nu - 1 \quad (2.1)$$

and

$$\underline{w}_j > 0 \quad \text{for } j = 0, \dots, \nu - 1, \quad (2.2)$$

where  $\underline{w}_{-1}$  is defined to be the zero vector. Further, by considering the Jordan decomposition of  $P$ , Assumption I also implies that:

(1) the right null space of  $P - \underline{\rho}I$  is one-dimensional; hence, each right eigenvector of  $P$  corresponding to  $\underline{\rho}$  is a scalar multiple of  $\underline{w}_0$ ;

(2) the left null space of  $P - \underline{\rho}I$  is one-dimensional; and

(3) if  $v$  is a left eigenvector of  $P$  corresponding to  $\underline{\rho}$ , then  $v^T \underline{w}_j = 0$  for  $j = 0, \dots, \nu - 2$  and  $v^T \underline{w}_{\nu-1} \neq 0$ .

It follows (by scaling an arbitrarily selected left eigenvector of  $P$  corresponding to  $\underline{\rho}$ ) that  $P$  has a unique eigenvector  $\underline{v}$  corresponding to  $\underline{\rho}$  satisfying

$$\underline{v}^T \underline{w}_{\nu-1} = 1; \quad (2.3)$$

further,  $\underline{v}$  satisfies

$$\underline{v}^T \underline{w}_j = 0 \quad \text{for } j = 0, \dots, \nu - 2. \quad (2.4)$$

As the Perron-Frobenius theorem assures that  $\underline{v}$  is a scalar multiple of a semipositive vector, the nonnegativity of  $\underline{w}_{\nu-1}$  combines with (2.3) to show that

$$\underline{v} > 0. \quad (2.5)$$

We are now ready to state our second assumption.

ASSUMPTION II. The matrix  $E$  satisfies  $\underline{v}^T E \underline{w}_0 > 0$ .

Assumption II is relaxed in Section 6 in the case where  $\nu = 1$ . When  $\nu \geq 2$ , (2.4) implies that  $\underline{v}^T P \underline{w}_0 = \underline{\rho} \underline{v}^T \underline{w}_0 = 0$ . Hence, the nonnegativity of

$\underline{v}$ , of  $\underline{w}_0$ , and of  $P + \varepsilon E$  for sufficiently small positive  $\varepsilon$  implies that for such  $\varepsilon$ ,  $0 \leq \varepsilon^{-1} \underline{v}^T (P + \varepsilon E) \underline{w}_0 = \underline{v}^T E \underline{w}_0$ . So, when  $\nu \geq 2$ , Assumption II is equivalent to the assertion that  $\underline{v}^T E \underline{w}_0 \neq 0$ .

Throughout the remainder of the paper, unless stated otherwise (in the extensions and discussion of Section 6), it is assumed that Assumptions I and II are in force.

### 3. A TARGET SYSTEM OF EQUATIONS AND ITS SOLUTION

Let  $f$  be a semipositive vector in  $R^n$ . It is shown in Eaves, Rothblum, and Schneider (1995, Theorem 4.4) that for some  $\gamma > 0$  the spectral radius  $\rho(\varepsilon)$  of the perturbed matrix  $P + \varepsilon E$  and the corresponding normalized eigenvector  $u(\varepsilon, f)$  have representations through converging fractional power series

$$\rho(P + \varepsilon E) = \sum_{k=0}^{\infty} \rho_k \varepsilon^{k/q} \quad \text{for } 0 < \varepsilon \leq \gamma \quad (3.1)$$

and

$$u(P + \varepsilon E, f) = \varepsilon^{-p/q} \sum_{k=0}^{\infty} u_k \varepsilon^{k/q} \quad \text{for } 0 < \varepsilon \leq \gamma, \quad (3.2)$$

respectively, where  $p$  is a nonnegative integer and  $q$  is a positive integer; we note here that  $q$  can be selected with  $q \leq N$  (see Eaves and Rothblum, 1995). Substituting these expressions into (1.1) and using the Cauchy formula for multiplying converging power series, it is observed in Eaves, Rothblum, and Schneider (1995, Theorem 4.6) that  $p$ ,  $q$ , and the coefficients  $\rho_0, \rho_1, \rho_2, \dots, u_0, u_1, \dots$  of the above representations satisfy the following set of (nonlinear) equations:

$$(P - \rho_0 I)u_k = \begin{cases} \sum_{s=0}^{k-1} \rho_{k-s} u_s & \text{for } 0 \leq k < q, \\ \sum_{s=0}^{k-1} \rho_{k-s} u_s - E u_{k-q} & \text{for } q \leq k, \end{cases} \quad (3.3)$$

and

$$f^T u_k = \begin{cases} 0 & \text{for } k \neq p, \\ 1 & \text{for } k = p. \end{cases} \quad (3.4)$$

The goal of the current section is to describe a recursive method for computing the general solution (over the complex field) of the target system (3.3)–(3.4) with  $\rho_0 = \rho$ ,  $f = \underline{v}$ ,  $q = \nu$ , and  $p = \nu - 1$ . Here and throughout, underlined symbols denote particular selections of vectors and scalars, whereas un-underlined symbols are used to denote variables in equations we take on solving. We emphasize that our solution technique does not rely on convergence properties of corresponding power series, or on the assertion that solutions should provide representations of the spectral radius and corresponding normalized eigenvector of the perturbed matrix.

The iterative procedure we are about to describe formally has the following structure. At the first stage,  $\rho_1$  is determined as one of  $\nu$  potential values, and  $u_0$  is determined in terms of  $\rho_1$ . At stage  $k \geq 2$ , the coefficients up to (but not including)  $u_{k-1}$  and  $\rho_k$  have been determined. These determined variables are then used to parametrically solve the subsystem consisting of equations  $k - 1$  through  $k + \nu - 2$  of (3.3). Next, feasibility of equation  $k + \nu - 1$  of (3.3) together with (normalization) condition  $k + \nu - 1$  of (3.4) is used to determine  $u_{k-1}$  and  $\rho_k$ .

In order to distinguish between variables and substituted values, we shall use underbars to denote specific selection of vectors and scalars. In particular, we continue to use the notation  $\underline{w}_0, \underline{w}_1, \dots, \underline{w}_{\nu-1}$ , for the Jordan chain of  $P$  corresponding to its spectral radius, and continue to use the notation  $\underline{\rho}$  and  $\underline{v}$  for the spectral radius of  $P$  and corresponding left eigenvector. Also, we continue to use the notation  $P$ ,  $E$ , and  $\nu$  without underlines for the given matrices and for the index of  $P$  with respect to its spectral radius  $\underline{\rho}$ .

With  $f = \underline{v}$  and the selection  $\rho_0 = \rho$ ,  $q = \nu$ , and  $p = \nu - 1$ , (3.3) and (3.4) reduce to the following system of (nonlinear) equations with variables  $\rho_1, \rho_2, \dots, \dots$  and  $u_0, u_1, \dots$ :

$$(P - \underline{\rho}I)u_k = \begin{cases} \sum_{s=0}^{k-1} \rho_{k-s}u_s & \text{for } 0 \leq k < \nu, \\ \sum_{s=0}^{k-1} \rho_{k-s}u_s - Eu_{k-\nu} & \text{for } \nu \leq k, \end{cases} \quad (\text{A})$$

and

$$\underline{v}^T u_k = \begin{cases} 0 & \text{for } k \neq \nu - 1, \\ 1 & \text{for } k = \nu - 1. \end{cases} \quad (\text{B})$$

For  $k = 0, 1, \dots$ , we denote by (A.k) and (B.k) the equations of (A) and (B), respectively, that determine  $(P - \rho_0 I)u_k$  and  $\underline{v}^T u_k$ .

The following lemma follows from standard results about solvability of linear systems. It is key for the forthcoming solution of (A)–(B). The lemma and its corollary are stated for arbitrary fields containing  $\underline{\rho}$  and the entries of  $P$ , but only the real and the complex fields are of interest.

LEMMA 3.1. *Let  $F$  be a field that contains the entries of  $P$  and  $\underline{\rho}$ , and let  $\underline{b} \in F^n$ . Then the system  $(P - \underline{\rho}I)x = \underline{b}$  is feasible over  $F$  if and only if  $\underline{v}^T \underline{b} = 0$ . Further, if  $\underline{x} \in F^n$  is a particular solution of the above system, then its general solution over  $F$  has the form  $x = \underline{x} + \gamma \underline{w}_0$  where  $\gamma$  is an arbitrary scalar in  $F$ .*

Standard results show that if  $F$  is a field which contains the entries of  $P$  and  $\underline{\rho}$ , then all entries of the  $\underline{w}_j$ 's and of  $\underline{v}$  are in  $F$ . Thus, we get from (2.1) the following corollary of Lemma 2.1.

COROLLARY 3.2. *Let  $F$  be a field that contains the entries of  $P$  and  $\underline{\rho}$ . If  $\alpha_0, \alpha_1, \dots, \alpha_{\nu-2} \in F$ , then the system  $(P - \underline{\rho}I)x = \sum_{s=0}^{\nu-2} \alpha_s \underline{w}_s$  is feasible, and its general solution is given by  $x = \sum_{s=0}^{\nu-2} \alpha_s \underline{w}_{s+1} + \gamma \underline{w}_0$  where  $\gamma$  is an arbitrary scalar in  $F$ .*

The next two lemmas show that truncations of (A)–(B) are feasible and allow for exactly  $\nu$  distinct values for  $\rho_1$ ; further, they identify variables that are uniquely determined by such truncations when augmented by the requirement that  $\rho_1$  takes any one of its  $\nu$  feasible values.

LEMMA 3.3. *The system consisting of (A.0)–(A. $\nu$ ) and (B.0)–(B. $\nu-1$ ) is feasible over the complex field, and every solution of this system has  $\rho_1$  as a (possibly complex)  $\nu$ -root of  $\underline{v}^T E \underline{w}$ . Further, if  $\rho_1$  is any specific  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , then the system (A.0)–(A. $\nu$ ), (B.0)–(B. $\nu-1$ ) augmented with  $\rho_1 = \underline{\rho}_1$ , with free variables  $u_0, u_1, \dots, u_\nu, \rho_1, \rho_2, \dots, \rho_\nu$ , is feasible, and  $u_0 = (\underline{\rho}_1)^{-\nu+1} \underline{w}_0 \neq 0$  for every solution of this system.*

*Proof.* Evidently, (A.0) is feasible and its general solution has the representation  $u_0 = \gamma_0 \underline{w}_0$  for some scalar  $\gamma_0$ . It follows that (A.1) reduces to

$$(P - \underline{\rho}I)u_1 = \rho_1 \gamma_0 \underline{w}_0,$$

and Corollary 3.2 assures that if  $\nu \geq 2$  this equation is feasible and its general solution has the representation

$$u_1 = \rho_1 \gamma_0 \underline{w}_1 + \gamma_1 \underline{w}_0$$

where  $\gamma_1$  is an arbitrary scalar. Next, Equation (A.2) reduces to

$$\begin{aligned} (P - \underline{\rho}I)u_2 &= \rho_2 \gamma_0 \underline{w}_0 + \rho_1 (\rho_1 \gamma_0 \underline{w}_1 + \gamma_1 \underline{w}_0) \\ &= (\rho_1)^2 \gamma_0 \underline{w}_1 + (\rho_2 \gamma_0 + \rho_1 \gamma_1) \underline{w}_0, \end{aligned}$$

and Corollary 3.2 implies that if  $\nu \geq 3$ , this equation is feasible and its general solution has the representation

$$u_2 = (\rho_1)^2 \gamma_0 \underline{w}_2 + (\rho_2 \gamma_0 + \rho_1 \gamma_1) \underline{w}_1 + \gamma_2 \underline{w}_0$$

where  $\gamma_2$  is an arbitrary scalar. It follows from a simple inductive argument that for  $j = 0, 1, \dots, \nu - 1$ , (A.0)–(A.j) are jointly feasible and their general solution has  $\rho_1, \rho_2, \dots, \rho_j$  arbitrary and

$$u_i = \sum_{t=0}^i \delta_{it} \underline{w}_t \quad \text{for } i = 0, 1, \dots, j, \quad (3.5)$$

where each  $\delta_{it}$  is a polynomial in  $\rho_1, \rho_2, \dots, \rho_j$  and in arbitrary scalars  $\gamma_0, \gamma_1, \dots, \gamma_j$ ; in particular,

$$\delta_{i0} = \gamma_i \text{ and } \delta_{ii} = (\rho_1)^i \gamma_0 \quad \text{for } i = 0, 1, \dots, j. \quad (3.6)$$

The representation of the general solution of (A.0)–(A. $\nu-1$ ) given in (3.5)–(3.6) combines with (2.3)–(2.4) to show that (B.0)–(B. $\nu-2$ ) are satisfied by all solutions of (A.0)–(A. $\nu-1$ ), while (B. $\nu-1$ ) is satisfied if and only if

$$1 = \underline{v}^T \sum_{t=0}^{\nu-1} \delta_{\nu-1,t} \underline{w}_t = \delta_{\nu-1,\nu-1} \underline{v}^T \underline{w}_{\nu-1} = (\rho_1)^{\nu-1} \gamma_0. \quad (3.7)$$

Also, by Lemma 3.1, (2.3)–(2.4), and (3.5)–(3.6), feasibility of (A. $\nu$ ) for the general solution of (A.0)–(A. $\nu-1$ ) is equivalent to the assertion

$$\begin{aligned} 0 &= \underline{v}^T (\rho_\nu u_0 + \rho_{\nu-1} u_1 + \dots + \rho_2 u_{\nu-2} + \rho_1 u_{\nu-1} - E u_0) \\ &= \rho_1 \underline{v}^T u_{\nu-1} - \underline{v}^T E u_0 = \rho_1 \delta_{\nu-1,\nu-1} \underline{v}^T \underline{w}_{\nu-1} - \gamma_0 \underline{v}^T E \underline{w}_0 \\ &= \gamma_0 (\rho_1)^\nu - \gamma_0 \underline{v}^T E \underline{w}_0. \end{aligned} \quad (3.8)$$

Of course, (3.7)–(3.8) holds if and only if  $\rho_1$  is a  $\nu$ -root of  $\underline{v}^T E \underline{w}$  and  $\gamma_0 = (\underline{\rho}_1)^{-\nu+1} \neq 0$  (recall that  $\underline{v}^T E \underline{w} > 0$  by Assumption II). We conclude that the system consisting of (A.0)–(A. $\nu$ ) and (B.0)–(B. $\nu-1$ ) is feasible and each solution of this system must have  $\rho_1$  as a  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , say  $\underline{\rho}_1$ , and  $u_0$  as  $\underline{u}_0 \equiv (\underline{\rho})^{-\nu+1} \underline{w}_0 \neq 0$  [obtained by selecting  $\gamma_0$  as  $(\underline{\rho}_1)^{-\nu+1}$ ]. ■

LEMMA 3.4. *Let  $\underline{\rho}_1$  be a (possibly complex)  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , and let  $k = 0, 1, \dots$ . Then there exist complex vectors  $\underline{u}_0 \neq 0, \underline{u}_1, \dots, \underline{u}_k$  and complex scalars  $\underline{\rho}_2, \dots, \underline{\rho}_{k+1}$  such that:*

(a) *with the substitution of  $u_0 = \underline{u}_0, u_1 = \underline{u}_1, \dots, u_k = \underline{u}_k, \rho_1 = \underline{\rho}_1, \rho_2 = \underline{\rho}_2, \dots, \rho_{k+1} = \underline{\rho}_{k+1}$ , the system consisting of (A.0)–(A. $k + \nu$ ) and (B.0)–(B. $k + \nu-1$ ) with the remaining free variables  $u_{k+1}, \dots, u_{k+\nu}$  and  $\rho_{k+2}, \dots, \rho_{k+\nu}$  is feasible, and*

(b) *every solution of the system consisting of (A.0)–(A. $k + \nu$ ), (B.0)–(B. $k + \nu-1$ ), and  $\rho_1 = \underline{\rho}_1$  has  $u_0 = \underline{u}_0, u_1 = \underline{u}_1, \dots, u_k = \underline{u}_k, \rho_1 = \underline{\rho}_1, \rho_2 = \underline{\rho}_2, \dots, \rho_{k+1} = \underline{\rho}_{k+1}$ .*

*Further, if  $\underline{\rho}_1$  is real, then so are  $\underline{u}_0, \underline{u}_1, \dots, \underline{u}_k, \underline{\rho}_2, \dots, \underline{\rho}_{k+1}$ .*

*Proof.* We prove the lemma by induction on the integer  $k$ . The case where  $k = 0$  follows directly from Lemma 3.3 with  $\underline{u}_0 \equiv (\underline{\rho}_1)^{-\nu+1} \underline{w}_0$ . Suppose that for integer  $k = 1, 2, \dots$ , the conclusion of the lemma holds for the integer  $k - 1$  replacing  $k$ , with determined vectors  $\underline{u}_0 \neq 0, \underline{u}_1, \dots, \underline{u}_{k-1}$  and determined scalars  $\underline{\rho}, \underline{\rho}_2, \dots, \underline{\rho}_k$ , and we will establish the conclusion of the lemma with the integer  $k$  itself. Our method is to examine equations (A. $k$ )–(A. $k + \nu-1$ ) in the remaining free variables  $u_k, \dots, u_{k+\nu-1}$  and  $\rho_{k+1}, \dots, \rho_{k+\nu-1}$  and determine the representation of the general (parametric) solution of this system. We then show that each such solution must satisfy (B. $k$ )–(B. $k + \nu-2$ ) and that Equation (B. $k + \nu-1$ ) together with feasibility of (A. $k + \nu$ ) determines  $u_k$  and  $\rho_{k+1}$ .

We next show, by a secondary inductive argument, that for each  $j = 0, \dots, 1, \dots, \nu - 1$ , the general solution of (A. $k$ )–(A. $k + j$ ) with the substitution

$$u_0 = \underline{u}_0, \quad u_1 = \underline{u}_1, \dots, u_{k-1} = \underline{u}_{k-1}, \quad \rho_1 = \underline{\rho}_1, \quad \rho_2 = \underline{\rho}_2, \dots, \quad \rho_k = \underline{\rho}_k \tag{3.9}$$

has

$$\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+j} \text{ arbitrary} \tag{3.10}$$

and

$$u_{k+i} = \underline{u}'_{k+i} + \sum_{t=0}^i \delta_{k+i,t} \underline{w}_t \quad \text{for } i = 0, 1, \dots, j, \quad (3.11)$$

where each  $\underline{u}'_{k+i}$  is a specified (computable) vector and each  $\delta_{k+i,t}$  is a (scalar) polynomial in  $\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+i}$  and arbitrary scalars  $\gamma_k, \gamma_{k+1}, \dots, \gamma_{k+j}$  (which are independent of  $i$  and of  $t$ ); in particular,

$$\delta_{k+i,0} = \gamma_{k+i} \quad \text{and} \quad \delta_{k+i,i} = (\underline{\rho}_1)^i \gamma_k + i \underline{\gamma}_0 (\underline{\rho}_1)^{i-1} \rho_{k+1}.$$

We first consider Equation (A.k) under the substitution (3.9); in particular, its only remaining free variable is  $u_k$ , and the equation reduces to

$$(P - \underline{\rho}I)u_k = \underline{\rho}_k u_0 + \underline{\rho}_{k-1} u_1 + \dots + \underline{\rho}_2 u_{k-2} + \underline{\rho}_1 u_{k-1} - E u_{k-\nu},$$

where  $\underline{u}_j$  is defined to be zero for  $j < 0$ . As our inductive assumption asserts that this equation is feasible, Lemma 3.1 implies that its general solution has the representation

$$u_k = \underline{u}'_k + \gamma_k \underline{w}_0$$

for some fixed vector  $\underline{u}'_k$  and arbitrary scalar  $\gamma_k$ . So, for  $j = 0$ , the representation (3.10)–(3.12) has been established for the solution of (A.k)–(A.k + j) under the substitution (3.9).

Next assume that for some  $j \in \{0, \dots, \nu - 2\}$ , the general solution of (A.k)–(A.k + j) with the substitution of (3.9) has the asserted representation (3.10)–(3.12). To establish the asserted representation with  $j + 1$  replacing  $j$ , we recall equation (A.k + j + 1) (without any substitution) has the form:

$$\begin{aligned} (P - \underline{\rho}I)u_{k+j+1} &= \rho_{k+j+1} u_0 + \rho_{k+j} u_1 + \dots + \rho_{k+1} u_j + \dots + \rho_1 u_{k+j} \\ &\quad - E u_{k+j+1-\nu}. \end{aligned} \quad (3.13)$$

Substituting (3.9), the general expressions (3.10)–(3.12) for  $u_k, \dots, u_{k+j}$  asserted by the secondary inductive assumption, and the representation of  $u_0, \dots, u_{\nu-1}$  given in (3.5)–(3.6), we see that (3.13) can be written as

$$(P - \underline{\rho}I)u_{k+j+1} = \underline{b}'_{k+j+1} + \sum_{t=0}^j \theta_{k+j+1,t} \underline{w}_t, \quad (3.14)$$

where

$$\underline{b}'_{k+j+1} \equiv \underline{\rho}_k \underline{u}_{j+1} + \cdots + \underline{\rho}_{j+2} \underline{u}_{k-1} + \underline{\rho}_{j+1} \underline{u}'_k + \cdots + \underline{\rho}_1 \underline{u}'_{k+j} - E \underline{u}_{k+j+1-\nu}, \quad (3.15)$$

and each  $\theta_{k+j+1,t}$  is a polynomial in  $\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+j+1}, \gamma_k, \gamma_{k+1}, \dots, \gamma_{k+j}$ ; in particular, we have that

$$\begin{aligned} \theta_{k+j+1,j} &= \rho_{k+1} \underline{\gamma}_0(\underline{\rho}_1)^j + \underline{\rho}_1 \delta_{k+j,j} \\ &= \rho_{k+1} \underline{\gamma}_0(\underline{\rho}_1)^j + \underline{\rho}_1 \left[ (\underline{\rho}_1)^j \gamma_k + j \underline{\gamma}_0(\underline{\rho}_1)^{j-1} \rho_{k+1} \right] \\ &= (\underline{\rho}_1)^{j+1} \gamma_k + (j+1) \underline{\gamma}_0(\underline{\rho}_1)^j \rho_{k+1}. \end{aligned}$$

As our primary inductive assumption asserts that (A.0)–(A. $k+j+1$ ) is feasible, Lemma 3.1 and (2.1) imply that the general representation of the solution  $u_{k+j+1}$  of (3.13) has the form

$$\begin{aligned} u_{k+j+1} &= \underline{u}'_{k+j+1} + \sum_{t=0}^j \theta_{k+j+1,t} \underline{w}_{t+1} \\ + \gamma_{k+j+1} \underline{w}_0 &= \underline{u}'_{k+j+1} + \sum_{t=0}^{j+1} \delta_{k+j+1,t} \underline{w}_t \end{aligned}$$

for corresponding  $\delta_{k+j+1,t}$ 's; in particular,  $\delta_{k+j+1,0} = \gamma_{k+j+1}$  and

$$\delta_{k+j+1,j+1} = \theta_{k+j+1,j} = (\underline{\rho}_1)^{j+1} \gamma_k + (j+1) \underline{\gamma}_0(\underline{\rho}_1)^j \rho_{k+1}.$$

This completes the secondary inductive argument, and verifies the representation (3.10)–(3.12) of the general solution of (A. $k$ )–(A. $k+j$ ) under the substitution (3.9) for each  $j = 0, 1, \dots, \nu-1$ .

We next observe that if  $j \leq \nu-2$ , (2.4) implies that  $v^T u_{k+j}$  is constant for all vectors  $u_{k+j}$  having the representation (3.11). As our primary inductive assumption asserts that the joint system (A.0)–(A. $k+\nu-1$ ) and (B.0)–(B. $k+\nu-2$ ) with the substitution (3.9) is feasible, it follows from Lemma 3.1 and the established representation of the general solution of (A.0)–(A. $k+\nu-2$ ) under

the substitution (3.9) that all such solutions satisfy (B.k)–(B.k +  $\nu$ -2). Further, we obtained the following representation of  $u_{k+\nu-1}$  in a general solution of (A.k)–(A.k +  $\nu$ -1) under the substitution (3.9):

$$\begin{aligned} u_{k+\nu-1} &= \underline{u}'_{k+\nu-1} + \sum_{t=0}^{\nu-1} \delta_{k+\nu-1,t} \underline{w}_t + \delta_{k+\nu-1,\nu-1} \underline{w}_{\nu-1} \\ &= \underline{u}'_{k+\nu-1} + \sum_{t=0}^{\nu-2} \delta_{k+\nu-1,t} \underline{w}_t \\ &\quad + \left[ (\underline{\rho}_1)^{\nu-1} \gamma_k + (\nu-1) \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-2} \rho_{k+1} \right] \underline{w}_{\nu-1}. \end{aligned}$$

As  $k + \nu - 1 \geq \nu$ , it follows from (2.3)–(2.4) that (B.k +  $\nu$ -1) is satisfied if and only if

$$\begin{aligned} 0 &= \underline{v}^T \left( \underline{u}'_{k+\nu-1} + \sum_{t=0}^{\nu-2} \delta_{k+\nu-1,t} \underline{w}_t \right. \\ &\quad \left. + \left[ (\underline{\rho}_1)^{\nu-1} \gamma_k + (\nu-1) \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-2} \rho_{k+1} \right] \underline{w}_{\nu-1} \right) \\ &= \underline{v}^T \underline{u}'_{k+\nu-1} + \left[ (\underline{\rho}_1)^{\nu-1} \gamma_k + (\nu-1) \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-2} \rho_{k+1} \right]. \quad (3.16) \end{aligned}$$

Next, the arguments used in the secondary inductive step to establish the reduction of (A.k +  $j + 1$ ) to (3.14)–(3.15) for  $j \in \{0, \dots, \nu - 2\}$  can also be used to show that (A.k +  $\nu$ ) with the substitutions of (3.9), with the general expressions (3.10)–(3.12) for  $u_k, \dots, u_{k+\nu-1}$ , and with the representations of  $u_0, \dots, u_{\nu-1}$  given in (3.5)–(3.6) reduces to

$$\begin{aligned} (P - \underline{\rho}I)u_{k+\nu} &= \underline{b}'_{k+\nu} + \sum_{t=0}^{\nu-2} \theta_{k+\nu,t} \underline{w}_t \\ &\quad + \left[ (\underline{\rho}_1)^\nu \gamma_k + \nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1} \rho_{k+1} \right] \underline{w}_{\nu-1} - \gamma_k E \underline{w}_0, \quad (3.17) \end{aligned}$$

where  $\underline{b}'_{k+\nu}$  is a specified (computable) vector and each  $\theta_{k+j+1,t}$  is a polynomial in  $\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+\nu}, \gamma_k, \gamma_{k+1}, \dots, \gamma_{k+\nu-1}$  [the last term in

the right hand side of (3.17) was missing from (3.14), as  $Eu_{k+j+1-\nu}$  was absorbed into the constant term  $\underline{b}'_{k+j+1}$ . Lemma 3.1 and (2.3)–(2.4) imply that feasibility of (3.17) is equivalent to the assertion

$$\begin{aligned}
 0 &= \underline{v}^T \left( \underline{b}'_{k+\nu} + \sum_{t=0}^{\nu-2} \theta_{k+\nu,t} \underline{w}_t \right. \\
 &\quad \left. + \left[ (\underline{\rho}_1)^\nu \gamma_k + \nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1} \rho_{k+1} \right] \underline{w}_{\nu-1} - \gamma_k E \underline{w}_0 \right) \\
 &= \underline{v}^T \underline{b}'_{k+\nu} + \left[ (\underline{\rho}_1)^\nu \gamma_k + \nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1} \rho_{k+1} \right] - \gamma_k \underline{v}^T E \underline{w}_0 \\
 &= \underline{v}^T \underline{b}'_{k+\nu} + \gamma_k \left[ (\underline{\rho}_1)^\nu - \underline{v}^T E \underline{w}_0 \right] + \nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1} \rho_{k+1} \\
 &= \underline{v}^T \underline{b}'_{k+\nu} + \nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1} \rho_{k+1}, \tag{3.18}
 \end{aligned}$$

where the last equality follows from the assumption that  $\underline{\rho}_1$  is a  $\nu$ -root of  $\underline{v}^T E \underline{w}_0$ . Thus, (3.18) characterizes feasibility of (A.  $k + \nu$ ) under the substitution (3.9).

We have show that the system consisting of (A.0)–(A.  $k + \nu$ ) and (B.0)–(B.  $k + \nu - 1$ ) with the substitution of (3.9) is feasible whenever (3.16) and (3.18) are satisfied. Further, the induction assumption shows that each solution of (A.0)–(A.  $k + \nu$ ) and (B.0)–(B.  $k + \nu - 1$ ) satisfies (3.9), and the above two paragraphs imply that each such solution must also satisfy (3.16) and (3.18). As (3.16) and (3.18) consist of two (linear) equations

$$0 = \underline{v}^T \underline{u}'_{k+\nu-1} + \left[ (\underline{\rho}_1)^{\nu-1} \gamma_k + (\nu - 1) \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-2} \rho_{k+1} \right] \tag{3.16'}$$

and

$$0 = \underline{v}^T \underline{b}'_{k+\nu} + \nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1} \rho_{k+1}, \tag{3.17'}$$

which uniquely determine  $\rho_{k+1}$  and  $\gamma_k$  and unique determination of  $\gamma_k$  uniquely determines  $u_k$ , the proof of the primary inductive step is complete.

We finally observe that if  $\underline{\rho}_1$  is real, the inductive argument can be extended to assert that the coefficients of (3.16')–(3.17') are real; consequently, so are the uniquely determined values of  $\rho_{k+1}$ ,  $\gamma_k$ , and  $u_k$ . ■

Note that the unique solution of the (linear) system (3.16')–(3.17') is given by

$$\rho_{k+1} = -\frac{\underline{v}^T \underline{b}'_{k+\nu}}{\nu \gamma_0 (\underline{\rho}_1)^{\nu-1}}, \quad (3.19)$$

$$\begin{aligned} \gamma_k &= -\frac{\underline{v}^T \underline{u}'_{k+\nu-1}}{(\underline{\rho}_1)^{\nu-1}} - \frac{[(\nu-1)\gamma_0 \rho_{k+1}]}{\underline{\rho}_1} \\ &= -\frac{\underline{v}^T \underline{u}'_{k+\nu-1}}{(\underline{\rho}_1)^{\nu-1}} - \left(1 - \frac{1}{\nu}\right) \frac{\underline{v}^T \underline{b}'_{k+\nu}}{(\underline{\rho}_1)^\nu}. \end{aligned} \quad (3.20)$$

As Assumption II asserts that  $\underline{v}^T E \underline{w} > 0$ , there is a unique selection of  $\underline{\rho}_1$  in Lemma 3.4 as a positive element. Hence, we have the following corollary of Lemma 3.4.

**COROLLARY 3.5.** *Let  $\underline{\rho}_1$  be the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , and let  $k = 0, 1, \dots$ . Then there exist real vectors  $\underline{u}_0 \neq 0, \underline{u}_1, \dots, \underline{u}_k$  and real scalars  $\underline{\rho}_2, \dots, \underline{\rho}_{k+1}$  such that:*

(a) *with the substitution of  $u_0 = \underline{u}_0, u_1 = \underline{u}_1, \dots, u_k = \underline{u}_k, \rho_1 = \underline{\rho}_1, \rho_2 = \underline{\rho}_2, \dots, \rho_{k+1} = \underline{\rho}_{k+1}$ , the system consisting of (A.0)–(A. $k + \nu$ ) and (B.0)–(B. $k + \nu - 1$ ) with the remaining free variables  $u_{k+1}, \dots, u_{k+\nu}$  and  $\rho_{k+2}, \dots, \rho_{k+\nu}$  is feasible, and*

(b) *every solution of the system consisting of (A.0)–(A. $k + \nu$ ), (B.0)–(B. $k + \nu - 1$ ) with  $\rho_1 > 0$  has  $u_0 = \underline{u}_0, u_1 = \underline{u}_1, \dots, u_k = \underline{u}_k, \rho_1 = \underline{\rho}_1, \rho_2 = \underline{\rho}_2, \dots, \rho_{k+1} = \underline{\rho}_{k+1}$ .*

*Proof.* By Lemma 3.3 every solution of (A.0)–(A. $\nu$ ) and (B.0)–(B. $\nu - 1$ ) has  $\rho_1$  as a  $\nu$ -root of  $\underline{v}^T E \underline{w}$ . As it is assumed that  $\underline{v}^T E \underline{w} > 0$ , there is exactly one such root which is positive. The remainder of the corollary now follows directly from Lemma 3.4. ■

The proofs of Lemmas 3.3 and 3.4 are constructive; hence, they yield an algorithm that will generate the  $\underline{u}_i$ 's and  $\underline{\rho}_i$ 's. For a formal description of a corresponding algorithm we recall the following standard fact about solvability of linear systems.

LEMMA 3.6. *There exists a matrix  $H^g$  such that for every vector  $\underline{b}$  for which the linear system  $(P - \underline{\rho}I)x = \underline{b}$  is feasible, the vector  $H^g\underline{b}$  is a particular solution of that system.*

The matrix  $H^g$  asserted in Lemma 3.6 is not unique. Such matrices belong to the class of generalized inverses of  $P - \underline{\rho}I$ ; a description of algorithms that compute them and further details are available in Campbell and Meyer (1979), for example.

We are now ready to present an algorithm that summarizes the inductive construction within the proofs of Lemmas 3.3 and 3.4. The presentation uses the solution of (3.16')–(3.17') given in (3.19)–(3.20). We note that the matrices  $P$  and  $E$  remain fixed and given.

ALGORITHM.

*Input:*  $\underline{\rho}_1$ —a (possibly complex)  $\nu$ -root of  $\underline{v}^T E \underline{w}_0$ .

*Step 0:*

Set

$$\begin{aligned} \underline{\rho}_j &= 0 \text{ for all integers } j < 0, \\ \underline{u}_j &= 0 \text{ for all integers } j < 0, \\ \underline{\gamma}_0 &= (\underline{\rho}_1)^{-\nu+1}, \text{ and} \\ \underline{u}_0 &= \underline{\gamma}_0 \underline{w}_0. \end{aligned}$$

*Step  $k$  for  $k = 1, 2, \dots$ :*

Set

$$\begin{aligned} \underline{b}'_k &= \underline{\rho}_k \underline{u}_0 + \underline{\rho}_{k-1} \underline{u}_1 + \dots + \underline{\rho}_2 \underline{u}_{k-2} + \underline{\rho}_1 \underline{u}_{k-1} - E \underline{u}_{k-\nu}, \text{ and} \\ \underline{u}'_k &= H^g \underline{b}'_k. \end{aligned}$$

For  $j = 0, 1, \dots, \nu - 2$ , set

$$\underline{b}'_{k+j+1} = \underline{\rho}_k \underline{u}_{j+1} + \dots + \underline{\rho}_{j+2} \underline{u}_{k-1} + \underline{\rho}_{j+1} \underline{u}'_k + \dots + \underline{\rho}_1 \underline{u}'_{k+j} - E \underline{u}_{k+j+1-\nu}, \text{ and } \underline{u}'_{k+j+1} = H^g \underline{b}'_{k+j+1}.$$

Set

$$\begin{aligned} \underline{b}'_{k+\nu} &= \underline{\rho}_k \underline{u}_\nu + \dots + \underline{\rho}_{\nu+1} \underline{u}_{k-1} + \underline{\rho}_\nu \underline{u}'_k + \dots + \underline{\rho}_1 \underline{u}'_{k+\nu-1} - E \underline{u}'_k, \\ \underline{\rho}_{k+1} &= - \frac{\underline{v}^T \underline{b}'_{k+\nu}}{\nu \underline{\gamma}_0 (\underline{\rho}_1)^{\nu-1}}, \text{ and} \end{aligned}$$

$$\underline{u}_k = \underline{u}'_k - \left[ \frac{\underline{v}^T \underline{u}'_{k+\nu-1}}{(\underline{\rho}_1)^{\nu-1}} + \left( 1 - \frac{1}{\nu} \right) \frac{\underline{v}^T \underline{b}'_{k+\nu}}{(\underline{\rho}_1)^\nu} \right] \underline{w}_0.$$

*Output:*  $\underline{\rho}_1, \underline{\rho}_2, \dots, \underline{u}_0, \underline{u}_1, \dots$

The next result summarizes properties of the output of the Algorithm. The proof is immediate from the proofs of Lemma 3.3 and 3.4 and Corollary 3.5.

## THEOREM 3.7.

(a) *The Algorithm has exactly  $\nu$  distinct inputs. The  $\nu$  outputs they determine are distinct.*

(b) *For each input, the output of the Algorithm is a solution of (A)–(B).*

(c) *For each solution of (A)–(B), there is an input of the Algorithm such that the resulting output is the given solution.*

(d) *If the input of the Algorithm is the positive  $\nu$ -root of  $\underline{v}^T E \underline{W}_0$ , then the output is the unique solution of (A)–(B) with  $\rho_1 > 0$ .*

We emphasize that though  $H^\#$  is not unique, the generated output of the Algorithm with any given input is unique; see Lemmas 3.3 and 3.4.

The next two corollaries are immediate from Theorem 3.7. The first shows that each of  $\nu$   $\nu$ -roots of  $\underline{v}^T E \underline{w}$  determines a unique solution of (A)–(B). The second spectralizes this observation to the selection of the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ .

**COROLLARY 3.8.** *Each solution of (A)–(B) has  $\rho_1$  as a (possibly complex)  $\nu$ -root of  $\underline{v}^T E \underline{w}$ . Further, under each selection of  $\underline{\rho}_1$  as a  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , the system (A)–(B) augmented with  $\rho_1 = \underline{\rho}_1$  has a unique solution.*

**COROLLARY 3.9.** *The system (A)–(B) augmented with  $\rho_1 > 0$  has a unique solution.*

## 4. FRACTIONAL POWER SERIES OVER FIELDS

In the next section we prove that when the input of the Algorithm (as described at the end of the previous section) is the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , the output yields converging power series representations of the spectral radius and corresponding normalized eigenvector of the perturbed matrices  $P + \varepsilon E$  for sufficiently small positive  $\varepsilon$ . Convergence of the power series generated by outputs of the Algorithm turns out to follow directly from (algebraic) transfer principles identified in Eaves, Rothblum, and Schneider (1995, Section 5) (see the last paragraph of the next section). More effort is devoted to demonstrating that the output of the algorithm with the particular (positive) input generates the spectral radius and corresponding normalized eigenvector. Here we find ourselves relying on results of Eaves and Rothblum (1996) to show that, with all  $\nu$  inputs, the power series generated by the Algorithm yield *all* the eigenvalues of the perturbed matrices.

In order to use to the above references and carry through our analysis, we have to refer to the formal ordered field of fractional power series over an arbitrary given (ordered) field  $F$ , and this section is devoted to the description of a construction of this field.<sup>1</sup> Our description concerns an arbitrary (ordered) field  $F$ , but in the forthcoming development only the cases where  $F$  is the field  $C$  of complex numbers or the ordered field  $R$  of the reals are used. Also, we let  $Z$  denote the ring of integers,  $Q$  denote the ordered field of rationals, and  $Z_+$  and  $Q_+$  denote the positive elements in  $Z$  and  $Q$ , respectively.

Throughout, let  $\omega$  denote an indeterminate symbol. Define  $F'[\omega]$  to be the collection of all triplets  $(r, p, a)$  where  $r \in Q_+$ ,  $p \in Z$ , and  $a: Z \rightarrow F$  where  $a_i = 0$  for all  $i \leq p$ . For  $(r, p, a) \in F'[\omega]$ , we define  $r$  to be the *exfactor* and  $p$  to be the *base*; also, with  $j \equiv \min\{i \in Z: a_i \neq 0\}$ , we define  $rj$  to be the *order* and  $a_j$  to be the *order coefficient*. If  $a \equiv 0$ , the order and order coefficient are defined to be  $+\infty$  and 0, respectively; in all other cases, the order is finite and the order coefficient is nonzero. For the sake of convenience, we denote a triple  $(r, p, a) \in F'[\omega]$  by the formal sum  $\sum_{i=p}^{\infty} a_i \omega^{ir}$ , or briefly  $\sum_p a_i \omega^{ir}$ , where  $\omega$  is an *indeterminate* symbol, and for  $i = \dots, -1, 0, 1, \dots$  we refer to the element  $a_i$  as the *ir-coefficient* of  $(r, p, a)$ .

We next introduce a relation  $\approx$  over  $F'[\omega]$ . For elements  $\sum_p a_i \omega^{ir}$  and  $\sum_q b_i \omega^{is}$  in  $F'[\omega]$ , we write  $\sum_p a_i \omega^{ir} \approx \sum_q b_i \omega^{is}$  if  $a_i = b_j$  for every pair of integers  $i$  and  $j$  with  $ir = js$ . It is easily seen that  $\approx$  is an equivalence relation. We denote the collection of corresponding equivalence classes which partition  $F'[\omega]$  by  $F[\omega]$ . It is easy to verify that the order, order coefficient, and 0-coefficient are invariant over equivalence classes in  $F'[\omega]$ . Consequently, the order, order coefficient, and 0-coefficient are well defined for an element  $\alpha$  in  $F[\omega]$ , and we denote them by  $\text{order}(\alpha)$ ,  $\text{ordercoeff}(\alpha)$ , and  $\alpha_0$ , respectively.

It is easily observed that the exfactor and the base are not invariant within equivalence classes of  $F'[\omega]$ , in fact, if  $\sum_p a_i \omega^{ir} \in F'[\omega]$ , then:

- (1) if  $q \in Z_+$  and  $q \leq p$ , then  $\sum_p a_i \omega^{ir} \approx \sum_q a_i \omega^{ir}$ , and
- (2) if  $s = r/k$  where  $k \in Z_+$ , then  $\sum_p a_i \omega^{ir} \approx \sum_q b_i \omega^{is}$ , where  $q = kp$  and

$$b_i = \begin{cases} a_{i/k} & \text{for } i = kp, k(p+1), k(p+2), \dots, \\ 0 & \text{otherwise.} \end{cases}$$

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<sup>1</sup>The construction described herein borrows from Eaves, Rothblum, and Schneider (1995) and Eaves and Rothblum (1996).

So the exfactor and the base for elements in  $F[\omega]$  are not uniquely defined. Still, given  $\alpha \in F[\omega]$ , we say that  $r \in Q_+$  is an *exfactor* of  $\alpha$  if  $r$  is the exfactor of some representative of  $\alpha$ . Similarly, we say that  $p \in Z$  is a *base* of  $\alpha$  if  $p$  is the base of some representative of  $\alpha$ . The above observation demonstrates that the base of an element in  $F[\omega]$  can be arbitrarily reduced and its exfactor can be divided by an arbitrary positive integer.

Consider a pair of elements  $\alpha$  and  $\beta$  in  $F[\omega]$  with representations  $\sum_p a_i \omega^{ir}$  and  $\sum_q b_i \omega^{is}$ , respectively. As  $r$  and  $s$  are positive rationals there exist  $e, f, g, h \in Z_+$  such that  $r = e/f$  and  $s = g/h$ . It then follows that both  $\alpha$  and  $\beta$  have representations with exfactor  $1/fh$  and base  $\min\{peh, qfg\}$ . Thus, every pair of elements in  $F[\omega]$  have representations with common base and exfactor. We define addition and multiplication of elements in  $F[\omega]$  by using such representations, namely, if  $\alpha$  and  $\beta$  are elements in  $F[\omega]$  with representations  $\sum_p a_i \omega^{ir}$  and  $\sum_p b_i \omega^{ir}$ , respectively, we let  $\alpha + \beta$  and  $\alpha\beta$  be the equivalence classes in  $F[\omega]$  of  $\sum_p (a_i + b_i) \omega^{ir}$  and  $\sum_{2p} (\sum_{j=p}^{i-p} a_j b_{i-j}) \omega^{ir}$ , respectively. It is easy to verify that these definitions of addition and multiplication in  $F[\omega]$  are well defined, that is, the outcome of these operations is independent of the selected representations.

The underlying field  $F$  is embedded in  $F[\omega]$  where an element  $u \in F$  is identified with the equivalence class of the element  $(1, 0, a) \in F'[\omega]$  with  $a_i = 0$  for all  $i \neq 0$  and  $a_0 = u$ . We shall identify the elements of  $F$  with the corresponding elements of  $F[\omega]$ , i.e., we consider  $F$  to be a subset of  $F[\omega]$ . In particular, the additive identity of  $F$  (zero) and the multiplicative identity of  $F$  (one) are considered to be elements of  $F[\omega]$ . The equivalence class of the element  $(r, 1, a) \in F'[\omega]$  with  $a_i = 0$  for all  $i \neq 1$  and  $a_1 = 1$  is denoted  $\omega^r$ , and if  $r = 1$ , we write  $\omega$  for  $\omega^1$ .

For  $r \in Q$ , we use the notation  $O(\omega^r)$  for an element in  $F[\omega]$  of order  $r$  or higher. Observing that for  $\alpha, \beta \in F[\omega]$  we have  $\text{order}(\alpha + \beta) = \min\{\text{order}(\alpha), \text{order}(\beta)\}$  and  $\text{order}(\alpha\beta) = \text{order}(\alpha) + \text{order}(\beta)$  [see Eaves and Rothblum (1995, Section 3)], it follows that for  $r, r' \in Q$

$$O(\omega^r) + O(\omega^{r'}) = O(\omega^{\min(r, r')}) \quad \text{and} \quad O(\omega^r)O(\omega^{r'}) = O(\omega^{r+r'}). \quad (4.1)$$

Next consider the case where  $F$  is an ordered field. In this case we define an order on  $F[\omega]$  by saying that a nonzero element  $\alpha \in F[\omega]$  is *positive*, written  $\alpha > 0$ , if its order coefficient is positive. It may be verified that addition and multiplication preserve positivity in  $F[\omega]$ .

We refer to  $F[\omega]$  as the *formal field of fractional power series over  $F$* . The name is justified by the next theorem.

**THEOREM 4.1.**  *$F[\omega]$  is a field with  $F$  as a subfield. If  $F$  is an ordered field,  $F[\omega]$  is an ordered field with  $F$  as a subordered field.*

We next turn our attention to the case where the underlying field is either the complex field  $C$  or the real field  $R$ . Let  $\iota \in C$  be a square root of  $-1$ . We observe that the representation  $C = R + \iota R$  extends to  $C[\omega]$  and we have that  $C[\omega] = R[\omega] + \iota R[\omega]$ .

An *eigenvalue* of a matrix  $A \in C[\omega]^{n \times n}$  is an element  $\lambda \in C[\omega]$  for which there is a nonzero vector  $u \in C[\omega]^n$  such that  $Au = \lambda u$ . In this case we say that  $u$  is an *eigenvector of  $A$  corresponding to  $\lambda$* . Of course,  $\lambda \in C[\omega]$  is an eigenvalue of  $A \in C[\omega]^{n \times n}$  if and only if  $\lambda$  is a root of the characteristic polynomial  $\chi_A(x) \equiv \det(xI - A)$ , which is a polynomial with coefficients in  $C[\omega]$ . If  $A \in R[\omega]^{n \times n}$ ,  $\lambda = \alpha + \iota\beta$ , and  $u = v + \iota w$ , we have that  $\lambda$  is an eigenvalue of  $A$  with corresponding eigenvector  $u$  if and only if either  $v \neq 0$  or  $w \neq 0$  and

$$Av = \alpha v - \beta w \quad \text{and} \quad Aw = \alpha w + \beta v.$$

## 5. SOLUTION OF THE TARGET SYSTEM AND EXPANSIONS OF THE SPECTRAL RADIUS AND CORRESPONDING EIGENVECTOR OF THE PERTURBED MATRICES

In the current section we relate outputs of the Algorithm to eigenvalues and corresponding normalized eigenvectors of the matrix  $P + \omega E$  whose elements are in the field  $R[\omega]$ . Using the Perron-Frobenius theorem over  $R[\omega]$ , we further show that the output corresponding to the (unique) positive input defines converging power series that yield the spectral radius and normalized eigenvector of the perturbed matrices  $P + \varepsilon E$  for sufficiently small positive  $\varepsilon$ .

We start by observing that solutions of (A)–(B) correspond to eigenvalues and corresponding normalized eigenvectors of  $P + \omega E$ . We conclude that outputs of the Algorithm generate such eigenvalues and corresponding normalized eigenvectors.

**LEMMA 5.1.** *Suppose  $\rho_1, \rho_2, \dots$  are complex scalars and  $\underline{u}_0, \underline{u}_1, \dots$  are vectors in  $C^n$ . Then (A)–(B) is satisfied by  $\underline{\rho}_1, \underline{\rho}_2, \dots, \underline{u}_0, \underline{u}_1, \dots$  if and only if  $\underline{\rho} + \sum_1 \underline{\rho}_k \omega^{k/\nu}$  is an eigenvalue of  $P + \omega E$  with eigenvector*

$\omega^{-(\nu-1)/\nu}(\sum_0 \underline{u}_k \omega^{k/\nu})$  that satisfies the normalization condition

$$\underline{v}^T \left[ \omega^{-(\nu-1)/\nu} \left( \sum_0 \underline{u}_k \omega^{k/\nu} \right) \right] = 1. \quad (5.1)$$

*Proof.* Let  $\underline{\rho}_0 \equiv \underline{\rho}$ . The rules for executing arithmetic operations in  $C[\omega]$  directly imply that (A) is equivalent to the equation (over  $C[\omega]$ )

$$\begin{aligned} (P + \omega E) \left[ \omega^{-(\nu-1)/\nu} \left( \sum_0 \underline{u}_k \omega^{k/\nu} \right) \right] \\ = \left( \sum_0 \underline{\rho}_k \omega^{k/\nu} \right) \left[ \omega^{-(\nu-1)/\nu} \left( \sum_0 \underline{u}_k \omega^{k/\nu} \right) \right], \end{aligned} \quad (5.2)$$

and (B) is equivalent to (5.1). ■

**COROLLARY 5.2.** *Let  $\underline{\rho}_1$  be an input of the Algorithm (that is,  $\underline{\rho}_1$  is a  $\nu$ -root of  $\underline{v}^T E \underline{w}$ ), and let  $\underline{\rho}_1, \underline{\rho}_2, \dots$  and  $\underline{u}_0, \underline{u}_1, \dots$  be the corresponding output. Then  $\underline{\rho} + \sum_1 \underline{\rho}_k \omega^{k/\nu}$  is an eigenvalue of  $P + \omega E$  with eigenvector  $\omega^{-(\nu-1)/\nu}(\sum_0 \underline{u}_k \omega^{k/\nu})$  that satisfies (5.1). Further, eigenvalues corresponding to distinct inputs are distinct.*

*Proof.* By Theorem 3.7, the output  $\underline{\rho}_1, \underline{\rho}_2, \dots$  and  $\underline{u}_0, \underline{u}_1, \dots$  of the Algorithm is a solution of (A)–(B); hence, by Lemma 5.1,  $\underline{\rho} + \sum_1 \underline{\rho}_k \omega^{k/\nu}$  is an eigenvalue of  $P + \omega E$  with eigenvector  $\omega^{-(\nu-1)/\nu}(\sum_0 \underline{u}_k \omega^{k/\nu})$  that satisfies (5.1). Of course, the eigenvalues generated by distinct inputs of the Algorithm are distinct, as their (selected)  $(1/\nu)$ -coefficients are distinct. ■

The assumption that  $P + \varepsilon E$  is nonnegative and irreducible for sufficiently small positive  $\varepsilon$  implies (in fact, is equivalent to) the assertion that  $P + \omega E$  is nonnegative and irreducible as a matrix in  $R[\omega]^{n \times n}$ . By Eaves, Rothblum, and Schneider (1995),  $R[\omega]$  is a real closed field (the formal definition is given therein), and (consequently) the Perron-Frobenius theorem holds over  $R[\omega]$ . The next proposition summarizes resulting properties of  $P + \omega E$ .

**PROPOSITION 5.3.** *There exists a positive element  $\rho$  in  $R[\omega]$  and a positive vector  $u$  in  $R[\omega]^n$  such that:*

(a)  $\rho$  is the unique eigenvalue of  $P + \omega E$  having a semipositive eigenvector,

(b)  $u$  is the unique eigenvector of  $P + \omega E$  corresponding to  $\rho$  that satisfies  $\underline{v}^T u = 1$ ,

(c)  $\text{order}(u) \geq 0$ , and

(d) for every eigenvalue  $\lambda \in C[\omega]$  of  $P + \omega E$  with representatoin  $\lambda = \alpha + \iota\beta$  we have that  $\alpha^2 + \beta^2 \leq \rho^2$ .

*Proof.* As  $\underline{v}$  is a semipositive vector in  $R^n \subseteq R[\omega]^n$  and  $P + \omega E$  is a semipositive, irreducible matrix in  $R[\omega]^{n \times n}$ , the conclusions of the proposition follow directly from Eaves, Rothblum, and Schneider (1995, Theorems 3.1, 4.2, and 4.6). ■

The (unique) positive element  $\rho \in R[\omega]$  and vector  $u \in R[\omega]^n$  identified in Proposition 5.1 will be denoted  $\rho(P + \omega E)$  and  $u(P + \omega E, \underline{v})$ , respectively (consistently with the notation used for real nonnegative matrices).

We next show that the outputs of the Algorithm with the (unique) positive input generates  $\rho(P + \omega E)$  and  $u(P + \omega E, \underline{v})$ . The analysis we need for this task is surprisingly elaborate.

**THEOREM 5.4.** *Let  $\underline{\rho}_1$  be the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , and let  $\underline{\rho}_1, \underline{\rho}_2, \dots$  and  $\underline{u}_0, \underline{u}_1, \dots$  be the output of the Algorithm with input  $\underline{\rho}_1$ . Then  $\underline{\rho} + \sum_1 \underline{\rho}_k \omega^{k/\nu} = \rho(P + \omega E)$  and  $\omega^{-(\nu-1)/\nu} \sum_0 \underline{u}_k \omega^{k/\nu} = u(P + \omega E, \underline{v})$ .*

*Proof.* Let  $Q \equiv P + \omega E \in R[\omega]^{n \times n}$ . We make four observations about the characteristic polynomials  $\chi_Q(x)$  of  $Q$  and  $\chi_P(x)$  of  $P$ :

(1)  $\chi_Q(x)$  can be written as  $\chi_Q(x) = \chi_P(x) + p(x)$  where all the coefficients of  $p(x)$  have positive order,

(2)  $\chi_Q(x)$  has degree  $\nu$ ,

(3) the leading coefficient of  $\chi_Q(x)$  (namely, the coefficient of  $x^\nu$ ) is 1, and

(4)  $\rho$  is a root of  $\chi_P(x)$  of multiplicity  $\nu$ .

These observations combine with the results of Eaves and Rothblum (1995, Section 6) to show that, allowing for multiplicities,  $\chi_Q(x)$  has exactly  $\nu$  roots whose 0-coefficient is  $\underline{\rho}$ , that is, allowing for multiplicities,  $Q$  has exactly  $\nu$  eigenvalues whose 0-coefficient is  $\underline{\rho}$ . Now, the Algorithm has exactly  $\nu$  inputs (namely, the  $\nu$  complex roots of  $\underline{v}^T E \underline{w}$ ), and by Corollary 5.2, the outputs corresponding to these inputs generate  $\nu$  distinct eigenvalues of  $Q$ . It follows that all eigenvalues of  $Q$  with 0-coefficient  $\underline{\rho}$  are generated by the Algorithm.

We next argue that  $\rho(Q)$  is generated by the Algorithm. As  $\rho(Q)$  is an eigenvalue of  $Q$ , it suffices (by the conclusion of the above paragraph) to

show that the 0-coefficient of  $\rho(Q)$ , say  $c$ , equals  $\underline{\rho}$ . We first observe that as  $\text{order}[\rho(Q)] \geq 0$  and  $\rho(Q) > 0$  (Proposition 5.3), we have that  $c \geq 0$ , and  $\rho(Q)$  can be written as  $\rho(Q) = c + O(\omega^t)t > 0$  where  $t \equiv \text{order}[\rho(Q) - c]$ . We also observe that  $c$  is a root of  $\chi_\rho(x)$  [e.g., Eaves and Rothblum (1996, Lemma 3.3)], that is,  $c$  is an eigenvalue of  $P$ ; hence,  $c^2 \leq \underline{\rho}^2$ . Next, consider the output of the Algorithm with the selection of  $\underline{\rho}_1$  as the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ , and let  $\lambda$  be the resulting eigenvalue of  $\underline{Q}$  (see Lemma 5.1). Then  $\lambda \in R[\omega]$  and  $\lambda = \underline{\rho} + O(\omega^{1/\nu})$ . Further, part (d) of Proposition 5.3 assures that  $[\rho(Q)]^2 \geq \lambda^2$ ; hence,

$$c^2 + O(\omega^t) = \rho(Q)^2 \geq \lambda^2 = \underline{\rho}^2 + O(\omega^{1/\nu}),$$

implying that  $c^2 \geq \underline{\rho}^2$ . So  $c^2 = \underline{\rho}^2$ . As both  $c$  and  $\underline{\rho}$  are nonnegative, it follows that  $c = \underline{\rho}$ .

We next argue that  $\rho(Q)$  is generated by the selection of  $\underline{\rho}_1$  as the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ . Suppose  $\lambda^1, \dots, \lambda^\nu$  be a list of all the eigenvalues of  $Q$  with 0-coefficient  $\underline{\rho}$ , that is (by our earlier conclusion), all the eigenvalues of  $Q$  that are generated by the Algorithm; in particular,  $\rho(Q)$  is in this list. Evidently, for each  $k = 1, \dots, \nu$ , the order of  $(\lambda^k - \underline{\rho})$  is  $1/\nu$ , and its order coefficient, say  $g^k$ , is the  $\nu$ -root of  $\underline{v}^T E \underline{w}$  that determines  $\lambda^k$ . Let  $a^k + \iota b^k$  be the representation of  $g^k$  where  $a^k$  and  $b^k$  are elements in  $R$ , and let  $\alpha^k + \iota \beta^k$  be the representation  $\lambda^k$  where  $\alpha^k$  and  $\beta^k$  are elements in  $R[\omega]$ . We then have that

$$\alpha^k = \underline{\rho} + a^k \omega^{1/\nu} + O(\omega^{2/\nu}),$$

$$\beta^k = b^k \omega^{1/\nu} + O(\omega^{2/\nu}),$$

and

$$(\alpha^k)^2 + (\beta^k)^2 = \underline{\rho}^2 + 2\underline{\rho}a^k \omega^{1/\nu} + O(\omega^{2/\nu}).$$

It immediately follows that  $(\alpha^k)^2 + (\beta^k)^2$  is uniquely maximized over  $k$  when  $\lambda^k$  is generated through the selection of  $\underline{\rho}_1$  as the positive  $\nu$ -root of  $\underline{v}^T E \underline{w}$ ; hence, by part (d) of Proposition 5.3, this eigenvalue is the spectral radius  $\rho(Q)$  of  $Q$ . It now follows from the uniqueness conclusion of part (b) of Proposition 5.3 that the generated normalized eigenvector corresponding to this eigenvalue is  $u(Q)$ . ■

**THEOREM 5.5.** *Let  $\underline{\rho}_1, \underline{\rho}_2, \dots$  and  $\underline{u}_0, \underline{u}_1, \dots$  be the output of the Algorithm with input  $\underline{\rho}_1 > \bar{0}$ . Then for all sufficiently small positive  $\varepsilon$  the*

power series  $\sum_{k=1}^{\infty} \underline{\rho}_k \varepsilon^{k/q}$  and  $\sum_{k=0}^{\infty} \underline{u}_k \varepsilon^{k/q}$  converge absolutely, and for such  $\varepsilon$ ,  $\rho(P + \varepsilon E) = \underline{\rho} + \sum_{k=1}^{\infty} \underline{\rho}_k \varepsilon^{k/q}$  and  $u(P + \varepsilon E, \underline{v}) = \varepsilon^{-p/q} \sum_{k=0}^{\infty} \underline{u}_k \varepsilon^{k/q}$ .

*Proof.* As  $\underline{\rho}_1, \underline{\rho}_2, \dots, \underline{u}_0, \underline{u}_1, \dots$  form a solution of (3.3) with  $\rho_0 = \rho$ ,  $f = \underline{v}$ ,  $q = \nu$ , and  $\underline{p} = \nu - 1$  where not all the  $\underline{u}_i$ 's are zero (Theorem 3.7), the first part of Eaves, Rothblum, and Schneider (1995, Theorem 5.3) (directly) assures that  $\sum_{k=1}^{\infty} \underline{\rho}_k \varepsilon^{k/q}$  converges absolutely for all sufficiently small positive  $\varepsilon$ . Further, as the  $\underline{u}_i$ 's satisfy (3.4) with  $f = \underline{v}$  and, by Theorems 5.4 and 5.2,  $\sum_0 \underline{u}_k \omega^{k/q} = u(P + \omega E, f) \gg 0$  (positive in  $R[\omega]$ ), the remaining conclusions of the theorem follow (directly) from the second part of Eaves, Rothblum, and Schneider (1995, Theorem 5.3). ■

Eaves, Rothblum, and Schneider (1995, Theorem 5.3) implies that for every solution of (A)–(B) we have that the power series  $\sum_{k=1}^{\infty} \underline{\rho}_k \varepsilon^{k/q}$  has a positive radius of convergence. Further, the proof of Theorem 5.5 implies that each of the  $\nu$  eigenvalues of  $P + \varepsilon E$  generated by the Algorithm is simple; thus, each has a single eigenvector satisfying the normalization condition (5.1). Consider the field  $C_+[\omega]$  of fractional power series over the complex numbers that have positive radius of convergence. Of course,  $C_+[\omega]$  is a subfield of  $C[\omega]$ ,  $P + \omega E \in (C_+[\omega])^{n \times n}$ , and the above argument shows that every eigenvalue determined by an execution of the Algorithm is in  $C_+[\omega]$ . As unique solvability of a linear system is invariant over any selected extension of the field containing the coefficients, we conclude that the unique eigenvectors corresponding to the determined eigenvalues are vectors with coefficients in  $C_+[\omega]$ . Thus, all eigenvalues and corresponding eigenvectors of  $P + \omega E$  over  $C[\omega]$  that are determined by the Algorithm yield complex power series with positive radius of convergence. This fact is not stated and derived formally, as our interest here is restricted to the spectral radii and corresponding normalized eigenvectors.

## 6. DISCUSSION AND EXTENSIONS

Herein, (3.3)–(3.4) with  $f = \underline{v}$  is solved under restrictive assumptions which assert, among other things, that the matrix  $P$  has a unique Jordan chain corresponding to its Perron-Frobenius eigenvalue. In particular, the obtained solution has  $q = \nu$  and  $\underline{p} = \nu - 1$ , where  $\nu$  is the index of  $P$  corresponding to the Perron-Frobenius eigenvalue of  $P$ . We recall that in Haviv, Ritov, and Rothblum (1992), (3.3)–(3.4) is solved under the assumption that  $P$  is belongs to a class of matrices with index 1 that contains the

class of irreducible matrices. The obtained solution has  $q = 1$  and  $p = 0$ . But the following example demonstrates that in general,  $q$  and  $p$  do not have such simple representation in terms of the index. Let

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The index of  $P$  is 3, but the spectral radius of  $P + \varepsilon E$  is  $1 + \sqrt{2} \varepsilon$ , implying that (3.3)–(3.4) does not have a solution with  $q = 3$ . Still, we conjecture that there exists a combinatorial/algebraic algorithm that computes corresponding integers  $q$  and  $p$  when Assumptions I and II are relaxed. In fact, we believe that ideas of the solution method developed in the restrictive case we consider can be modified and applied to producing a solution of (3.3)–(3.4) in the general case.

We also note that in the solutions of (3.3)–(3.4) obtained herein and in Haviv, Ritov, and Rothblum (1992), the coefficients of the fractional power series of the Perron-Frobenius normalized eigenvector turn out to yield a “preferred basis” of the generalized eigenspace corresponding to the Perron-Frobenius eigenvalue of  $P$ ; see Rothblum (1975), Richman and Schneider (1978), and Schneider (1986) for formal definitions, and see the introduction for a discussion of this property. Though this interesting phenomenon does not extend in general (note the above example), there seems to much spectral information in the coefficients of the fractional power series expansions of the Perron-Frobenius eigenvector of the perturbation of a given nonnegative matrix.

We next discuss the need of the restrictive assumptions for our analysis. First, Assumption I is used to verify the existence of a solution to the (recursive) system  $(P - \rho I)x = \underline{b}'_{k+j+1}$ . While a matrix  $H^\varepsilon$  satisfying the conclusion of Lemma 3.6 can be identified when Assumption I fails to hold, its application to  $\underline{b}'_{k+j+1}$  will not produce a solution to  $(P - \rho I)x = \underline{b}'_{k+j+1}$  when the system is infeasible. Also, Assumption II asserting that  $\underline{v}^T E \underline{w} \neq 0$  assures that the Algorithm has  $\nu$  distinct input and thereby generates  $\nu$  eigenvalues of  $P - \varepsilon I$ .

When  $\nu = 1$ , our analysis does carry through even if Assumption II is relaxed. It is easily verified that in this particular case Lemma 3.3 is valid with the modification that we let  $\underline{u}_0 = \underline{w}_0$  [that is,  $(\rho_1)^{\nu-1}$  is defined to be 1 even if  $\rho_1 = 0$  (which occurs when  $\underline{v}^T E \underline{w} = 0$ )]. The remaining results of Section 3 then remain unchanged. In particular, we get a single solution of (A)–(B), and the analysis of Section 5 implies that this solution yields the expansions of the spectral radius of  $P$  and corresponding eigenvector. We

note that Assumption I together with the assumption that  $\nu = 1$  yields a special instance of the situation studied and solved in Haviv, Ritov, and Rothblum (1992).

Our analysis focused on nonnegative irreducible linear perturbations of a given matrix  $P$ , i.e., perturbations of the form  $P + \varepsilon E$  for sufficiently small positive  $\varepsilon$ . But, we next argue that the methods we develop generalize to polynomial perturbation of the form  $P(\varepsilon) \equiv P + \sum_{i=1}^m \varepsilon^i E_i$ . First, Assumption I remains unchanged, and Assumption II translates to the condition that for some  $i$ ,  $\underline{v}^T E_i \underline{w} \neq 0$ . Proposition 5.3 extends to the more general context, asserting the existence of the spectral radius of  $P + \sum_{i=1}^m \omega^i E_i$  in  $R[\omega]$ . The resulting system of equations for the coefficients is obviously more complicated than (3.3)–(3.4), and so is the resulting variant of (A)–(B) obtained by letting  $\rho_0 = \rho(P)$ ,  $q = \nu$ , and  $p = \nu - 1$ . Still, our solution method can be modified to find the general solution of the variant of (A)–(B), and one of the solutions will yield an expansion of the spectral radius and corresponding normalized eigenvector of the perturbed matrices  $P(\varepsilon)$ . Further, any solution of the modification of (A)–(B) defines a fractional power series with a positive radius of convergence; see the paragraph following Theorem 5.5.

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