Math 847 Notes
Modular Forms

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# Contents

I. **Modular forms**

1.1 2014-01-22: Introduction to modular forms ................................................................. 7
   1.1.1 Modular forms ........................................................................................................ 7
   1.1.2 Eisenstein series .................................................................................................... 8
1.2 2014-01-24: Applications of modular forms ............................................................... 9
   1.2.1 Sums of squares ..................................................................................................... 9
   1.2.2 Theta functions ...................................................................................................... 9
   1.2.3 More general thetas ............................................................................................... 10
1.3 2014-01-27: Modular curves ......................................................................................... 11
   1.3.1 Example: Fundamental domain for $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ .................... 11
   1.3.2 $\text{SL}_2(\mathbb{R})$-structure of the upper half-plane .............................................. 12
1.4 2014-01-29: Construction of modular curves ............................................................ 12
   1.4.1 The action is properly discontinuous .................................................................... 12
   1.4.2 Make $Y(\Gamma)$ a complex manifold ................................................................. 13
1.5 2014-01-31: Elliptic points and compactification ....................................................... 14
   1.5.1 Elliptic points ........................................................................................................ 14
   1.5.2 Compactification ................................................................................................... 14
   1.5.3 The topology on $\mathcal{H}$ ...................................................................................... 15
   1.5.4 The topology of $X(\Gamma)$ ................................................................................... 15
   1.5.5 Complex structure at the cusps ............................................................................ 16

II. **Dimensions of spaces of modular forms** .................................................................. 17

II.1 2014-02-03: Dimension formulas .............................................................................. 17
   II.1.1 Motivational example .......................................................................................... 17
   II.1.2 Maps between compact Riemann surfaces and the Riemann–Hurwitz formula .............................................................................................................. 18
II.2 2014-02-05: Riemann–Hurwitz formula ....................................................................... 19
   II.2.1 Remarks on Riemann–Hurwitz .......................................................................... 19
   II.2.2 Prototypical ramified cover ............................................................................... 19
II.3 2014-02-07: Ramification at elliptic points ............................................................... 20
   II.3.1 The local picture ................................................................................................... 20
   II.3.2 Ramification indices ............................................................................................ 20
   II.3.3 Genus of $X(\Gamma)$ .............................................................................................. 22
II.4 2014-02-10: Meromorphic differentials on Riemann surfaces ................................... 22
   II.4.1 Meromorphic modular forms ............................................................................ 22
II.4.2 Differential forms ........................................... 22
II.4.3 Pullbacks of differential forms ............................... 23
II.4.4 Gluing differential forms ..................................... 24
II.5 2014-02-12: Meromorphic modular forms and differentials .... 24
II.6 2014-02-14: Divisors and Riemann–Roch ..................... 25
II.6.1 Meromorphic modular forms and differentials .......... 25
II.6.2 Divisors on Riemann surfaces ............................. 26
II.6.3 The Riemann–Roch theorem .................................. 27
II.7 2014-02-16: Riemann–Roch .................................. 27
II.7.1 Consequences of Riemann–Roch ............................. 27
II.7.2 Application to meromorphic modular forms .......... 28
II.7.3 Order of differentials ......................................... 29
II.8 2014-02-19: Computing the dimensions ...................... 29
II.8.1 Computing the dimension of \( M_k(\Gamma) \) .................. 29
II.8.2 Computing the dimension of \( S_k(\Gamma) \) ................... 31
II.8.3 Negative weight modular forms ............................ 31
II.8.4 Example: modular forms for \( \text{SL}_2(\mathbb{Z}) \) .......... 31
II.8.5 Odd weight modular forms .................................. 32

III Eisenstein series .............................................. 33
III.1 2014-02-21 [missing] ........................................... 33
III.2 2014-02-24 ..................................................... 33
III.2.1 Cusps of \( \Gamma(N) \) ........................................... 33
III.2.2 Cocycle relations ............................................ 33
III.3 2014-02-26: Eisenstein series ............................... 35
III.4 2014-02-28: The \( q \)-expansion ............................. 37
III.5 2014-03-03 ..................................................... 39
III.5.1 \( G_4 \) and \( G_6 \) generate modular forms .............. 40
III.5.2 Congruences ................................................... 41
III.6 2014-03-05: Dirichlet characters ............................. 41
III.6.1 Dirichlet characters ......................................... 41
III.6.2 Gauss sums ................................................... 42
III.6.3 \( L \)-functions ................................................ 43
III.7 2014-03-07: The functional equation ........................ 44
III.7.1 The Mellin transform ....................................... 45
III.7.2 Proof of the functional equation ......................... 46
III.8 2013-03-10: Eisenstein series of a character ............... 47
III.8.1 Fourier expansions ........................................... 48
III.9.1 Weight 2 Eisenstein series .................................. 50
III.9.2 Families of Eisenstein series ............................... 51
CONTENTS

IV Hecke operators 53
IV.1 2014-03-26: Hecke operators 53
IV.1.1 Motivation 53
IV.1.2 Connection to modular forms 54
IV.1.3 Hecke operators 54
IV.2 2014-03-28: Abstract Hecke algebras 55
IV.2.1 Concrete example 57
IV.3 2014-03-31: Hecke actions 58
IV.3.1 Petersson inner product 58
IV.4 2014-04-09: Newforms 59
IV.4.1 Atkin–Lehner–Li theory 61
IV.5 2014-04-11: More about newforms 61
IV.5.1 Euler products 62
IV.6 2014-04-14: L-functions 62
IV.6.1 Euler products 63
IV.6.2 Analytic continuation and functional equation 64
IV.7 2014-04-16 65
IV.8 2014-04-21 68
IV.8.1 CM forms 69

V Automorphic forms 71
V.1 2014-04-23: Automorphic forms and representations 71
V.1.1 Automorphic forms 74
V.2 2014-04-25: Adelic stuff 74

VI Elliptic curves 77
VI.1 2014-04-28: Algebro-geometric perspective 77
VI.2 2014-04-30: Elliptic curves 78
VI.2.1 Geometric definition of modular forms of level $1$ 78
VI.2.2 Tate curves 79
VI.2.3 Moduli 79
VI.3 2014-05-05: Galois representations attached to modular forms 80
VI.4 2014-05-07: Galois representations of higher weight 82

VII Generalizations of modular forms 85
VII.1 2014-05-08, continued 85
VII.1.1 Hilbert modular forms 85
VII.2 2014-05-09: Siegel modular forms, CFT, and Langlands 87
VII.2.1 Siegel modular forms 87
VII.2.2 Class field theory 88
VII.2.3 Langlands reciprocity 88
Chapter I

Modular forms

I.1 2014-01-22: Introduction to modular forms

I.1.1 Modular forms

Vaguely: modular forms are holomorphic functions \( f : \mathfrak{h} \rightarrow \mathbb{C} \) (where \( \mathfrak{h} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \)) that

- satisfy symmetry properties with respect to certain finite index subgroups \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \), where \( \text{GL}_2^+(\mathbb{R}) = \{ g \in \text{GL}_2(\mathbb{R}) \mid \det g > 0 \} \) acts on \( \mathfrak{h} \) by
  \[
  \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.
  \]

- behave well near \( i\infty \).

One symmetry: for all \( \Gamma \), if \( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \) for some \( h \in \mathbb{Z} \), then \( f(z + h) = f(z) \). So we get a Fourier expansion:

\[
f(z) = \sum_{n \in \mathbb{Z}} a_n(f) q_h^n,
\]

where \( q_h = e^{2\pi i z / h} \). As \( q_h \to 0 \), \( z \to i\infty \).

**Definition I.1.1** (holomorphic at \( \infty \)). We say \( f \) is “holomorphic at \( \infty \)” if

\[
f = \sum_{n \geq 0} a_n(f) q_h^n.
\]

If also \( a_0(f) = 0 \), we say \( f \) “vanishes at \( \infty \)” and say \( f \) is *cuspidal*.

**Definition I.1.2** (automorphy factor). Recall the action \( \text{GL}_2^+(\mathbb{R}) \circlearrowleft \mathfrak{h} \). The *automorphy factor* for \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \) and \( z \in \mathfrak{h} \) is

\[
j(\alpha, z) = cz + d.
\]
Definition I.1.3 (weight \(k\) action). For all \(k \in \mathbb{Z}\), define the \(weight \, k\) action \(GL_2^+(\mathbb{R}) \circ \{ f : \mathfrak{h} \to \mathbb{C} \}\) by
\[(f|_k \alpha)(z) = \det(\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha \cdot z).\]

Definition I.1.4 (congruence subgroups). Let \(N \in \mathbb{Z}_{\geq 1}\). Define the \(principal\) congruence subgroup \(\Gamma(N) \leq SL_2(\mathbb{Z})\) by
\[\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv I \pmod{N} \} = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})).\]
This is a finite-index subgroup. A subgroup \(\Gamma \leq SL_2(\mathbb{Z})\) is called a congruence subgroup (of level \(N\)) if \(\Gamma(N) \leq \Gamma\). For example:
\[
\begin{align*}
\Gamma_0(N) &= \left\{ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}, \\
\Gamma_1(N) &= \left\{ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.
\end{align*}
\]

Definition I.1.5 (modular form). Fix a congruence subgroup \(\Gamma\). A map \(f : \mathfrak{h} \to \mathbb{C}\) is a modular form of weight \(k\) and level \(\Gamma\) if
- \(f\) is holomorphic;
- \(f|_k \gamma = f\) for all \(\gamma \in \Gamma\);
- \((f|_k \alpha)(z)\) is holomorphic at \(\infty\) for all \(\alpha \in SL_2(\mathbb{Z})\).
Denote the space of modular forms of weight \(k\) and level \(\Gamma\) by \(M_k(\Gamma)\).

Fact I.1.6. \(\dim M_k(\Gamma) < \infty\).

Definition I.1.7 (cusp form). If in addition \((f|_k \alpha)\) vanishes at \(\infty\) for all \(\alpha \in SL_2(\mathbb{Z})\), then \(f\) is a cusp form. Denote the space of cusp forms by \(S_k(\Gamma)\).

Remark I.1.8. The only weight 0 forms are the constant functions.

We can think of \(f\) as a function on \(\Gamma \backslash \mathfrak{h}\). The spaces \(\Gamma \backslash \mathfrak{h}\) are known as modular curves.

I.1.2 Eisenstein series
Let
\[\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \leq SL_2(\mathbb{Z}) = \Gamma.\]
Then \(j(\gamma, z) = 1\) for all \(\gamma \in \Gamma_\infty\), so \(1|_k \gamma = j(\gamma, z)^{-k} \cdot 1 = 1\). Consider the sum
\[\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma.\]
If $k \geq 4$ is even, then this converges absolutely, and we can verify explicitly that this is a modular form. This is an example of an Eisenstein series. This is, up to a constant factor, equal to

$$E_k = \frac{3(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n)q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

**Fact I.1.9.** $M(\Gamma) = \bigoplus_k M_k(\Gamma)$ is freely generated (as an algebra) by $E_4$ and $E_6$.

**Example I.1.10.** $\Delta(z) = q \prod_{n \geq 1} (1-q^n)^{24}$.

### I.2 2014-01-24: Applications of modular forms

Guest lecture by Jordan Ellenberg.

#### I.2.1 Sums of squares

(Serre, *Course in Arithmetic*, part III.)

Let $r_k(n)$ be the number of representations of $n$ as a sum of $k$ squares:

$$r_k(n) = \# \{ (a_1, \ldots, a_k) \in \mathbb{Z}^k \mid a_1^2 + \cdots + a_k^2 = n \}$$

$$= \# \{ a \in \mathbb{Z}^k \mid \|a\| = \sqrt{n} \}.$$

What can we say about $r_k(n)$ (e.g., asymptotics as $n \to \infty$ with $k$ fixed)? For example,

$$\sum_{n=1}^{N} r_2(n) = \# \{ \text{lattice points at distance } \leq \sqrt{N} \text{ from } 0 \} \sim \text{area of circle } = \pi N.$$

More generally,

$$\sum_{n=1}^{N} r_k(n) = \# \{ \text{lattice points inside a } k\text{-ball of radius } \sqrt{N} \} = \text{constant}_k \cdot N^{k/2}.$$

So, on average, $r_k(n) \sim cN^{\frac{k}{2}-1}$.

**Theorem I.2.1** (Lagrange’s theorem). $r_4(n) > 0$ for all $n \geq 0$.

#### I.2.2 Theta functions

We can put $r_k(n)$ in a generating function

$$\sum_{n=0}^{\infty} r_k(n)q^n.$$
Define
\[ \theta := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots. \]

Note that
\[ \theta^k = \sum_n r_k(n) q^n. \]

Taking \( q = e^{2\pi i \tau} \), we can think of \( \theta \) as a function on \( \mathfrak{h} \), the upper half-plane, satisfying
\[ \theta(\tau) = \theta(\tau + 1). \]

This is a “modular form” of weight 1/2. In the definition of modular form, this requires us to say something like
\[ f|_{(a \ b \ c \ d)} = (c\tau + d)^{-1/2} f(\tau). \]

To remove quotes, raise to the 4-th power: \( \theta^4 \) is a modular form of weight 2 under a congruence subgroup of SL_2(\mathbb{Z}) of level 4.

Recall:
\[ E_4 \sim \sum d_3(n) q^n, \]
\[ E_6 \sim \sum d_5(n) q^n, \]
where \( d_i(n) = \sum_{k|n} k^i \). More generally, \( \theta^k \) is a modular form of weight \( k/2 \) for all even \( k \).

We can decompose
\[ \theta^k = E_{k/2} + f, \]
where \( E_{k/2} \) is an Eisenstein series and \( f \) is a cusp form. For \( k \geq 5 \), the \( n \)-th coefficient of \( E_{k/2} \) is very close to \( n^{k/2} - 1 \), and the \( n \)-th coefficient of \( f \) is \( o(n^{k/2} - 1) \).

Remark I.2.2 (For those who know Hecke operators). \( \theta^4 \) is actually a Hecke eigenform, which is why there’s a formula for \( r_4(n) \).

I.2.3 More general thetas

Let \( Q \) be an arbitrary positive-definite quadratic form in \( k \) variables over \( \mathbb{Z} \). Then
\[ \theta_Q = \sum_{n \in \mathbb{Z}^k} q^{Q(n)} \]
is again a modular form for some subgroup of SL_2(\mathbb{Z}) (depending on \( Q \)).

More generally still: Let \( Q \) be a positive definite symmetric matrix. Then we define
\[ \sum_{A \in M_m \times k(\mathbb{Z})} q^{A^T Q A}. \]
(Note that \( A^T Q A \) is an \( m \times m \) positive semidefinite symmetric matrix.) This can be thought of as a formal sum in a power series ring, or as a function on the Siegel upper half plane. This is still a Siegel modular form, which we also call an automorphic form on \( \text{Sp}_{2m} \).

This gives you a way of addressing questions like: Is every positive definite \( 2 \times 2 \) matrix the sum of \( k \) “squares” \( A^T A \), where \( A \) is a \( 1 \times 2 \) matrix?
I.3 2014-01-27: Modular curves

Earlier, we defined congruence subgroups \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \) and modular forms \( M_k(\Gamma) \), which are "functions on \( \Gamma \backslash \mathfrak{h} \)." We want to think of the modular curve \( \Gamma \backslash \mathfrak{h} \) as a geometric object, a Riemann surface. (cf. Diamond–Shurman, chapters 2 & 3)

Our goal is to turn \( \Gamma \backslash \mathfrak{h} \) into a Riemann surface \( Y(\Gamma) \), then compactify to get a compact Riemann surface \( X(\Gamma) \).

I.3.1 Example: Fundamental domain for \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \)

**Definition I.3.1.** A fundamental domain for \( \Gamma \) acting on \( \mathfrak{h} \) is a subset \( F \subseteq \mathfrak{h} \) such that:

1. \( F \) is a connected closed domain in \( \mathfrak{h} \).
2. \( \mathfrak{h} = \bigcup_{\gamma \in \Gamma} \gamma \cdot F \).
3. \( \gamma \cdot F \cap F = \emptyset \) for all \( \gamma \in \Gamma \) with \( \gamma \neq \pm I \), where \( F \) is the interior of \( F \).

**Theorem I.3.2.** (i) \( F = \{ z \in \mathfrak{h} : |\text{Re}(z)| \leq \frac{1}{2}, |z| \geq 1 \} \) is a fundamental domain for \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \).

(ii) If \( z_1 \in F \) such that \( z_2 = \gamma \cdot z_1 \in F \) for some \( \gamma \neq \pm I \), then one of the following is true:

(a) \( \text{Re}(z_1) = \pm \frac{1}{2} \) and \( \gamma \cdot z_1 = z_1 \mp 1 \), or
(b) \( |z_1| = 1 \) and \( \gamma \cdot z_1 = -\frac{1}{z_1} \).

**Proof.** Let \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), so \( S \cdot z = -\frac{1}{z} \) and \( T^{\pm 1} \cdot z = z \pm 1 \).

(i) We define an algorithm that takes \( z \in \mathfrak{h} \) as input, and outputs \( \gamma \in \langle S,T \rangle \subseteq \text{SL}_2(\mathbb{Z}) \) such that \( \gamma \cdot z \in F \).

**Step 1:** Apply \( T \) or \( T^{-1} \) until \( |\text{Re}(z)| \leq \frac{1}{2} \).

**Step 2:** If \( |z| \geq 1 \), then we are done. Otherwise, apply \( S \). Note that, if \( \text{Im}(z) > 1 \), then

\[
\text{Im} \left( \frac{-1}{z} \right) = \text{Im} \left( \frac{z}{|z|^2} \right) > \text{Im}(z).
\]

**Step 3:** If \( |\text{Re}(z)| \leq \frac{1}{2} \), then we are done; otherwise, go to step 1.

This process must eventually terminate. Indeed, note that if \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \), then

\[
\text{Im}(\alpha \cdot z) = \frac{\text{Im}(z)}{|cz + d|^2}.
\]

Given \( z \), there are only finitely many pairs \( (c,d) \in \mathbb{Z}^2 \) such that \( |cz + d| < 1 \). Moreover,

\[
T^{\pm 1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.
\]

But step 2 strictly increases \( \text{Im}(z) \), so it can only occur finitely many times.

(ii) The proof is left as an exercise. \( \square \)
I.3.2 $\text{SL}_2(\mathbb{R})$-structure of the upper half-plane

Theorem I.3.3. As $\text{SL}_2(\mathbb{R})$-sets, $\mathfrak{h} \cong \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$.

Proof. This follows from the following two facts:

(i) $\text{SL}_2(\mathbb{R})$ acts transitively on $\mathfrak{h}$.

(ii) There exists $z \in \mathfrak{h}$ such that $\text{Stab}(z) = \text{SO}_2(\mathbb{R})$.

These can both be verified by direct computation:

(i) $\frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i = x + iy$.

(ii) $\frac{ai + b}{ci + d} = i \iff ai + b = -c + di$, so $a = d$ and $b = -c$, whence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SO}_2(\mathbb{R})$. $\square$

Remark I.3.4.

(i) $\text{SL}_2(\mathbb{R})$ is a locally compact topological group, and $\text{SL}_2(\mathbb{Z})$ is a discrete subgroup.

(ii) The action $\text{SL}_2(\mathbb{R}) \times \mathfrak{h} \to \mathfrak{h}$ is continuous.

(iii) The map $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}$ given by $\alpha \cdot \text{SO}_2(\mathbb{R}) \mapsto \alpha \cdot i$ is a homeomorphism.

Next time, we will prove that the action is “properly discontinuous”.

I.4 2014-01-29: Construction of modular curves

I.4.1 The action is properly discontinuous

Proposition I.4.1. For all $z_1, z_2 \in \mathfrak{h}$, there are open neighborhoods $U_i \ni z_i$ such that, for all $\gamma \in \text{SL}_2(\mathbb{Z})$, if $\gamma(U_1) \cap U_2 \neq \emptyset$, then $\gamma \cdot z_1 = z_2$. (That is, the action is “properly discontinuous”.)

We prove this proposition in several steps.

Claim I.4.2. Given any neighborhoods $U_i' \ni z_i$ such that $\overline{U_i'}$ is compact, $\gamma(U_1') \cap U_2' \neq \emptyset$ for only finitely many $\gamma \in \text{SL}_2(\mathbb{Z})$.

Proof. Let $\text{SL}_2(\mathbb{R}) \xrightarrow{\pi} \mathfrak{h}$ be the map $\alpha \mapsto \alpha \cdot i$, and consider the closed subsets $S_i = \pi^{-1}(\overline{U_i'}) \subseteq \text{SL}_2(\mathbb{R})$. Let $\bigcup_j V_{i,j} \supseteq S_i$ be open covers with $V_{i,j}$ compact. Then $\overline{U_i'} \subseteq \bigcup_j \pi(V_{i,j})$. Since $\overline{U_i'}$ is compact, we can take $\overline{U_i'} \subseteq \bigcup_{j=1}^{n_i} \pi(V_{i,j})$. Hence

$$S_i \subseteq \bigcup_{j=1}^{n_i} \pi^{-1}(\pi(V_{i,j})) \subseteq \bigcup_{j=1}^{n_i} V_{i,j} \text{SO}_2(\mathbb{R}),$$

so $S_i$ is compact. Observe that

$$\{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma(U_1') \cap U_2' \neq \emptyset \} \subseteq \text{SL}_2(\mathbb{Z}) \cap S_2S_1^{-1}.$$

Since $\text{SL}_2(\mathbb{Z})$ is discrete and $S_2S_1^{-1}$ is compact, the above set is finite. $\square$
Let \( F = \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma(U'_1) \cap U'_2 \neq \emptyset \} \). By Claim [I.4.2], \( F \) is finite. The following completes the proof of the proposition.

Claim I.4.3. Let \( U_i, \gamma \) be disjoint neighborhoods of \( \gamma \cdot z_1 \) and \( z_2 \), respectively. Then for \( i = 1, 2 \),

\[ U_i = U'_i \cap \bigcap_{\gamma \in F} \gamma^{-1}(U_i, \gamma) \]

are the neighborhoods stated to exist in Proposition [I.4.1].

This is easily verified, completing the proof of the proposition.

**Corollary I.4.4.** If \( \Gamma \) is any congruence subgroup, then \( Y(\Gamma) = \Gamma \setminus \mathfrak{h} \) is Hausdorff.

### I.4.2 Make \( Y(\Gamma) \) a complex manifold

To give \( Y(\Gamma) \) the structure of a complex manifold, we find for all \( y \in Y(\Gamma) \) a neighborhood \( \tilde{U}_y \subseteq Y(\Gamma) \) and a homeomorphism \( \varphi_y : \tilde{U}_y \xrightarrow{\sim} V_y \subseteq \mathbb{C} \) with holomorphic transition maps.

Proposition [I.4.1] says that, if \( z \in \mathfrak{h} \) is only fixed by \( \Gamma \cap \{ \pm I \} \), then \( \pi_\Gamma : \mathfrak{h} \to Y(\Gamma) \) is a homeomorphism near \( z \).

**Definition I.4.5.** For \( z \in \mathfrak{h} \), let \( \Gamma_z := \text{Stab}_z(\Gamma) = \{ \gamma \in \Gamma \mid \gamma \cdot z = z \} \). We call \( z \) an **elliptic point** for \( \Gamma \) if \( \pm I \cdot \Gamma_z \supseteq \{ \pm I \} \). We also call \( \pi_\Gamma(z) \in Y(\Gamma) \) and \( \gamma \in \Gamma_z, \gamma \neq \pm I \) **elliptic**.

**Example I.4.6.** If \( z = i \), then

\[
\text{SL}_2(\mathbb{Z})_i = \left\{ \pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} = \langle S \rangle.
\]

If \( z = \rho = e^{i\pi/3} = 1 + e^{2\pi i/3} \), then \( \text{SL}_2(\mathbb{Z})_\rho = \langle TS \rangle \), where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Here is a convenient transformation: given \( \tau \in \mathfrak{h} \), let

\[
F_\tau(z) = \begin{pmatrix} 1 & -\tau \\ 1 & -\tau \end{pmatrix} \cdot z = \frac{z - \tau}{z - \overline{\tau}}.
\]

So \( F_\tau(\tau) = 0 \) and \( F_\tau(\overline{\tau}) = \infty \).

[See §2.2 of Diamond–Shurman for details.]

More generally:

**Proposition I.4.7.** For all \( \Gamma \) and elliptic \( \tau \), \( \Gamma_\tau \) is finite cyclic. Hence,

\[
h_\tau = \begin{cases} 
\#\Gamma_\tau & \text{if } -I \notin \Gamma, \\
\frac{1}{2}\#\Gamma_\tau & \text{if } -I \in \Gamma.
\end{cases}
\]
I.5 2014-01-31: Elliptic points and compactification

I.5.1 Elliptic points

A few remarks:

- The only elliptic points for $\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$ are $i$ and $\rho = e^{i\pi/3}$.
- For any congruence subgroup $\Gamma$, there are only finitely many elliptic points in $\Gamma \backslash \mathfrak{h}$.
- $\alpha \in \text{GL}^+_2(\mathbb{R})$ is called elliptic iff $\text{tr}(\alpha)^2 < 4 \det(\alpha)$.
- $\gamma \in \text{SL}_2(\mathbb{Z})$ is elliptic iff $\gamma$ is conjugate in $\text{SL}_2(\mathbb{Z})$ to one of the following:
  
  \[
  \begin{pmatrix}
  0 & 1 \\
  -1 & -1
  \end{pmatrix} \pm \frac{1}{\sqrt{3}}, \quad
  \begin{pmatrix}
  0 & -1 \\
  1 & 0
  \end{pmatrix} \pm \frac{1}{\sqrt{3}}, \quad
  \begin{pmatrix}
  0 & -1 \\
  1 & 1
  \end{pmatrix} \pm \frac{1}{\sqrt{3}}.
  \]

The following have no elliptic elements:

(i) $\Gamma(N)$ for $N \geq 2$.
(ii) $\Gamma_1(N)$ for $N \geq 4$.
(iii) $\Gamma_0(N)$ if there is prime $p \mid N$ such that $p \equiv 11 \pmod{12}$.

I.5.2 Compactification

Earlier, we showed that $Y(\Gamma)$ is a Riemann surface. We would like to work with a compact Riemann surface instead; to obtain one, we need to compactify!

Let us move from $\mathfrak{h}$ to the Poincaré disc $D := \{|z| < 1\}$. This is another model of the hyperbolic plane. We can move from the upper half-plane to the Poincaré disc via the Cayley transformation

$$C : \mathfrak{h} \to D,$$

$$C(z) = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \cdot z = \frac{z - i}{z + i}.$$ 

Note that $C(0) = -1$, $C(\infty) = 1$, $C(i) = 0$, $C(1) = -i$, and $C(\rho) = \frac{-i}{2 + \sqrt{3}}$. Moreover, the boundary $\partial D = \{|z| = 1\}$ corresponds to the projective real line $\mathbb{R} \cup \{\infty\} = \mathbb{P}^1(\mathbb{R})$.

The fundamental domain $F$ corresponds to a hyperbolic triangle in $D$, with a missing point on $\partial D$. “Filling in” this point gives a compactification.

More generally, if $\Gamma$ is any congruence subgroup, then there is a fundamental domain which is a convex polygon in hyperbolic geometry, with finitely many cusps on the boundary to be filled in. The idea is to add points to $\mathfrak{h}$ to get $\mathfrak{h}^*$, then look at $\Gamma \backslash \mathfrak{h}^* =: X(\Gamma)$.

For $X(1) = X(\Gamma(1)) = X(\text{SL}_2(\mathbb{Z}))$, we just need to add a single point $\infty$ to $Y(\Gamma)$, so we can take $\mathfrak{h}^* = \mathfrak{h} \cup \text{SL}_2(\mathbb{Z}) \cdot \infty$, where we think of $\mathfrak{h}$ as a subset of

$$\mathbb{P}^1(\mathbb{C}) = \{[\alpha : \beta] \mid (\alpha, \beta) \neq (0, 0)\} / ([\alpha : \beta] \sim [\lambda \alpha : \lambda \beta] \forall \lambda \in \mathbb{C}^\times).$$
Note that $[\alpha : \beta] \in \mathbb{P}^1(\mathbb{C})$ corresponds to $\frac{\alpha}{\beta}$ if $\beta \neq 0$, and to $\infty = [1 : 0]$ if $\beta = 0$. We extend the $\text{SL}_2(\mathbb{Z})$-action by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [\alpha : \beta] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{c\alpha + b}{c\beta + d}.
\]

Observe that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q})$.

So, let $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$.

Remark I.5.1. Let $\Gamma \leq \text{SL}_2(\mathbb{R})$ be any discrete subgroup. An element $\gamma \in \Gamma$ is called parabolic if $\text{tr}(\gamma)^2 = 4$, so $\gamma$ has a unique fixed point $x_\gamma \in \mathbb{P}^1(\mathbb{C})$, and $x_\gamma \in \mathbb{P}^1(\mathbb{R})$. Here, take $h^* = h \cup \{x_\gamma \mid \gamma \in \Gamma \text{ is parabolic}\}$.

For $\text{SL}_2(\mathbb{Z})$, $\infty = x(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$.

I.5.3 The topology on $\mathfrak{h}^*$

We want $\Gamma \setminus \mathfrak{h}^*$ to be Hausdorff.

Example I.5.2. $\Gamma_0(11) \setminus \mathfrak{h}$ has two cusps, at $\infty$ and 0. These are indeed different cusps:

\[
\begin{pmatrix} a & b \\ 11c & d \end{pmatrix} \cdot \infty = \begin{pmatrix} a & b \\ 11c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{a}{11c},
\]

which is 0 iff $a = 0$, which is impossible because then the determinant of $\begin{pmatrix} a & b \\ 11c & d \end{pmatrix}$ would be $11bc \neq 1$. However, $\frac{a}{11c}$ can get arbitrarily close to 0, so we need to be careful if we want the topology to be Hausdorff!

In general, here is how we define the topology on $\mathfrak{h}^*$: The neighborhoods of $\infty$ are of the form

\[
U_M := \{z \in \mathfrak{h} \mid \text{Im}(z) > M\} \cup \{\infty\}
\]

for all $M > 0$. The neighborhoods of $\frac{a}{c} \in \mathbb{Q}$ are $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot U_M$ for all $M > 0$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. (These neighborhoods look like circles in the upper half-plane, tangent to the real line $\mathbb{R}$ at the point $\frac{a}{c}$.)

I.5.4 The topology of $X(\Gamma)$

Give $X(\Gamma) = \Gamma \setminus \mathfrak{h}^*$ the quotient topology, and let $\pi_\Gamma : \mathfrak{h}^* \to X(\Gamma)$ be the natural projection.

Theorem I.5.3. $X(\Gamma)$ is Hausdorff, connected, and compact.

Proof. To prove compactness, consider the fundamental domain

\[
\mathcal{F}^* = \{z \in \mathfrak{h} : |z| \geq 1, \ \text{Re}(z) \leq \frac{1}{2}\} \cup \{\infty\} \subseteq \mathfrak{h}^*.
\]

This is a closed subset of $\mathbb{P}^1(\mathbb{C})$, hence is compact. (See [Diamond–Shurman] for details.)
I.5.5 Complex structure at the cusps

Let $s \in \mathbb{P}^1(\mathbb{Q})$. Choose $\gamma_s \in \text{SL}_2(\mathbb{Z})$ such that $\gamma_s \cdot s = \infty$. Consider

$$h_{s,\Gamma} = \# (\text{SL}_2(\mathbb{Z})_s / \{ \pm I \} \Gamma_s),$$

the “width of $s$ in $\Gamma$”. Take $U_M$ (for, say, $M = 2$) to $V = \psi(U_M) \subseteq \mathbb{C}$ via $\psi(z) = e^{2\pi iz/h}$, then descend to $X(\Gamma)$ via $\pi_\Gamma$ and proceed similarly to with elliptic points.
Chapter II

Dimensions of spaces of modular forms

II.1 2014-02-03: Dimension formulas

Guest lecture by Nigel Boston.

Now we have the Riemann surfaces $Y(\Gamma)$ and $X(\Gamma)$. Our goals for today:

(1) Understand modular forms as functions on $X(\Gamma)$.

(2) Use this to find the dimensions of spaces of modular forms $M_k(\Gamma)$ and $S_k(\Gamma)$.

II.1.1 Motivational example

If $f \in M_k(\Gamma)$, then $f : \mathfrak{h} \to \mathbb{C}$ satisfies $f(\gamma \cdot z) = j(\gamma, z)^k f(z)$ for all $\gamma \in \Gamma$. Thus, $f$ is well-defined as a function $Y(\Gamma) \to \mathbb{C}$ if and only if $k = 0$.

What do we do if $k \neq 0$? Say $k = 2$, and consider the differential $dz$ on $\mathfrak{h}$. Then for all $\alpha \in \text{GL}_2^+(\mathbb{R})$,

$$d(\alpha \cdot z) = d \left( \frac{az + b}{cz + d} \right) = \frac{(cz + d)a - (az + b)c}{(cz + d)^2} \, dz = \frac{ad - bc}{(cz + d)^2} \, dz = \frac{\det \alpha}{(cz + d)^2} \, dz.$$  

So if $f \in M_2(\Gamma)$, let $\omega_f(z) = f(z) \, dz$ (a differential on $\mathfrak{h}$). Then

$$\omega_f(\gamma \cdot z) = f(\gamma \cdot z) \, d(\gamma \cdot z) = \frac{(cz + d)^2 f(z) \, dz}{(cz + d)^2} = f(z) \, dz = \omega_f(z),$$

i.e., $\omega_f(z)$ is $\Gamma$-invariant. Thus, $\omega_f(z)$ is well-defined on $Y(\Gamma) = \Gamma \backslash \mathfrak{h}$.

The differential $dz$ has a “simple pole” at $\infty$, but if $f(z) \in S_2(\Gamma)$, then $f(z)$ vanishes at $\infty$, so $f(z) \, dz$ is a holomorphic differential on $X(\Gamma)$. In fact, we shall see that

$$\Omega_{\text{hol}}(X(\Gamma)) \cong S_2(\Gamma)$$

as $\mathbb{C}$-vector spaces. (Here, $\Omega_{\text{hol}}$ denotes the space of holomorphic differentials.) As a consequence,

$$\dim_{\mathbb{C}} S_2(\Gamma) = \dim_{\mathbb{C}} \Omega_{\text{hol}}(X(\Gamma)) = \text{genus of } X(\Gamma).$$
Example II.1.1. Recall the fundamental domain for $\Gamma = \text{SL}_2(\mathbb{Z})$ acting on $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$. Take the three points $i, \rho, \infty$, together with a fourth point $z_0$ in the interior; joining these together gives a triangulation of $X(\Gamma)$. The Euler characteristic is thus

$$\chi(X(\Gamma)) = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) = 4 - 6 + 4 = 2.$$ 

But $\chi(X(\Gamma)) = 2 - 2 \cdot \text{genus}(X(\Gamma))$, so $\text{genus}(X(\Gamma)) = 0$.

Thus, $\dim \mathbb{C} S_2(\text{SL}_2(\mathbb{Z})) = 0$, and so there are no nontrivial weight 2 cusp forms for $\text{SL}_2(\mathbb{Z})$.

Example II.1.2. Recall the fundamental domain for $\Gamma = \Gamma_0(11)$ on $\mathfrak{h}^*$. Using Sage, we can compute that the fundamental domain is isomorphic to a torus. So $\dim \mathbb{C} S_2(\Gamma_0(11)) = \text{genus}(X_0(11)) = 1$.

II.1.2 Maps between compact Riemann surfaces and the Riemann–Hurwitz formula

More generally, for all congruence subgroups $\Gamma \leq \text{SL}_2(\mathbb{Z})$, there is a natural quotient map $f_\Gamma : X(\Gamma) \to X(1) = X(\text{SL}_2(\mathbb{Z}))$.

The Riemann–Hurwitz formula applied to $f_\Gamma$ will relate $\text{genus}(X(\Gamma))$ to $\text{genus}(X(1)) = 0$, yielding a formula for $\dim \mathbb{C} S_2(\Gamma)$.

For $k \neq 0, 2$, we will need tensor products of differentials, and then we can use Riemann–Roch to compute the dimensions of $M_k(\Gamma)$ and $S_k(\Gamma)$.

Let $f : X \to Y$ be a holomorphic map of (connected) compact Riemann surfaces.

Fact II.1.3. $f$ is constant or surjective.

Proof sketch. $f(X)$ is the continuous image of a compact, connected set, so $f(X)$ is compact and connected, and hence is closed in $Y$. If $f$ is nonconstant, then it is open (by the open mapping theorem). So if $f$ is nonconstant, then $f(X)$ is a connected, clopen subset of $Y$. Since $Y$ is connected, $f(X) = Y$. \hfill \Box

Fact II.1.4. If $f$ is nonconstant, then $f$ has a well-defined degree $\deg(f) \geq 1$ such that, for all but finitely many $y \in Y$, $\# f^{-1}(y) = \deg(f)$.

For each $x \in X$, the ramification degree of $f$ at $x$, denoted $e_x \geq 1$, is the multiplicity with which $f$ takes $x$ to 0 in local coordinates; i.e., $f(z) = f(x) + \sum_{i=e_x}^\infty c_i (z - x)^i$, where $c_{e_x} \neq 0$. Then for all $y \in Y$,

$$\deg f = \sum_{x \in f^{-1}(y)} e_x.$$ 

Theorem II.1.5 (Riemann–Hurwitz). If $f : X \to Y$ is nonconstant, then

$$2g(X) - 2 = (\deg f)(2g(Y) - 2) + \sum_{x \in X} (e_x - 1).$$

One can also determine that $X(\text{SL}_2(\mathbb{Z}))$ has genus zero in several other ways. For example, the $j$-function gives an explicit isomorphism with the Riemann sphere.
II.2 2014-02-05: Riemann–Hurwitz formula

Guest lecture by Jordan Ellenberg.

II.2.1 Remarks on Riemann–Hurwitz

Let $X, Y$ be compact Riemann surfaces. Recall the Riemann–Hurwitz formula:

$$2g(X) - 2 = \deg(f) \cdot (2g(Y) - 2) + \sum_{x \in X} (e_x - 1),$$

where $e_x$ is the ramification degree of $f$ at $x$. (Since $e_x = 1$ for all but finitely many $x \in X$, the above sum is finite.)

**Definition II.2.1.** $\chi(X) = 2g(X) - 2$ is the Euler characteristic of $X$.

When there’s no ramification, $f : X \to Y$ is a covering map, and

$$\chi(X) = \deg(f) \cdot \chi(Y).$$

This is also a property of the Euler characteristic in higher dimension.

Topologically, being unramified means every $y \in Y$ has a neighborhood $U_y$ such that $f^{-1}(U_y)$ is the disjoint union of the right number of discs $\coprod U_x$.

**Example II.2.2.** Consider a cover of $\mathbb{P}^1(\mathbb{C})$ given by adjoining a square root $\sqrt{f(x)}$ of a polynomial of degree $2n$. Continuously winding around the zeros of $f$ “transports” between the two branches. This yields a Riemann surface $X$ of genus $n-1$, which has an involution $\gamma : X \to X$ such that $X/\gamma \cong \mathbb{P}^1(\mathbb{C})$. So we have a degree 2 map

$$X \xrightarrow{\pi} X/\gamma = \mathbb{P}^1(\mathbb{C}),$$

ramified exactly at the roots of $f$. By Riemann–Hurwitz,

$$2g(X) - 2 = \deg(\pi) \cdot (2g(\mathbb{P}^1(\mathbb{C})) - 2) + \sum_{6 \text{ points}} (e_x - 1)$$

$$= 2 \cdot (2 \cdot 0 - 2) + \sum_{6 \text{ points}} (2 - 1) = -4 + 6,$$

so $g(X) = 2$.

II.2.2 Prototypical ramified cover

Consider the map

$$\mathfrak{h}^*/\Gamma(p) \to \mathfrak{h}^*/\Gamma(1) \cong \mathbb{P}^1(\mathbb{C}).$$

Notice that $\Gamma(p)$ is a normal subgroup of $\Gamma(1) = \text{SL}_2(\mathbb{Z})$, and $\Gamma(1)/\Gamma(p) = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm 1\}$. So

$$G = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm 1\} \circ \mathfrak{h}^*/\Gamma(p) = X(p),$$
and $X(p)/G \cong X(1)$. Thus, the degree of the map is $|G| = \frac{1}{2}(p^3 - p)$.

Write $Y = X/G$. For each $y \in Y$, $\pi^{-1}(y)$ is an orbit of $G$ on $X$. A point $x \in X$ is ramified iff $|Gx| < |G|^2$.

To compute $g(X(p))$, we need to find the ramification points in $X(p) \to X(1)$. These are points $[z] \in \mathfrak{h}^*/\Gamma(p)$ where $g_1[z] = g_2[z]$ for some distinct $g_1, g_2 \in G = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm 1\}$, i.e., $g_1z = g_2z$ for $g_1, g_2 \in \text{SL}_2(\mathbb{Z})/\{\pm 1\}$ which lie in different cosets of $\Gamma(p)$.

The ramification comes from stabilizers in the upper half-plane: points which are fixed by nontrivial elements of $\text{SL}_2(\mathbb{Z})/\{\pm 1\}$. In other words, ramification only occurs above the cusps and the elliptic points.

II.3 2014-02-07: Ramification at elliptic points

II.3.1 The local picture

We have the map $f_{\Gamma} : X(\Gamma) \to X(1)$, and we want to compute the ramification index $e_x$ of $f_{\Gamma}$ for $x$ above $i, \rho, \infty$.

Say $f_{\Gamma}(x) = y \in X(1)$. We can find $\tau \in \mathfrak{h}^*$ such that $\pi_{\Gamma}(\tau) = x$ and $\pi_{\text{SL}_2(\mathbb{Z})}(\tau) = y$, and a neighborhood $U \ni \tau$ and charts $\varphi_x, \varphi_y$ near $x$ and $y$ giving a commutative diagram

\[
\begin{array}{ccc}
V_x & \xrightarrow{f_{\text{loc}}} & V_y \\
\varphi_x \downarrow & & \downarrow \varphi_y \\
F_\tau(U) & \xrightarrow{F_\tau} & F_\tau(U) \\
\pi_{\Gamma}(U) & \xrightarrow{f_{\Gamma}} & \pi_{\text{SL}_2(\mathbb{Z})}(U) \\
F_\tau(U) \xrightarrow{\pi_{\Gamma}} U & \xrightarrow{id} & U \\
\end{array}
\]

with $f_{\text{loc}}$ defined by the above diagram, and where

\[
P_h = \begin{cases} 
  z \mapsto e^{2\pi i z/h} & \text{near a cusp,} \\
  z \mapsto z^h & \text{near an elliptic point}
\end{cases}
\]

and

\[
h_x = \begin{cases} 
  \text{width of the cusp } \tau & \text{if } \tau \text{ is a cusp of } \Gamma, \\
  \text{period of the elliptic } \tau & \text{if } \tau \text{ is elliptic for } \Gamma, \\
  1 & \text{otherwise,}
\end{cases}
\]

and similarly with $\text{SL}_2(\mathbb{Z})$ instead of $\Gamma$.

II.3.2 Ramification indices

Let $\psi_x = P_{h_x} \circ F_{\tau}$ and $\psi_y = P_{h_y} \circ F_{\tau}$. Two cases:

\footnote{For those who like stacks, this has an interpretation in terms of stacks.}
(i) $\tau \in \mathfrak{h}$, i.e., $\tau$ is not a cusp. In this case, $P_h(z) = z^h$, so $f_{\text{loc}} \circ \psi_x = \psi_y$. Then $f_{\text{loc}}(z) = z^{h_y/h_x}$. We know $h_x, h_y \in \{1, 2, 3\}$, and $f_{\text{loc}}$ is holomorphic, so $h_y/h_x \in \mathbb{Z}$. Hence, either $h_x = 1$ or $h_x = h_y$. Thus, by definition,

$$e_x = \frac{h_y}{h_x} = \begin{cases} h_y & \text{if } \tau \text{ is elliptic for } \text{SL}_2(\mathbb{Z}) \text{ but not } \Gamma, \\ 1 & \text{otherwise} \end{cases} = [\text{SL}_2(\mathbb{Z})_\tau : \{\pm I\} \Gamma_\tau].$$

(ii) $\tau \in \mathfrak{h}^* \setminus \mathfrak{h} = \mathbb{P}^1(\mathbb{Q})$, i.e., $\tau$ is a cusp. Then $P_h(z) = e^{2\pi i z/h}$, so $f_{\text{loc}}(z) = z^{h_x/h_y}$. Thus,

$$e_x = \frac{h_x}{h_y} = [\text{SL}_2(\mathbb{Z})_\tau : \{\pm I\} \Gamma_\tau].$$

But $y = \infty$, so $h_y = 1$, hence $e_x = h_x$.

So, we have elliptic points

$$y_2 := \text{SL}_2(\mathbb{Z}) \cdot i \in X(1),$$
$$y_3 := \text{SL}_2(\mathbb{Z}) \cdot \rho \in X(1)$$

and the cusp

$$y_\infty := \text{SL}_2(\mathbb{Z}) \cdot \infty \in X(1),$$

with $h_{y_2} = 2$, $h_{y_3} = 3$, and $h_{y_\infty} = 1$.

Every elliptic point of $\Gamma$ of period 2 is in $f^{-1}_\Gamma(y_2)$, of period 3 in $f^{-1}_\Gamma(y_3)$, and every cusp of $\Gamma$ is in $f^{-1}_\Gamma(y_\infty)$. Let

$$\nu_2 := \# \text{ of elliptic points of } X(\Gamma) \text{ of period 2},$$
$$\nu_3 := \# \text{ of elliptic points of } X(\Gamma) \text{ of period 3},$$
$$\nu_\infty := \# \text{ of cusps of } X(\Gamma).$$

For $h \in \{2, 3\}$,

$$\deg(f_\Gamma) = \sum_{x \in f^{-1}_\Gamma(y_h)} e_x = \nu_h \cdot 1 + (\# f^{-1}_\Gamma(y_h) - \nu_h) \cdot h.$$

Hence,

$$\sum_{x \in f^{-1}_\Gamma(y_h)} (e_x - 1) = (h - 1) \left( \# f^{-1}_\Gamma(y_h) - \nu_h \right) = \frac{h - 1}{h} \left( \deg(f_\Gamma) - \nu_h \right).$$

Likewise, $\deg(f_\Gamma) = \sum_{x \in f^{-1}_\Gamma(y_\infty)} e_x$ and $\nu_\infty = \# f^{-1}_\Gamma(y_\infty)$, so

$$\sum_{x \in f^{-1}_\Gamma(y_\infty)} (e_x - 1) = \deg(f_\Gamma) - \nu_\infty.$$
II.3.3 Genus of $X(\Gamma)$

Let $\mu := \deg(f_{\Gamma}) = [\text{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$.

**Theorem II.3.1.** The genus of $X(\Gamma)$ is

$$g(X(\Gamma)) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$  

**Proof.** By the Riemann–Hurwitz formula,

$$2g(X(\Gamma)) - 2 = \mu \cdot (2 \cdot 0 - 2) + \sum_{x \in X(\Gamma)} (e_x - 1),$$

so

$$g(X(\Gamma)) = 1 - \mu + \frac{1}{2} \left( \frac{1}{2}(\mu - \nu_2) + \frac{2}{3}(\mu - \nu_3) + \mu - \nu_\infty \right).$$  

II.4 2014-02-10: Meromorphic differentials on Riemann surfaces

II.4.1 Meromorphic modular forms

**Definition II.4.1.** Let $k \in \mathbb{Z}$, and let $\Gamma \leq \text{SL}_2(\mathbb{Z})$ be a congruence subgroup. A meromorphic modular form of weight $k$ and level $\Gamma$ is a function $f : \mathfrak{h} \to \mathbb{P}^1(\mathbb{C})$ such that:

(i) $f$ is meromorphic;

(ii) $f|_{k\gamma} = f$ for all $\gamma \in \Gamma$;

(iii) $f|_{k\alpha}$ is meromorphic at $\infty$ for all $\alpha \in \text{SL}_2(\mathbb{Z})$.

Denote the $\mathbb{C}$-vector space of all such forms by $A_k(\Gamma)$. Then

$$S_k(\Gamma) \subseteq M_k(\Gamma) \subseteq A_k(\Gamma).$$

II.4.2 Differential forms

Given a Riemann surface $X$, the “best” way to think about differential forms on $X$ is to construct the cotangent bundle $p : \Omega_X \to X$; this is a line bundle.

**Definition II.4.2.** A differential form $\omega$ is a section of $p : \Omega_X \to X$, i.e., a map $\omega : X \to \Omega_X$ such that $p \circ \omega = \text{id}$.

Concretely: $X$ is covered by open sets $U$ equipped with maps $\varphi : U \to V \subseteq \mathbb{C}$. So, to describe differential forms on $X$, we need to define $\Omega(V)$ for $V \subseteq \mathbb{C}$ an open subset (the local picture), and to see what independence of the chart for differential forms on $X$ implies for the transition maps on overlaps.

Let $V \subseteq \mathbb{C}$ be open.
• $dz$ is in $\Omega(V)$.

• Because $V$ is 1-dimensional, $\Omega(V)$ should be 1-dimensional over

$$\mathcal{M}(V) = \{\text{meromorphic functions } f : V \to \mathbb{C}\},$$

the “ring of functions on $V$”.

Hence, we define

$$\Omega^1(V) = \{ f(z) \, dz \mid f \in \mathcal{M}(V) \}.$$

For $n \in \mathbb{Z}_{\geq 0}$, define

$$\Omega^\otimes n(V) := \Omega^1(V) \otimes_{\mathcal{M}(V)} \cdots \otimes_{\mathcal{M}(V)} \Omega^1(V) = \{ f(z) \, (dz)^n \mid f \in \mathcal{M}(V) \}.$$

Then we define the ring

$$\Omega(V) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Omega^\otimes n(V),$$

where $(dz)^n \cdot (dz)^m = (dz)^{n+m}$. This is the local picture.

### II.4.3 Pullbacks of differential forms

Next, let us study how gluing works. In particular, we need to understand the relation between $\Omega(V_1)$ and $\Omega(V_2)$ given a holomorphic map $\varphi : V_1 \to V_2$.

$n = 0$ $\Omega^\otimes 0(V) = \mathcal{M}(V)$, and a holomorphic map $\varphi : V_1 \to V_2$ induces a “pull-back” map

$$\varphi^* : \mathcal{M}(V_2) \to \mathcal{M}(V_2)$$

$$f \mapsto f \circ \varphi.$$

Given $V_1 \overset{\varphi_1}{\longrightarrow} V_2 \overset{\varphi_2}{\longrightarrow} V_3$, we have $(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$.

$n = 1$ If $f(z_2) \, dz_2$ on $V_2$, then

$$\Omega^1(V_1) \ni \varphi^*(f(z_2) \, dz_2) = f(\varphi(z_1)) \, d(\varphi(z_1)) = (f(\varphi(z_1)) \varphi'(z_1)) \, dz_1.$$

**General** $n$ The pullback map is given by

$$\varphi^* : \Omega^\otimes n(V_2) \to \Omega^\otimes n(V_1)$$

$$f(z_1)(dz_2)^n \mapsto f(\varphi(z_1)) (\varphi'(z_1))^n (dz_1)^n.$$

**Remark II.4.3.**

(i) If $\varphi : V_1 \hookrightarrow V_2$ is an inclusion, then $\varphi^*(\omega) = \omega|_{V_1}$.

(ii) If $\varphi : V_1 \to V_2$ is surjective, then $\varphi^*$ is injective.

(iii) If $\varphi : V_1 \overset{\cong}{\rightarrow} V_2$ is an isomorphism, then $(\varphi^{-1})^* = (\varphi^*)^{-1}$. 
II.4.4 Gluing differential forms

Consider a Riemann surface $X$ with coordinate charts $\varphi_j : U_j \to V_j \subseteq \mathbb{C}$ (where $j \in J$). Given two coordinate charts $\varphi_j, \varphi_k$, we have holomorphic transition maps $\varphi_{j,k}$ and $\varphi_{k,j}$.

**Definition II.4.4.** A *global* meromorphic differential form $\omega$ on $X$ of degree $n \geq 0$ is a *compatible* system $(\omega_j)_{j \in J} \in \prod_{j \in J} \Omega^n(V_j)$, i.e.,

$$\varphi_{k,j}^* (\omega_k|_{V_{k,j}}) = \omega_j|_{V_{j,k}}.$$

II.5 2014-02-12: Meromorphic modular forms and differentials

Recall that

$$\Omega^n(X) = \{ \omega = (\omega_j)_{j \in J} \text{ on } X \text{ of deg } n \geq 0 \mid \omega_j \in \Omega^n(V_j) \}.$$

Our goal is to show that, for $k \in \mathbb{Z}_{\geq 0}$ even, $\Gamma$ a congruence subgroup, there is an isomorphism of $\mathbb{C}$-vector spaces

$$\omega : A_k(\Gamma) \xrightarrow{\cong} \Omega^{k/2}(X(\Gamma))$$

$$f \mapsto \omega(f).$$

The map $\pi_\Gamma : h^* \to X(\Gamma)$ induces a map of differentials

$$\pi^*_\Gamma : \Omega^n(X(\Gamma)) \to \Omega^n(h).$$

**Claim II.5.1.** A collection $\omega_j \in \prod_{j \in J} \Omega^n(V_j)$ is compatible iff the $\psi^*_j(\omega_j)$ to $h$ are the restriction of a global meromorphic differential $f(\tau)(d\tau)^n \in \Omega^n(h)$, where $f \in A_{2n}(\Gamma)$.

**Claim II.5.2.** For all $f \in A_{2n}(\Gamma)$, there exists $(\omega_j)_{j \in J}$ with $\omega_j \in \Omega^n(V_j)$ such that $\psi^*_j(\omega_j) = f(\tau)(d\tau)^n|_{U_j}$.

Define $\pi^*_\Gamma$ given $\omega = (\omega_j)_{j \in J} \in \Omega^n(X(\Gamma))$. Let $U'_j = U_j \cap h$, $V'_j = \psi_j(U'_j)$, and $\omega'_j = w_j|_{V'_j}$. Define $\tilde{\omega}_j := \psi^*_j(\omega_j)$ on $U'_j$. These $\tilde{\omega}_j$ are compatible:

$$\tilde{\omega}_k|_{U'_j \cap U'_k} = \psi^*_k(\omega_k|_{V'_{k,j}}) = \psi^*_j(\varphi^*_{k,j}(\omega_k|_{V'_{k,j}})) = \tilde{\omega}_j|_{U'_j \cap U'_k}.$$ 

Hence, the $\tilde{\omega}_j$ give $\pi^*_\Gamma(\omega) := \tilde{\omega} = f(\tau)(d\tau)^n \in \Omega^n(h)$.

**Claim II.5.3.** $f \in A_{2n}(\Gamma)$. 

Proof. For all $\gamma \in \Gamma$, we have $\gamma : \mathfrak{h} \to \mathfrak{h}$, $\tau \mapsto \gamma \tau$, so $\gamma^*: \Omega^\otimes n(\mathfrak{h}) \to \Omega^\otimes n(\mathfrak{h})$. We have

$$
\begin{array}{c}
\mathfrak{h} \\
\downarrow \gamma \\
\pi_\Gamma \\
\downarrow \\
X(\Gamma),
\end{array}
$$

i.e., $\pi_\Gamma \circ \gamma = \pi_\Gamma$, so $\gamma^* \circ \pi_\Gamma^* = \pi_\Gamma^*$. Thus,

$$
\pi_\Gamma^*(\omega) = \gamma^*(\pi_\Gamma^*(\omega)) = \gamma^*(f(\tau)(d\tau)^n) = f(\gamma \cdot \tau) \left((\gamma \cdot \tau)^n(d\tau)^n\right) = f(\gamma \cdot \tau) j(\gamma, \tau)^{-2n}(d\tau)^n = f(\tau)(d\tau)^n,
$$

i.e., $(f|_{2n}\gamma) = f$. It remains to check $f|_{2n}\alpha$ is meromorphic at $\infty$ for all $\alpha \in \text{SL}_2(\mathbb{Z})$. \qed

Claim II.5.4. Let $f \in A_{2n}(\Gamma)$, $\psi_j : U_j \cap V_j$, and $U'_j = U_j \cap \mathfrak{h}$. Then there is $\omega_j \in \Omega^\otimes n(V_j)$ such that $\psi_j^*(\omega_j|_{V'_j}) = f(\tau)(d\tau)^n|_{U'_j}$.

Proof. Consider $\delta_j = \begin{pmatrix} 1 & -\tau_j \\ 1 & -\overline{\tau}_j \end{pmatrix} \in \text{GL}_2(\mathbb{C})$. Write $\delta_j \cdot \tau = f_{\tau_j}(\tau)$. Since $F_{\tau_j}$ is a bijection, there exists

$$
\lambda_j := \left(F_{\tau_j}^{-1}\right)^*(f(\tau)(d\tau)^n|_{U'_j})
$$

such that $f(\tau)(d\tau)^n|_{U'_j} = F_{\tau_j}^*(\lambda_j)$. Let $\alpha = \delta_j^{-1}$. Then

$$
\lambda_j = \alpha^*(f(\tau)(d\tau)^n) = f(\alpha \cdot \tau) \frac{\det(\alpha)^n}{j(\alpha, \overline{\alpha})^{2n}}(d\tau)^n = (f|_{2n}\alpha)(\tau)(d\tau)^n.
$$

So for any $\gamma \in \Gamma$,

$$
(\alpha^{-1}\gamma\alpha)^* \lambda_j = (f|_{2n}\alpha\alpha^{-1}\gamma\alpha)(\tau)(d\tau)^n = (f|_{2n}\alpha)(\tau)(d\tau)^n = \lambda_j.
$$

Hence, $\lambda_j$ is invariant under $\delta_j \Gamma \delta_j^{-1}$. \qed

II.6 2014-02-14: Divisors and Riemann–Roch

II.6.1 Meromorphic modular forms and differentials

Continuing from last time: $\lambda_j$ is $\delta_j \Gamma \delta_j^{-1}$-invariant. If $\tau_j$ is not a cusp, then $\delta_j : \tau_j \mapsto 0$, and

$$
h_j = \# \{ \pm I \} (\delta_j \Gamma \delta_j^{-1})_0 / \{ \pm I \}
$$

is cyclic generated by $r_{h_j} : z \mapsto \mu_{h_j} z$, where $\mu_{h_j} = e^{2\pi i h_j}$. We have

$$
(f|_{2n}\alpha)(\tau)(d\tau)^n = \lambda_j = r_{h_j}^*(\lambda_j) = (f|_{2n}\alpha)(\mu_{h_j} z)(d(\mu_{h_j} z))^n = (f|_{2n}\alpha)(\mu_{h_j} z)\mu_{h_j}^n(dz)^n,
$$

so there exists $g_j \in \mathfrak{M}(V'_j)$ such that

$$
(\mu_{h_j} z)^n(f|_{2n}\alpha)(\mu_{h_j} z) = z^n(f|_{2n}\alpha)(z) = g_j(z^{h_j}) = g_j(q),
$$
where \( q = z^{h_j} \). Define

\[
\omega'_j := \frac{g_j(q)}{(h_jq)^n}(dq)^n.
\]

Hence

\[
\psi^*_j(\omega'_j) = F^*_{\tau j}(\rho^*_j(\omega'_j)).
\]

We want to show that \( \rho^*_j(\omega'_j) = \lambda_j \). Indeed,

\[
\rho^*_j(\omega'_j) = \frac{g_j(z^{h_j})}{(h_j z^{h_j})^n}(h_j z^{h_j-1})^n(dz)^n = \frac{g_j(z^{h_j})}{z^n} (dz)^n = (f|_n \alpha)(z)dz = \lambda_j.
\]

At a cusp, \( \delta_j : \tau_j \mapsto \infty \), \( h_j \) is the width, and \( \rho_j(z) = e^{2\pi iz/h_j} \). We now instead have

\[
\omega'_j = \frac{g_j(q)}{(2\pi i q/h_j)^n}(dq)^n,
\]

and the rest proceeds similarly.

This completes the proof of the isomorphism

\[
\omega : A_k(\Gamma) \xrightarrow{\sim} \Omega^{\otimes k/2}(X(\Gamma))
\]

for \( k \) even.

### II.6.2 Divisors on Riemann surfaces

Let \( X \) be a compact Riemann surface.

**Definition II.6.1.** The *divisor group* on \( X \), denoted \( \text{Div}(X) \), is the free abelian group on the set of points in \( X \). In other words, a divisor \( D \) on \( X \) is a finite formal sum \( D = \sum_{x \in X} n_x x \), where \( n_x \in \mathbb{Z} \) and all but finitely many \( n_x \) are zero.

- Write \( D \geq D' \) if \( n_x \geq n'_x \) for all \( x \in X \).
- There is a natural homomorphism

\[
\deg : \text{Div}(X) \to \mathbb{Z},
\]

\[
\sum_{x \in X} n_x x \mapsto \sum_{x \in X} n_x.
\]

**Example II.6.2 (Principal divisors).** For any \( f \in \mathcal{M}(X) \) and \( x \in X \), define the *order of vanishing* \( \text{ord}_x(f) \) by locally writing \( f_x(z) = \sum_{n \in \mathbb{Z}} a_n(z - x)^n \), and setting

\[
\text{ord}_x(f) = \begin{cases} 
\min \{n \in \mathbb{Z} : a_n \neq 0\} & \text{if } f \neq 0, \\
\infty & \text{if } f = 0.
\end{cases}
\]

For \( f \neq 0 \), define the *principal divisor*

\[
\text{div}(f) := \sum_{x \in X} \text{ord}_x(f)x.
\]

This gives a homomorphism

\[
\mathcal{M}(X) \to \text{Div}(X).
\]
II.7. 2014-02-16: RIEMANN–ROCH

Note II.6.3.  • By a general theorem of complex analysis, \( \deg(\text{div}(f)) = 0 \).

• \( \text{ord}_x(f_1 + f_2) \geq \min(\text{ord}_x f_1, \text{ord}_x f_2) \).

Definition II.6.4. Let \( D \) be a divisor. The linear space of \( D \) is

\[
L(D) := \{ f \in \mathcal{M}(X) \mid \text{div}(f) + D \geq 0 \} \cup \{0\}.
\]

Example II.6.5. (1) If \( D = -2 \cdot 0 + 3 \cdot 1 \), then \( L(D) \) consists of functions with a zero of order at least 2 at 0, and a pole of order at most 3 at 1.

(2) If \( g \in \mathcal{M}(X)^* \), then \( L(\text{div}(g)) = \{ f \mid fg \text{ is holomorphic} \} \).

Fact II.6.6.  • \( L(D) \) is a \( \mathbb{C} \)-vector space.

• \( \ell(D) := \dim_{\mathbb{C}} L(D) < \infty \).

If \( \omega \in \Omega^n(X) \) with \( \omega \neq 0 \), and \( x \in U \xrightarrow{\varphi} V \subseteq \mathbb{C} \) is a local chart, set \( \omega = \varphi^*(\omega_x) \) with \( \omega_x = f_x(q)(dq)^n \). Define

\[
\text{ord}_x(\omega) := \text{ord}_{q=\varphi(x)} f_x
\]

and

\[
\text{div}(\omega) = \sum_{x \in X} \text{ord}_x(\omega)x.
\]

Then

\[
\text{div}(\omega_1 \omega_2) = \text{div}(\omega_1) + \text{div}(\omega_2).
\]

Definition II.6.7. If \( \lambda \in \Omega^1(X) \) with \( \lambda \neq 0 \), then \( \text{div}(\lambda) \) is called a canonical divisor.

II.6.3 The Riemann–Roch theorem

Theorem II.6.8 (Riemann–Roch). For all \( D \in \text{Div}(X) \),

\[
\ell(D) = \deg(D) - g(X) + 1 + \ell(\text{div}(\lambda) - D),
\]

where \( \text{div}(\lambda) \) is a canonical divisor and \( g(X) \) is the genus of \( X \).

II.7 2014-02-16: Riemann–Roch

II.7.1 Consequences of Riemann–Roch

Let \( \lambda \in \Omega^1(X) \) be nonzero and \( D \in \text{Div}(X) \) be arbitrary.

Theorem II.7.1 (Riemann–Roch). \( \ell(D) = \deg(D) - g(X) + 1 + \ell(\text{div}(\lambda) - D) \).

Corollary II.7.2. (a) \( \ell(\text{div}(\lambda)) = g \).

(b) \( \deg(\text{div}(\lambda)) = 2g - 2 \).

(c) If \( \deg(D) < 0 \), then \( \ell(D) = 0 \).
(d) If $\deg(D) > 2g-2$, then $\ell(D) = \deg(D) - g + 1$. (This is the “simple form” of Riemann–Roch.)

Proof. (a) If $D = 0$, then $\ell(D) = 1$. Plug in $D = 0$.

(b) Plug in $D = \text{div}(\lambda)$.

(c) If $\ell(D) > 0$, then there exists $f$ such that $\text{div}(f) \geq -D$. So $0 = \deg(\text{div}(f)) \geq -\deg(D)$.

(d) In this case, $\deg(\text{div}(\lambda - D)) < 0$, so the result follows from (c).

II.7.2 Application to meromorphic modular forms

Let $k \geq 2$ be even. Here is our plan:

1. Show there exists nonzero $f_0 \in A_k(\Gamma)$.

2. Show $A_k(\Gamma) = M(X(\Gamma)) \cdot f_0$.

3. For any $g \in A_k(\Gamma)$, we will make sense of $\text{div}(g) \in \text{Div}_Q(X(\Gamma)) := \text{Div}(X(\Gamma)) \otimes \mathbb{Z} \otimes \mathbb{Q}$.

4. $M_k(\Gamma) = L(\text{div}(f_0))$, and similarly for $S_k(\Gamma)$.

5. Relate $\text{ord}_x(\omega(f))$ to $\text{ord}_\tau(f)$.

For any $f \in A_k(\Gamma)$ and $\gamma \in \Gamma$,

$$(f|k\gamma)(z) = (cz + d)^k f(z).$$

Note that $(cz + d)^k \notin \{0, \infty\}$ for all $z \in h$, so $\text{ord}_x f = \text{ord}_{\gamma, x} f$.

Let $\tau \in h$ and $q_0 = \pi_\Gamma(\tau) \in X(\Gamma)$. Then

$$f(z) = \sum_{n \geq \text{ord}_x f} a_n (z - \tau)^n.$$

In local coordinates, $(q - q_0)$ is $(z - \tau)^{h_x}$. So

$$f(q) = \sum_{n \geq \text{ord}_x f} a_n ((z - \tau)^{h_x})^{n/h_x} = \sum_{n \geq \text{ord}_x f} a_n (q - q_0)^{n/h_x}.$$

Hence, we can define

$$\text{ord}_{q_0}(f) := \frac{\text{ord}_x f}{h_x} \in \frac{1}{2} \mathbb{Z} \cup \frac{1}{3} \mathbb{Z}.$$

Say $x = \pi_\Gamma(s)$, where $s \in \mathbb{P}^1(\mathbb{Q})$. Let $h_x$ be the width of $x$. If $s = \infty$, then

$$\{\pm I\} \Gamma_\infty = \{\pm I\} \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix},$$

but since $-I$ might not be in $\Gamma$, this only implies $\Gamma_\infty$ is one of the following:

$$\{\pm I\} \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix}.$$
If \( x = \pi_{\Gamma}(s) \) with \( s \in \mathbb{P}^1(\mathbb{Q}) \), choose \( \alpha \in \text{SL}_2(\mathbb{Z}) \) such that \( \alpha \cdot \infty = s \). If
\[
(\alpha^{-1}\Gamma\alpha)_{\infty} = \left\langle -\begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right\rangle,
\]
then \( x \) is called an \textit{irregular cusp}, and then \( \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \notin \alpha^{-1}\Gamma\alpha \). So if \( k \) is odd, then
\[
\left( (f|_{k\alpha})|_k - \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right)(z) = j \left( -\begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix}, z \right)^k (f|_{k\alpha})(z) = (-1)^k (f|_{k\alpha})(z).
\]
so \( h_x \) is not the period. Thus,
\[
\text{ord}_x f := \begin{cases} 
\text{ord}_s f & \text{if } x \text{ is regular or } k \text{ is even}, \\
\frac{1}{2} \text{ord}_s f & \text{if } x \text{ is irregular and } k \text{ is odd}.
\end{cases}
\]

II.7.3 Order of differentials

Let \( \omega \in \Omega_{\text{het}}^n(X(\Gamma)) \), so \( \omega = \omega(f) \) for some \( f \in A_{2n}(\Gamma) \).

If \( x \) is not a cusp, let \( g(q) = z^{h_x} f(z) \), where \( q = z^{h_x} \). Then
\[
\text{ord}_x \omega = \text{ord}_0 \frac{g(q)}{(h_x q)^{k/2}} = \text{ord}_0 g(q) - \frac{k}{2} = \text{ord}_x(f) + \frac{k}{2} \cdot \frac{1}{h_x} - \frac{k}{2}.
\]
In particular, if \( h_x = 1 \), then \( \text{ord}_x(\omega) = \text{ord}_x(f) \).

If \( x \) is a cusp, let \( g(q) = f(e^{2\pi i q/h_x}) \). Then
\[
\text{ord}_x \omega = \text{ord}_0 \frac{g(q)}{(2\pi i q/h_x)^{k/2}} = \text{ord}_x f - \frac{k}{2}.
\]

We formally define
\[
\text{div}(d_\tau) := -\sum_i \frac{1}{2} x_{2,i} - \sum_i \frac{2}{3} x_{3,i} - \sum_i x_{\infty,i} \in \text{Div}_\mathbb{Q}(X),
\]
where \( x_{2,i}, x_{3,i}, x_{\infty,i} \) range over the order 2 elliptic points, order 3 elliptic points, and cusps, respectively.

II.8 2014-02-19: Computing the dimensions

II.8.1 Computing the dimension of \( M_k(\Gamma) \)

Assume\( \text{3} \) there exists \( f_0 \neq 0 \) in \( A_k(\Gamma) \) for \( k \geq 0 \).

Proposition II.8.1. For \( k \geq 0 \), \( A_k(\Gamma) = \mathcal{M}(X(\Gamma)) \cdot f_0 \).

\( \text{3} \)For now; we’ll prove this later.
CHAPTER II. DIMENSIONS OF SPACES OF MODULAR FORMS

Proof. Let \( f \in A_k(\Gamma) \). Then \( f/f_0 \in A_0(\Gamma) \) is \( \Gamma \)-invariant, so descends to \( X(\Gamma) \). 

So 

\[
M_k(\Gamma) = \{ ff_0 \mid f \in \mathfrak{M}(X(\Gamma)), \ f f_0 = 0 \text{ or } \text{div}(f f_0) \geq 0 \} \\
\cong \{ f \in \mathfrak{M}(X(\Gamma)) \mid f = 0 \text{ or } \text{div}(f) + \text{div}(f_0) \geq 0 \} = L(\text{div}(f_0)).
\]

The problem is, \( \text{div}(f_0) \) is usually not integral. To fix this, for \( D = \sum n_x x \in \text{Div}_\mathbb{Q}(X(\Gamma)) \), define \( \lfloor D \rfloor := \sum \lfloor n_x \rfloor x \in \text{Div}(X(\Gamma)) \). Since \( \text{div}(f) \) is integral for \( f \in M(X(\Gamma)) \), 

\[
M_k(\Gamma) \cong L(\lfloor \text{div}(f_0) \rfloor).
\]

For \( k \geq 0 \) even, \( \omega(f_0) = f_0(\tau)(d\tau)^{k/2} \), so 

\[
\text{div}(\omega(f_0)) = \text{div}(f_0) + \frac{k}{2} \text{div}(d\tau)
\]

is integral. Hence, 

\[
\lfloor \text{div} f_0 \rfloor = \text{div}(\omega(f_0)) + \sum_i \left\lfloor \frac{k}{4} \right\rfloor x_{2,i} + \sum_i \left\lfloor \frac{k}{3} \right\rfloor x_{3,i} + \sum_i \frac{k}{2} x_{\infty,i}.
\]

We need to know \( \deg(\text{div}(\omega(f_0))) \).

If \( \lambda \in \Omega^1(X(\Gamma)) \) (so \( \lambda^{k/2} \in \Omega^{\cdot k/2}(X(\Gamma)) \)), then 

\[
\deg(\text{div}(\lambda^{k/2})) = \frac{k}{2} \deg(\text{div}(\lambda)) = \frac{k}{2}(2g - 2) = k(g - 1).
\]

For all \( \omega \in \Omega^{\cdot k/2}(X(\Gamma)) \in \mathfrak{M}(X(\Gamma)) \cdot \lambda^{k/2} \), writing \( \omega = f \cdot \lambda^{k/2} \), we have 

\[
\text{div}(\omega) = \text{div}(f) + \text{div}(\lambda^{k/2}),
\]

so 

\[
\deg(\text{div}(\omega)) = \deg(\text{div}(f)) + \deg(\text{div}(\lambda^{k/2})) = \deg(\text{div}(\lambda^{k/2})).
\]

Hence, 

\[
\deg(\lfloor \text{div} f_0 \rfloor) = k(g - 1) + \left\lfloor \frac{k}{4} \right\rfloor \nu_2 + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + \frac{k}{2} \nu_\infty \\
\geq \frac{k}{2}(2g - 2) + \left( \frac{k - 2}{4} \right) \nu_2 + \left( \frac{k - 2}{3} \right) \nu_3 + \frac{k}{2} \nu_\infty \\
= 2g - 2 + \frac{k - 2}{2} \left( 2g - 2 + \frac{\nu_2}{2} + \frac{2\nu_3}{3} + \nu_\infty \right) + \nu_\infty \\
= \mu > 0 \text{ by the genus formula} \\
\geq 2g - 2 + \nu_\infty \quad \text{(if } k \geq 2 \text{)} \\
> 2g - 2
\]

because there are always cusps. Therefore, we can use the simple version of Riemann–Roch: if \( k \geq 2 \) is even, then 

\[
\dim_\mathbb{C} M_k(\Gamma) = (k - 1)(g - 1) + \left\lfloor \frac{k}{4} \right\rfloor \nu_2 + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + \frac{k}{2} \nu_\infty.
\]
II.8.2 Computing the dimension of $S_k(\Gamma)$

Similarly:

$$S_k(\Gamma) \cong L \left( \left\lfloor \div f_0 \right\rfloor - \sum_i x_{\infty, i} \right).$$

If $k = 2$, then

$$\left\lfloor \div f_0 \right\rfloor = \div (\omega(f_0)) + 0 + 0 + \sum_i x_{\infty, i},$$

so $S_2(\Gamma) \cong L(\div(\omega(f_0)))$, so $\dim_{\mathbb{C}} S_2(\Gamma) = g(X(\Gamma))$.

If $k \geq 4$, then $\frac{k-2}{2} > 0$, so

$$2g - 2 + \frac{k-2}{2} \left( \frac{\mu}{6} \right) > 2g - 2.$$ 

By the simple version of Riemann–Roch, for $k \geq 4$,

$$\dim_{\mathbb{C}} S_k(\Gamma) = \dim_{\mathbb{C}} M_k(\Gamma) - \nu_{\infty}.$$ 

II.8.3 Negative weight modular forms

If $k = 0$, then

$$M_0(\Gamma) = \{ f \text{ holomorphic} \} = \text{constants} = \mathbb{C}.$$

So dim $M_0(\Gamma) = 1$ and dim $S_0(\Gamma) = 0$.

Suppose $k < 0$. Recall $\Delta \in S_{12}(SL_2(\mathbb{Z}))$, given by

$$\Delta = q \prod_{n \geq 0} (1 - q^n)^{24}.$$ 

So $\Delta(\tau) \neq 0$ for all $\tau \in \mathfrak{h}$. Let $f \in M_k(\Gamma)$. Then $f^{12} \Delta^k \in S_0(\Gamma) = 0$, so $f = 0$. Thus,

$$M_k(\Gamma) = S_k(\Gamma) = 0$$

for $k < 0$.

II.8.4 Example: modular forms for $SL_2(\mathbb{Z})$

Let $\Gamma = SL_2(\mathbb{Z})$, and let $k \geq 4$ be even. Then

$$\dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z})) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor - 1 & \text{if } k \equiv 2 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{otherwise}, \end{cases}$$

$$\dim_{\mathbb{C}} M_k(SL_2(\mathbb{Z})) = \dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z})) + 1.$$
II.8.5 Odd weight modular forms

Let \( k > 0 \) be odd. If \(-I \in \Gamma\), then \( j(-I, z)^k = -1 \), so

\[
f(\tau) = (f|_k - I)(\tau) = -f(\tau),
\]

and there are no nontrivial modular forms of weight \( k \).

Assume \(-I \notin \Gamma\). Given \( f \in A_k(\Gamma) \), look at \( \omega(f^2) \).

**Theorem II.8.2.** Suppose \( k \geq 3 \) is odd and \(-I \notin \Gamma\). Then

\[
\dim_{\mathbb{C}} M_k(\Gamma) = (k - 1)(g - 1) + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + \frac{k}{2} \nu_{\text{reg}} + \frac{k - 1}{2} \nu_{\text{irr}},
\]

\[
\dim_{\mathbb{C}} S_k(\Gamma) = \dim_{\mathbb{C}} M_k(\Gamma) - \nu_{\text{reg}}.
\]

Now consider \( k = 1 \):

- If \( \nu_{\text{reg}} > 2g - 2 \), then
  \[
  \dim_{\mathbb{C}} M_1(\Gamma) = \frac{\nu_{\text{reg}}}{2},
  \]
  \[
  \dim_{\mathbb{C}} S_1(\Gamma) = 0.
  \]

- Otherwise:
  \[
  \dim_{\mathbb{C}} M_1(\Gamma) \geq \frac{\nu_{\text{reg}}}{2},
  \]
  \[
  \dim_{\mathbb{C}} S_1(\Gamma) = \dim_{\mathbb{C}} M_1(\Gamma) - \frac{\nu_{\text{reg}}}{2}.
  \]
Chapter III

Eisenstein series

III.1 2014-02-21 [missing]

III.2 2014-02-24

III.2.1 Cusps of $\Gamma(N)$

Recall: the cusps of $\Gamma(N)$ are given by

\[
\begin{pmatrix} a \\ c \end{pmatrix} \equiv \begin{pmatrix} a' \\ c' \end{pmatrix} \pmod{N}
\]

with $\gcd(a, c) = 1$, so the cusps of $\Gamma(N)$ are represented by $\frac{a}{c}$ with $\gcd(a, c) = 1$, i.e., by elements of $(\mathbb{Z}/N)^2$ of order exactly $N$.

**Remark III.2.1.**

\[
\nu_\infty(\Gamma(N))) = \begin{cases} 
\frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) & \text{if } N > 2, \\
3 & \text{if } N = 2.
\end{cases}
\]

III.2.2 Cocycle relations

We have the following cocycle relation for $j(\gamma, z)$:

**Lemma III.2.2.** For all $\alpha, \beta \in \text{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{h}$,

\[
j(\alpha \beta, z) = j(\alpha, \beta z)j(\beta, z).
\]

Let $x$ be a cusp of $\Gamma$, and let $\sigma \in \text{SL}_2(\mathbb{Z})$ such that $\sigma \cdot x = \infty$. Then:

**Lemma III.2.3.** Let $k \geq 0$. Then

\[
j(\sigma \gamma \sigma^{-1}, z)^k = 1 \quad \forall \gamma \in \Gamma_x
\]

iff $k$ is even or $-I \notin \Gamma$ or $x$ is regular.
Definition III.2.4. Let $k \geq 3$, and take $\Gamma = \Gamma(N)$. Let $x_i \in \mathfrak{h}^*$ be cusps of $\Gamma(N)$, and let $\sigma_i \in \text{SL}_2(\mathbb{Z})$ such that $\sigma_i \cdot x_i = \infty$. (If $k$ is odd, assume the $x_i$ are regular.) Suppose $x_0 = \infty$ and $\sigma_0 = I$. Define:

$$g_i(z) = g_i(z; k, N) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma \setminus \Gamma} j(\sigma_i, z)^{-k}|_{k}\gamma.$$ 

Our goal is to show that $\{g_i(z)\}$ is a basis of the Eisenstein space $E(\Gamma(N))$ for all $k \geq 3$, i.e.:

(i) $g_i \in M_k(\Gamma(N))$.

(ii) The $g_i$ are linearly independent.

Note III.2.5. $-I \in \Gamma(N)$ iff $N \in \{1, 2\}$.

Lemma III.2.6. There is a bijection

$$\Gamma_\infty \setminus \Gamma \leftrightarrow \begin{cases} (c, d) \in \mathbb{Z}^2 & \gcd(c, d) = 1, \\
(c, d) \equiv (0, 1) \pmod{N}, & \text{if } N \in \{1, 2\}, \text{ then } d > 0 \end{cases}$$

Note that

$$\pm \begin{pmatrix} 1 & Ne \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + cNe & b + dNe \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

with $a'd - b'c = 1$ and $xd - yc = 1$. So $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and $N | a - a', N | b - b'$.

Let $C_N = 1$ if $N \geq 3$, and $C_N = \frac{1}{2}$ if $N = 1, 2$. So

$$g_0(z) = C_n \sum_{(c, d) \in \mathbb{Z}^2} \sum_{\substack{(c, d) \equiv (0, 1) \\ \gcd(c, d) = 1}} \frac{1}{(cz + d)^k}.$$ 

For $v = (c, d) \mod N$ of order $N$ in $(\mathbb{Z}/N)^2$, let

$$G_k^v(z) := C_N \sum_{\substack{(c, d) \in \mathbb{Z}^2 \\ \gcd(c, d) = 1}} \frac{1}{(cz + d)^k}.$$ 

Let

$$\tilde{E}_k(z) := \sum_{\substack{(c, d) \in \mathbb{Z}^2 \\ (c, d) \neq 0}} \frac{1}{(cz + d)^k}.$$ 

Proposition III.2.7. For all $k \geq 3$, $\tilde{E}_k(z)$ converges absolutely and uniformly on compact subsets of $\mathfrak{h}$. 
Proof. Let \( z \in \mathfrak{h} \), and denote \( L_z := \{cz + d \mid c, d \in \mathbb{Z} \} \subseteq \mathbb{C} \). For \( n \in \mathbb{Z}_{>0} \), let \( M_n \) be the boundary of the parallelogram whose vertices are \( \pm nz \pm n \). Then \( \bigcup_{n \geq 1} (L \cap M_n) = L \setminus \{0\} \), and \( \#(L \cap M_n) = 8n \).

Let \( r(z) \) be the distance from the origin to \( M_1 \); this is a continuous function of \( z \). If \( \omega \in L \cap M_n \), then \( |\omega| \geq r(z) \cdot n \). Thus,

\[
\sum_{\begin{subarray}{c}(c,d) \in \mathbb{Z}^2 \\ (c,d) \neq 0 \end{subarray}} \frac{1}{|cz + d|^k} = \sum_{n=1}^{\infty} \frac{1}{|\omega|^k} \leq \sum_{n=1}^{\infty} 8n \cdot \frac{1}{(r(z)n)^k} = 8r(z)^{-k} \zeta(k - 1). \quad \square
\]

Lemma III.2.8. (i) Let \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) such that \( (c,d) \equiv v \pmod{N} \). Then

\[
G_k^v(z) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} j(\gamma, z)^{-k}.
\]

(ii) For all \( \gamma \in \text{SL}_2(\mathbb{Z}) \),

\[
(G_k^v|k\gamma) = G_k^{v\gamma}.
\]

Proof. (i) If \( (c', d') \equiv (0, 1) \pmod{N} \), then

\[
\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ c & d \end{pmatrix} \pmod{N}.
\]

(ii) We have

\[
(G_k^v|k\gamma)(z) = C_N j(\gamma, z)^{-k} \sum_{\gamma' \in \Gamma_{\infty} \setminus \Gamma} j(\gamma', \gamma' \cdot z)^{-k}
= C_N \sum_{\gamma' \in \Gamma_{\infty} \setminus \Gamma} j(\gamma', z)^{-k}
= C_N \sum_{\gamma'' \in \Gamma_{\infty} \setminus \Gamma \gamma} j(\gamma'', z)^{-k} = G_k^{v\gamma}(z). \quad \square
\]

III.3 2014-02-26: Eisenstein series

Let \( \bar{\nu} \in (\mathbb{Z}/N\mathbb{Z})^2 \). Define

\[
G_k^{\bar{\nu}}(z) = C_N \sum_{\begin{subarray}{c}(c,d) \in \mathbb{Z}^2 \\ \gcd(c,d) = 1 \\ (c,d) \equiv \bar{\nu} \pmod{N} \end{subarray}} \frac{1}{(cz + d)^k},
\]

where \( C_N = \frac{1}{2} \) if \( N = 1, 2 \), and \( C = 1 \) otherwise. This set of functions (with \( \bar{\nu} \) varying and \( k \) fixed) is permuted by \( \text{SL}_2(\mathbb{Z}) \), whence \( G_k^{(0,1)}(z) \) is preserved by \( \text{Stab}_{(0,1)} \text{SL}_2(\mathbb{Z}) = \Gamma_1(N) \).
CHAPTER III. EISENSTEIN SERIES

How does $G_k^\nu(z)$ behave as we move towards the boundary of $\mathfrak{h}^*$, e.g., what happens to $G_k$ as $z \to i\infty$? (Note that if we wanted to describe behavior as $z \to p/q = \gamma \infty$, we can just study $G_k^\nu(z) = c \cdot G_k^{\nu'}(z)$ as $z \to \infty$.)

When $k > 2$, the sum $\sum (cz + d)^{-k}$ is uniformly convergent. So

$$
\lim_{z \to i\infty} C_N \sum (cz + d)^k = \lim_{z \to i\infty} \frac{1}{(cz + d)^k}.
$$

Note that $\lim_{z \to i\infty} (cz + d)^{-k} = 0$ whenever $c \neq 0$. Since $\gcd(c, d) = 1$, if $c = 0$, then $d = \pm 1$. Thus,

$$
C_N \sum_{z \to i\infty} \frac{1}{(cz + d)^k} = C_N \sum_{d \in \{\pm 1\}} \frac{1}{d^k} = \begin{cases}
1 & \text{if } \nu = (0, 1) \text{ and } N \geq 3, \\
(-1)^k & \text{if } \nu = (0, -1) \text{ and } N \geq 3, \\
1 & \text{if } \nu = (0, 1) \text{ and } N \in \{1, 2\} \text{ and } k \text{ even}, \\
0 & \text{otherwise}.
\end{cases}
$$

To sum up, $\lim_{z \to i\infty} G_k^\nu(z)$ is one of $0, 1, (-1)^k$.

We can use this computation to show these forms are a basis of the space of Eisenstein series. We have an exact sequence

$$
0 \to S_k(\Gamma(N)) \to M_k(\Gamma(N)) \xrightarrow{E} \mathbb{C}^{\{\text{cusps of } \Gamma(N)\}} \xrightarrow{f \mapsto \lim_{z \to s} f(z)}.
$$

What we have just done is computed $E(G_k^{(0, 1)}).

Consider

$$
E_k^\nu(z) = \sum_{(c, d) \in \mathbb{Z}^2} \frac{1}{(cz + d)^k}
= \sum_{n=1}^{\infty} \sum_{\gcd = n} \frac{1}{(cz + d)^k}
= \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{\gcd = n} \frac{1}{(cz + d)^k}
$$

(where $c = nc', d = nd'$, and $\gcd(c', d') = 1$)

$$
= \sum_n \frac{1}{n^k} G_k^{x^{-1}\nu}(z)
= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \left( \sum_{n \equiv x (N)} \frac{1}{n^k} G_k^{x^{-1}\nu}(z) \right)
= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \zeta_x(k) G_k^{x^{-1}\nu}(z).
$$
III.4 2014-02-28: The $q$-expansion

Recall $\tilde{E}_k^\theta$ from last time: Fix $(c_v, d_v) \in \mathbb{Z}^2$ such that $\gcd(c_v, d_v) = 1$ and $(c_v, d_v) \equiv \bar{v} \pmod{N}$. Then

$$
\tilde{E}_k^\theta = \sum \frac{1}{(cz+d)^k} = \sum_{c \equiv c_v, d \in \mathbb{Z}} \frac{1}{(cz+d_v + N^k)^k} = \frac{1}{N^k} \sum_{c \equiv c_v, d \in \mathbb{Z}} \frac{1}{(\frac{cz+d_v}{N} + d)^k}.
$$

The $c = 0$ term gives a constant term

$$
\text{constant term} = \begin{cases} 
0 & \text{if } c_v \not\equiv 0 \pmod{N}, \\
\sum d \equiv d_v \frac{1}{d^k} & \text{if } c_v \equiv 0. 
\end{cases}
$$

If $c > 0$, then $\tau := \frac{cz+d_v}{N} \in \mathfrak{h}$.

Lemma III.4.1. We have

$$
\sum_{d \in \mathbb{Z}} \frac{1}{(\tau + d)^k} = \Omega_k \sum_{m=1}^{\infty} m^{k-1} q^m,
$$

where $\tau \in \mathfrak{h}$, $k \geq 2$, $q = e^{2\pi i \tau}$, and $\Omega_k := \frac{(-2\pi i)^k}{(k-1)!}$.

Proof. Use Poisson summation: let $\mathfrak{h} : \mathbb{R} \to \mathbb{C}$ such that

- $\int_{-\infty}^{\infty} |h(x)| \, dx < \infty$, and
- $\sum_{d \in \mathbb{Z}} h(x+d)$ converges absolutely and uniformly on compact subsets, and is $C^\infty$.

Then

$$
\sum_{d \in \mathbb{Z}} h(x+d) = \sum_{m \in \mathbb{Z}} \hat{h}(m) e^{2\pi imx},
$$

where $\hat{h}$ is the Fourier transform

$$
\hat{h}(x) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i xt} \, dt.
$$

Let $\tau := x + iy$ for fixed $y > 0$. Applying Poisson summation to $h(x) = \frac{1}{\tau^k} = \frac{1}{(x+iy)^k}$ for $k \geq 2$, it suffices to show that

$$
\hat{h}(m) = \begin{cases} 
0 & \text{if } m \leq 0, \\
\Omega_k m^{k-1} & \text{if } m > 0. 
\end{cases}
$$

For $m = 0$, observe that

$$
\hat{h}(0) = \int_{-\infty}^{\infty} h(t) \, dt = \int_{-\infty}^{\infty} \frac{1}{(x+iy)^k} = \frac{1}{(1-k)(x+iy)^{k-1}} \bigg|_{-\infty}^{\infty} = 0.
$$
If \( m > 0 \), then replacing \( t \) with \( t - iy \) by a change of variables,

\[
\hat{h}(m) = \int_{-\infty}^{\infty} \frac{1}{(t + iy)^k} e^{-2\pi i m t} dt = e^{2\pi i m y} \int_{iy - \infty}^{iy + \infty} \frac{1}{t^k} e^{-2\pi i m t} dt.
\]

Let \( f_m(z) = z^{-k} e^{-2\pi i m z} \). This is meromorphic with a pole at \( z = 0 \). We can check that

\[
-2\pi i \text{Res}_{z=0} f_m(z) = \Omega_k m^{k-1},
\]

and contour integration shows this is indeed the value of the above integral.

If \( m < 0 \), then

\[
\hat{h}(m) = \int_{-\infty}^{\infty} \frac{1}{(t + iy)^k} e^{-2\pi i m t} dt
\]

\[
= \int_{\infty}^{-\infty} \frac{1}{(-t + iy)^k} e^{-2\pi i |m| t} dt
\]

\[
= (-1)^{k+1} \int_{-\infty}^{\infty} \frac{1}{(t - iy)^k} e^{-2\pi i |m| t} dt.
\]

We can now do a contour integral containing no poles, so by the residue theorem, the integral is zero.

Continuing with the Eisenstein series from before,

\[
\tilde{E}_k^0(z) = \frac{1}{N_k} \sum_{c \equiv c_0} \sum_{d \in \mathbb{Z}} \frac{1}{(\frac{cz + d}{N} + d)^k}.
\]

If \( c < 0 \), \( \tau = \frac{cz + d}{N} \), \( \mu_N = e^{2\pi i / N} \), and \( q_N = e^{2\pi i \tau / N} \), then

\[
e^{2\pi i \tau m} = e^{2\pi i mcz / N} e^{2\pi i d_m / N} = q_N^{c_m} \mu_N^{d_m},
\]

so the \( c > 0 \) part is equal to

\[
\frac{\Omega_k}{N_k} \sum_{c \equiv c_0} \sum_{c > 0} \sum_{m=1}^\infty \mu_N^{d_m} q_N^{c_m} = \frac{\Omega_k}{N_k} \sum_{n=1}^\infty \sum_{m \mid n} \mu_N^{d_m} q_N^n.
\]

If \( c < 0 \), \( \tau = -(\frac{cz + d}{N}) \in \mathfrak{h} \), then the \( c < 0 \) part is

\[
(-1)^k \frac{\Omega_k}{N_k} \sum_{c \equiv c_0} \sum_{c < 0} \sum_{m=1}^\infty \mu_N^{-d_m} q_N^{-c_m} = \frac{\Omega_k}{N_k} \sum_{n=1}^\infty \sum_{m \mid n} \mu_N^{d_m} q_N^n.
\]

This computation yields the following:
Theorem III.4.2. For \( k \geq 3 \) and \( \bar{v} \in (\mathbb{Z}/N)^2 \) of order \( N \),

\[
\tilde{E}_k^0(z) = \delta(c_0) \zeta_{d_0}^{\pm}(k) + \frac{\Omega_k}{N^k} \sum_{n=1}^{\infty} \sigma_{k-1}^0(n) q_n^k,
\]

where

\[
\delta(c_0) = \begin{cases} 
1 & \text{if } c_0 \equiv 0 \pmod{N}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\zeta_{d_0}^{\pm}(k) = \sum_{d \in \mathbb{Z}} \chi_{d_0}(d) \frac{1}{d^k} = \begin{cases} 
0 & \text{if } k \text{ is odd}, \\
2\zeta(k) & \text{if } k \text{ is even},
\end{cases}
\]

\[
\sigma_{k-1}^0(n) = \sum_{d \mid m \mid n} sgn(m) m^{k-1} \mu_{d,m}^N.
\]

Remark III.4.3. If \( \Gamma \) is a congruence subgroup such that \( \Gamma(N) \leq \Gamma \), let

\[
\tilde{E}_{k,\Gamma} := \sum_{\gamma \in \Gamma(N) \setminus \Gamma} \tilde{E}_k^{0}|_{k \gamma} \in M_k(\Gamma).
\]

Example III.4.4. For \( N = 1 \) and \( v = (0, 1) \), we have

\[
\zeta_{d_0}^{\pm}(k) = \sum_{d \neq 0} \frac{1}{d^k} = \begin{cases} 
0 & \text{if } k \text{ is odd}, \\
2\zeta(k) & \text{if } k \text{ is even},
\end{cases}
\]

and

\[
\sigma_{k-1}^0(n) = \sum_{m \mid n} sgn(m) m^{k-1} = \begin{cases} 
0 & \text{if } k \text{ is odd}, \\
2\sigma_{k-1}(n) & \text{if } k \text{ is even}.
\end{cases}
\]

Also, \( \sigma_{k-1}(n) = \sum_{d \mid n, d > 0} d^{k-1} \), so we get \( \tilde{E}_k(z) = 0 \) if \( k \) is odd, and

\[
\tilde{E}_k(z) = 2\zeta(k) + 2\Omega_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q_n^k
\]

if \( k \) is even.

III.5 2014-03-03

Let

\[
E_k(z) = \frac{1}{2\zeta_k} \tilde{E}_k(z) = \begin{cases} 
\zeta(1-k) \frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q_n^k, & k \text{ even}, \\
0, & k \text{ odd}.
\end{cases}
\]

Another useful normalization:

\[
G_k(z) = \frac{1}{2} \left( \sum_{m=1}^{\infty} \frac{\mu(m)}{m^k} \right) \tilde{E}_k(z).
\]
Fact III.5.1. \( \sum_{m=1}^{\infty} \frac{\mu(m)}{m^k} = \frac{1}{\zeta(k)} \), so if \( k \) is even, then

\[
G_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.
\]

Define:

\[
\Delta(z) := \frac{1}{1728} \left( G_4^3 - G_6^2 \right).
\]

Then \( \Delta \in S_{12}(SL_2(\mathbb{Z})) \).

Consider the \( j \)-invariant:

\[
j(z) = \frac{G_4^3}{\Delta} = \frac{1}{q} + 744 + 196884q + \cdots \in A_0(X(1)) = \mathcal{M}(X(1)) \cdot j.
\]

**Theorem III.5.2.** \( j'(z) \in A_2(SL_2(\mathbb{Z})) \), so there exists \( f_0 \neq 0 \) in \( A_k(SL_2(\mathbb{Z})) \) for all \( k \geq 0 \) even (namely, \( f_0 = (j'|_2)^{k/2} \)).

**Proof.** For all \( \gamma \in SL_2(\mathbb{Z}) \), consider the map \( \gamma \cdot j \rightarrow j \). We have

\[
j(z) = (j \circ \gamma)(z) = j'(\gamma z) \cdot \gamma'(z) = j'(\gamma z) \frac{1}{j(\gamma, z)^2} = (j'|_2)(z).
\]

This is holomorphic on \( \mathfrak{h} \) and meromorphic at \( \infty \), so \( j' \in A_2(SL_2(\mathbb{Z})) \). \(\square\)

### III.5.1 \( G_4 \) and \( G_6 \) generate modular forms

**Theorem III.5.3.** \( M(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} M_k(SL_2(\mathbb{Z})) \) is freely generated by \( G_4 \) and \( G_6 \).

We will soon show that \( G_4 \) and \( G_6 \) are algebraically independent. Assuming this, \( \langle G_4, G_6 \rangle \) is a freely generated subring of \( M(SL_2(\mathbb{Z})) \). Recall that

\[
dim M_k(SL_2(\mathbb{Z})) = \left\lfloor \frac{k}{12} \right\rfloor + 1 \quad \text{if} \quad k \not\equiv 2 \pmod{12},
\]

\[
dim M_k(SL_2(\mathbb{Z})) = \left\lfloor \frac{k}{12} \right\rfloor \quad \text{if} \quad k \equiv 2 \pmod{12}.
\]

For example:

<table>
<thead>
<tr>
<th>( k )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{k}{12} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \frac{k}{12} )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Note that \( \dim S_{k+12} + 1 = \dim M_{k+12} = \dim M_k + 1 \). Consider the map

\[
\delta : M_k \xrightarrow{\sim} S_{k+12}, \quad f \mapsto f \cdot \Delta.
\]

So we want to show there exists a non-cusp form in \( \langle G_4, G_6 \rangle \) for all even \( k \geq 4 \). Indeed:
III.6. 2014-03-05: DIRICHLET CHARACTERS

- If \( k \equiv 0, 4, 8 \pmod{12} \), we have \( G_4^{k/4} \).
- If \( k \equiv 6 \pmod{12} \), we have \( G_6^{k/6} \).
- If \( k \equiv 2, 10 \pmod{12} \), we have \( G_4^{k-6} \cdot G_6 \).

**Lemma III.5.4.** If \( f_1, f_2 \in M_k(\Gamma) \) are non-zero and \( f_1 \neq \lambda f_2 \) for \( \lambda \in \mathbb{C} \), then \( f_1 \) and \( f_2 \) are algebraically independent.

**Proof.** If \( F(x,y) \in \mathbb{C}[x,y] \) such that \( F(f_1, f_2) = 0 \), then \( F(x,y) = \sum_{d \geq 0} F_d(x,y) \), where \( F_d \) is homogeneous of degree \( d \). For each \( d \), there exists \( P_d(t) \) such that

\[
\frac{F_d(f_1, f_2)}{f_2} = P_d(f_1/f_2).
\]

Since \( P_d \) is a polynomial in one variable, it has finitely many roots. Thus, \( f_1/f_2 \) is constant, contrary to what was assumed. \qed

Thus, \( G_4^3 \) and \( G_6^2 \) are algebraically independent, and hence so are \( G_4 \) and \( G_6 \).

**III.5.2 Congruences**

Observe that

\[
E_{12} = \frac{691}{2 \cdot 12 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13} + \ldots,
\]
\[
E_6 = \frac{-1}{2 \cdot 6 \cdot 2 \cdot 3 \cdot 7} + \ldots,
\]

so

\[
\Delta = \frac{65}{65 + 691} E_{12} - \frac{691}{3} E_6^2.
\]

We can write \( \Delta = \sum_{n=1}^{\infty} \tau(n)q^n \), and

\[
\Delta \equiv E_{12} \pmod{691},
\]

i.e., if \( p \) is prime, then

\[
\tau(p) \equiv 1 + p^{11} \pmod{691}.
\]

**III.6 2014-03-05: Dirichlet characters**

**III.6.1 Dirichlet characters**

Let \( N \) be a positive integer. A \textit{Dirichlet character} mod \( N \) is a homomorphism \( \chi : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times \).

Given two Dirichlet characters \( \chi, \eta : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times \), we can form the product

\[
(\chi \eta)(a) = \chi(a)\eta(a).
\]
There is also an identity character \( \chi_N = 1 \) defined by \( \chi_N(a) = 1 \), and inverses are given by complex conjugation. Hence, the Dirichlet characters form a group \((\mathbb{Z}/N)^\times\), the dual group of \((\mathbb{Z}/N)^\times\).

Dirichlet characters satisfy the usual orthogonality relations.

**Note III.6.1.** If \( d \mid N \) and \( \chi \) is a Dirichlet character mod \( d \), then \( \chi \) induces a Dirichlet character \( \chi_N \) mod \( N \).

The *conductor* \( c_\chi \) of \( \chi \) is the least positive integer such that \( \chi \) factors through \((\mathbb{Z}/c_\chi)^\times\).

**Example III.6.2.** Let \( p \) be an odd prime. Then

\[
a \mapsto \left( \frac{a}{p} \right) = \begin{cases} 
1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution}, \\
-1 & \text{otherwise}
\end{cases}
\]

is a Dirichlet character mod \( p \).

If \( c_\chi = N \), then \( \chi \) is called *primitive*.

**Remark III.6.3.** By convention, we extend \( \chi \) to \( \mathbb{Z}/N\mathbb{Z} \) by defining \( \chi(0) = 0 \), and to \( \mathbb{Z} \) via \( \mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z} \), so

\[
\chi(a) = \chi(a \mod N).
\]

**Note III.6.4.**

- \( \chi(ab) = \chi(a)\chi(b) \) for all \( a, b \in \mathbb{Z} \).
- \( \chi_N(a) = \begin{cases} 
1, & \gcd(a, N) = 1, \\
0, & \gcd(a, N) \neq 1.
\end{cases} \)
- \( \chi(0) = \begin{cases} 
0, & N > 1, \\
1, & N = 1.
\end{cases} \)

### III.6.2 Gauss sums

Concretely: let \( \mu_N := e^{2\pi i/N} \), a primitive \( N \)-th root of 1. Let \( \chi \) be a Dirichlet character mod \( N \). Then

\[
\tau(\chi) := \sum_{a \in \mathbb{Z}/N} \chi(a)\mu_N^a \in \mathbb{C}.
\]

Look at

\[
\sum_{a \in \mathbb{Z}/N} \chi(a)(\mu_N^m)^a = \sum_{a \in \mathbb{Z}/N} \chi(a)\psi_m(a).
\]

Conceptually: The map

\[
\psi_m : \mathbb{Z}/N \rightarrow \mathbb{C} \\
a \mapsto (\mu_N^m)^a
\]

is an *additive character* of \( \mathbb{Z}/N \). Note: \( \psi_m(a) = \psi_1(ma) \).
Given any additive character \( \psi : \mathbb{Z}/N \to \mathbb{C} \) and any \( f : \mathbb{Z}/N \to \mathbb{C} \), we define the \textit{Fourier transform}\(^1\)

\[
\hat{f}(m) := \sum_{a \in \mathbb{Z}/N} f(a) \psi_m(-a).
\]

Note that \( \tau(\chi) = \tau(\chi, \psi_{-1}) = \hat{\chi}(-1) \).

Facts:

- If \( \chi \) is primitive mod \( N \), then for all \( m \in \mathbb{Z} \),
  \[
  \tau(\chi, \psi_m) = \overline{\chi(m)} \tau(\chi),
  \]
  and \( |\tau(\chi)| = \sqrt{N} \).

- Consider \( \chi \) mod \( N \) of conductor \( N_0 \) coming from \( \chi_0 \) mod \( N_0 \). Then
  \[
  \tau(\chi) = \mu(N/N_0)\chi_0(N/N_0)\tau(\chi_0).
  \]

- \( \tau(\overline{\chi}) = \chi(-1)\tau(\chi) \).

- If \( \chi \) and \( \chi' \) are characters mod \( N \) and \( N' \), respectively, and \( \gcd(N, N') = 1 \), then
  \[
  \tau(\chi \chi') = \chi(N')\chi'(N)\tau(\chi)\tau(\chi').
  \]
  More generally, if \( (N, N') \geq 1 \), then
  \[
  \tau(\chi, \chi') = \frac{\tau(\chi)\tau(\chi')}{J(\chi, \chi')},
  \]
  where \( J(\chi, \chi') \) is the \textit{Jacobi sum}.

- \( \chi(-1) = \pm 1 \). If \( \chi(-1) = 1 \) (resp. \( \chi(-1) = -1 \)), then \( \chi \) is \textit{even} (resp. \textit{odd}). If \( N \geq 3 \), then exactly half the Dirichlet characters are even and half are odd.

### III.6.3 \( L \)-functions

**Definition III.6.5.** The \textit{Dirichlet \( L \)-function} of a character \( \chi \) is the function defined for \( \text{Re}(s) > 1 \) by

\[
L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.
\]

Since \( |\chi(n)| = 1 \), this converges absolutely for \( \text{Re}(s) > 1 \).

**Note III.6.6.** If \( \chi = 1_1 \), then \( L(s, \chi) = \zeta(s) \).

\(^1\)The name makes sense: If \( f : \mathbb{R} \to \mathbb{C} \), the additive characters of \( \mathbb{R} \) are \( \psi_{y}(x) = e^{2\pi iyx} \), and

\[
\hat{f}(y) = \int_{\mathbb{R}} f(x)\psi_{y}(-x) \, dx
\]

is the usual Fourier transform.
**Theorem III.6.7 (Euler).** Let $F(n)$ be a multiplicative function $\mathbb{Z}_{\geq 1} \to \mathbb{C}$ (i.e., $F(ab) = F(a)F(b)$ if $(a, b) = 1$) such that $\sum_{n=1}^{\infty} F(n)$ is absolutely convergent. Then

$$
\sum_{n=1}^{\infty} F(n) = \prod_{p \text{ prime}} \left( \sum_{m=0}^{\infty} F(p^m) \right),
$$

and the product is absolutely convergent. If $F$ is completely multiplicative, then $F(p^m) = F(p)^m$, so

$$
\sum_{n=1}^{\infty} F(n) = \prod_{p \text{ prime}} \frac{1}{1 - F(p)}.
$$

Applying this to $F(n) = \chi(n)n^{-s}$, we obtain

$$
L(s, \chi) = \prod_{p} \frac{1}{1 - \chi(p)p^{-s}}.
$$

**III.7 2014-03-07: The functional equation**

Today, we will prove the functional equation for $L(s, \chi)$.

Here’s the idea for proving the functional equation: use “modular forms” of half-integer weight (theta series) and harmonic analysis for $\mathbb{R}^\times > 0$.

As motivation, consider the function $\Gamma$ defined for $\Re(s) > 0$ by

$$
\Gamma(s) = \int_{0}^{\infty} e^{-y} y^{s} \frac{dy}{y}.
$$

This generalizes the factorial function, in the sense that $\Gamma(s + 1) = s\Gamma(s)$ and $\Gamma(1) = 1$, so $\Gamma(n) = (n - 1)!$ for all $n \in \mathbb{Z}_{\geq 1}$.

Using the change of variables $y \mapsto \pi n^2 y$, we have

$$
\int_{0}^{\infty} e^{-\pi n^2 y} y^{2} \frac{dy}{y} = \pi^{-s} \Gamma(s) \frac{1}{n^{2s}}.
$$

Summing over all $n \geq 1$,

$$
\pi^{-s} \Gamma(s) \zeta(2s) = \int_{0}^{\infty} \left( \sum_{n=1}^{\infty} e^{-\pi n^2 y} \right) y^{s} \frac{dy}{y}.
$$

Recall the function

$$
\theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z} = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi n^2 z}.
$$

The above function $\theta$ satisfies a functional equation coming from the fact that $\theta$ is a modular form. Define

$$
\Lambda(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).
$$

We will use “modularity” of $\theta$ to show that $\Lambda(s) = \Lambda(1 - s)$. 
III.7.1 The Mellin transform

Let us compare the topological groups \((\mathbb{R}, +)\) and \((\mathbb{R}_{\geq 0}, \cdot)\):

<table>
<thead>
<tr>
<th>Haar measure:</th>
<th>locally compact</th>
<th>Haar measure:</th>
<th>locally compact</th>
</tr>
</thead>
<tbody>
<tr>
<td>(dx)</td>
<td></td>
<td>(d^\times y = \frac{dy}{y})</td>
<td></td>
</tr>
<tr>
<td>Characters:</td>
<td>(\psi_m(x) = e^{2\pi imx}, m \in \mathbb{R})</td>
<td>Characters:</td>
<td>(\psi_s(y) = y^s, s \in \mathbb{C})</td>
</tr>
<tr>
<td>“Fourier” transform:</td>
<td>(f(m) = \int_{\mathbb{R}} f(x)e^{-2\pi imx}dx)</td>
<td>“Fourier” transform:</td>
<td>(C(y) = \int_{0}^{\infty} F(y) y^s dy)</td>
</tr>
</tbody>
</table>

**Definition III.7.1.** The Mellin transform of a continuous function \(F : \mathbb{R}_{\geq 0} \rightarrow \mathbb{C}\) is

\[
\mathcal{M}(F)(s) := \int_{0}^{\infty} (F(y) - F(\infty)) y^s dy.
\]

**Example III.7.2.** \(\Gamma(s) = \mathcal{M}(e^{-y})(s)\).

**Theorem III.7.3.** Let \(F, G : \mathbb{R}_{\geq 0} \rightarrow \mathbb{C}\) be continuous functions such that

\[
F(y) = a_F + O(e^{-cy^\alpha}), \\
G(y) = a_G + O(e^{-cy^\alpha})
\]
as \(y \rightarrow \infty\). Suppose there exist \(k \in \mathbb{R}_{>0}\) and \(C \in \mathbb{C}\) such that

\[
F(\frac{1}{y}) = Cy^k G(y).
\]

Then:

(i) \(\mathcal{M}(F)(s)\) and \(\mathcal{M}(G)(s)\) converge absolutely and uniformly on compact subsets of \(\text{Re}(s) > k\), and they have an analytic continuation to \(\mathbb{C} \setminus \{0, k\}\).

(ii) \(\mathcal{M}(F)(s)\) and \(\mathcal{M}(G)(s)\) have simple poles at 0 and \(k\) with residues as follows:

\[
\begin{array}{c|cc}
\mathcal{M}(F) & 0 & k \\
\mathcal{M}(G) & -a_F & Ca_G \\
\end{array}
\]

(iii) \(\mathcal{M}(F)(s) = C\mathcal{M}(G)(k - s)\).

**Proof.** As \(s\) varies over compact subsets of \(\mathbb{C}\), for \(y \geq 1\), \(e^{-cy^\alpha} y^{\text{Re}(s)+1}\) is \(O(1)\)-independent of \(\text{Re}(s)\). Thus, for some constant \(\beta\),

\[
\int_{1}^{\infty} \left| (F(y) - a_F) y^{s-1} \right| dy \leq \int_{1}^{\infty} \beta e^{-cy^\alpha} y^{\text{Re}(s)+1} \frac{dy}{y^2}.
\]

So for \(\text{Re}(s) > k\), \(\mathcal{M}(F)(s) = \int_{1}^{\infty} + \int_{0}^{1}\), and the \(\int_{1}^{\infty}\) term converges absolutely and uniformly on compact subsets. Now we use the functional equation to handle the \(\int_{0}^{1}\) term. Applying the change of variables \(y \mapsto \frac{1}{y} = u\), \(du = -\frac{1}{y^2} dy\),

\[
\int_{1}^{\infty} F\left(\frac{1}{y}\right) y^{-s} dy \left| y^{-1}\right|_0 = \frac{a_F}{s} + C \int_{1}^{\infty} G(y) y^{k-s} dy
\]

\[
= \frac{a_F}{s} - \frac{Ca_G}{k-s} + C \int_{1}^{\infty} (G(y) - a_G) y^{k-s} dy.
\]
CHAPTER III. EISENSTEIN SERIES

Note that \( \int_1^\infty (G(y) - a_G) y^{k-s} \frac{dy}{y} \) converges absolutely and uniformly. The other two terms can give poles at 0 and \( k \). In particular:

\[
\mathcal{M}(F)(s) = \frac{-a_F}{s} + \frac{Ca_G}{s - k} + \mathcal{F}(s),
\]

where

\[
\mathcal{F}(s) = \int_1^\infty (F(y) - a_F) y^s \frac{dy}{y} + C \int_1^\infty (G(y) - a_G) y^{k-s} \frac{dy}{y}
\]
is a holomorphic function that converges absolutely and uniformly. Applying the same argument for \( G \), we have

\[
\mathcal{M}(G)(s) = \frac{-a_G}{s} + \frac{C^{-1}a_F}{s - k} + \mathcal{G}(s),
\]
where \( \mathcal{G}(s) \) is a holomorphic function that converges absolutely and uniformly. Clearly,

\[
\mathcal{F}(s) = C\mathcal{G}(k-s),
\]

from which the functional equation follows immediately. \( \square \)

III.7.2 Proof of the functional equation

Let \( \chi \) be a primitive Dirichlet character mod \( N \). We want to find \( F_\chi \) and \( G_\chi \) such that

\[
\mathcal{M}(F_\chi)(s) = \Lambda(s, \chi) N^{\frac{s+\delta}{2}} =: \tilde{\Lambda}(s, \chi), \]

\[
\mathcal{M}(G_\chi)(s) = \Lambda(s, \overline{\chi}) N^{\frac{s+\delta}{2}} =: \tilde{\Lambda}(s, \overline{\chi}).
\]

Here, we will have \( C = \frac{\tau(\chi)}{i^\delta \sqrt{N}} \).

Observe that

\[
\Lambda(s, \chi) = \Lambda \left( \frac{s + \delta}{2} \right) L(s, \chi) \pi^{-\frac{s+\delta}{2}} \chi(-1)
\]

and

\[
\Gamma \left( \frac{s + \delta}{2} \right) = \int_0^\infty e^{-y} y^{\frac{s+\delta}{2}} \frac{dy}{y}.
\]

Taking the change of variables \( y \mapsto \frac{\pi n^2}{N} y \), we obtain

\[
\left( \frac{N}{\pi} \right)^{\frac{s+\delta}{2}} \Gamma \left( \frac{s + \delta}{2} \right) \frac{1}{n^s} = \int_0^\infty n^\delta e^{-\pi n^2 y/N} y^{\frac{s+\delta}{2}} \frac{dy}{y},
\]

so

\[
\Lambda(s, \chi) = \left( \frac{N}{\pi} \right)^{\frac{s+\delta}{2}} \Gamma \left( \frac{s + \delta}{2} \right) L(s, \chi)
\]

\[
= \int_0^\infty \left( \sum_{n=1}^{\infty} \chi(n) n^\delta e^{-\pi n^2 y/N} \right) y^{\frac{s+\delta}{2}} \frac{dy}{y}
\]

\[
= \int_0^\infty \frac{1}{2} \left( \theta_\chi(iy) - \chi(0) \right) y^{\frac{s+\delta}{2}} \frac{dy}{y}.
\]
where
\[ \theta_\chi(z) := \sum_{n \in \mathbb{Z}} \chi(n) n^\delta e^{i\pi n^2 z / N}. \]

**Theorem III.7.4.** The function \( \theta_\chi \) satisfies the functional equation
\[
\theta_\chi\left(\frac{-1}{z}\right) = \frac{\tau(\chi)}{i^\delta \sqrt{N}} \left(\frac{z}{i}\right)^{\delta + \frac{1}{2}} \theta_\chi(z).
\]

Assuming the above theorem for now, let \( \alpha_\chi := \left(\frac{\pi}{N}\right)^{\delta / 2} \). Then
\[ F_\chi(y) = \theta_\chi(iy) \cdot \frac{\alpha_\chi}{2}, \]
\[ G_\chi(y) = \theta_\chi(iy) \cdot \frac{\alpha_\chi}{2}. \]

Thus,
\[ F_\chi\left(\frac{1}{y}\right) = \frac{\alpha_\chi}{2} \theta_\chi\left(\frac{i}{y}\right) = \frac{\alpha_\chi}{2} \theta_\chi\left(\frac{-1}{cy}\right) = w(\chi) y^{\delta + 1 / 2} \theta_\chi(iy) \frac{\alpha_\chi}{2}, \]
where \( w(\chi) := \frac{\tau(\chi)}{i^\delta \sqrt{N}} \) is the root number. Note that \(|w(\chi)| = 1\).

By the previous theorem, assuming \( \chi \neq 1 \),
\[ \tilde{\Lambda}(s, \chi) = M(F_\chi) \left(\frac{s + \delta}{2}\right) = w(\chi)M(G_\chi) \left(\delta + \frac{1}{2} - \frac{s + \delta}{2}\right) = w(\chi)\tilde{\Lambda}(1 - s, \chi). \]

### III.8 2013-03-10: Eisenstein series of a character

Today, we’ll talk more about the Eisenstein series \( E_k(\Gamma_0(N)) \) and \( E_k(N, \chi) \), where \( \chi \) is a Dirichlet character mod \( N \).

Recall that
\[ M_k(\Gamma_1(N)) = \bigoplus_{\chi \in (\mathbb{Z}/N)^\times} M_k(N, \chi), \]
with an action of \( \Gamma_0(N) / \Gamma_1(N) \).

Suppose \( k \geq 3 \). Recall that for \( \bar{v} \in (\mathbb{Z}/N)^2 \) of order \( N \),
\[ \tilde{E}_k^\varnothing(\tau) = \delta(\bar{v}) \zeta_{d_0}^\pm(k) + \frac{\Omega_k}{N^k} \sum_{n=1}^\infty \sigma_{k-1}^\varnothing(n) q_N^n, \]
where
\[ \sigma_{k-1}^\varnothing = \sum_{m|m|n} \operatorname{sgn}(m) m^{k-1} \mu_N^{d_0 m}. \]

**Example III.8.1.** Consider \( \sum_{d \in (\mathbb{Z}/N)^\times \atop (d, N) = 1} E_k^{(0,d)}(\tau) \). Note that for \( \gamma = \left(\begin{array}{cc} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{array}\right) \in \Gamma_0(N), \)
\[ (0,d)\gamma = (0,dd_\gamma). \]
CHAPTER III. EISENSTEIN SERIES

Notation:

- \( \varphi \) is a Dirichlet character mod \( M_\varphi \).
- \( \psi \) is a Dirichlet character mod \( M_\psi \).
- \( \varphi \psi \) is a Dirichlet character mod \( M_\varphi M_\psi \).

Let \( \varphi \) be primitive, and \( \psi \) such that \( (\varphi \psi)(-1) = (-1)^k \).

One can check that, for \( \gamma \in \Gamma_0(N) \) (hence \( c_\gamma \equiv 0 \) and \( a_\gamma d_\gamma \equiv 1 \) mod \( N \)),

\[
(cM_\varphi, d + eM_\varphi)\gamma = (c'M_\varphi, d' + e'M_\varphi),
\]

where \( c' = ca_\gamma \), \( d' = dd_\gamma \), and \( e' = (e + ca_\gamma b_\gamma)d_\gamma \). Moreover,

\[
\psi(c')\varphi(d') = \psi(c)\psi(a_\gamma)\varphi(d)\varphi(d_\gamma) = (\psi \varphi)(d_\gamma)^{-1}\psi(c)\varphi(d).
\]

This suggests that we take the following “symmetrized twisted sum”:

**Definition III.8.2.** \( \tilde{E}^{\psi,\varphi}_k(\tau) := \sum_{c \pmod{M_\varphi}} \sum_{d \pmod{M_\psi}} \sum_{e \pmod{M_\psi}} \psi(c)\varphi(d) \tilde{E}^{(cM_\varphi, d + eM_\varphi)}_k(\tau) \).

If \( \gamma \in \Gamma_0(N) \), then

\[
(\tilde{E}^{\psi,\varphi}_k |_{k\gamma})(\tau) = (\psi \varphi)(d_\gamma) \tilde{E}^{\psi,\varphi}_k(\tau).
\]

Hence, \( \tilde{E}^{\psi,\varphi}_k \in M_k(N, \psi \varphi) \).

### III.8.1 Fourier expansions

We’ll split this up into the constant and non-constant parts.

The non-constant part of \( \tilde{E}^{\psi}_k \) can be rewritten as

\[
\frac{\Omega_k}{N^k} \sum_{m > 0 \atop n \equiv M_\varphi} \text{sgn}(m)m^{k-1} \mu(d) m^m q_N^m.
\]

For \( \tilde{E}^{\psi,\varphi}_k \), we get

\[
\frac{\Omega_k}{N^k} \sum_{c} \sum_{d} \sum_{e} \psi(c)\varphi(d) \sum_{mn > 0 \atop n \equiv M_\varphi} \text{sgn}(m)m^{k-1} \mu(d + eM_\varphi)m^m q_N^m
\]

\[
= \frac{\Omega_k}{N^k} \sum_{c} \sum_{d} \psi(c)\varphi(d) \sum_{mn > 0 \atop n \equiv M_\varphi} \text{sgn}(m)m^{k-1} \mu(d) m^m \left( \sum_{e} \mu(M_\psi) \right)^m q_N^m.
\]
Note that $\sum_e \mu_{M\phi}^{cm} = M\psi$ if $M\psi \mid m$, and 0 otherwise. Applying the change of variables $m \mapsto mM\psi$ and $n \mapsto nM\phi$,

$$
= \frac{\Omega_k}{M^k\phi} \sum_c \sum_d \psi(c)\varphi(d) \sum_{mn>0 \mod (M\phi)} \text{sgn}(m)m^{k-1}\mu_{M\phi}^{dn}q^{mn}
$$

$$
= \frac{\Omega_k}{M^k\phi} \sum_c \psi(c) \sum_{mn>0 \mod (M\phi)} \text{sgn}(m)m^{k-1} \left( \sum_d \varphi(d)\mu_{M\phi}^{dn} \right)
$$

$$
= \frac{\Omega_k}{M^k\phi} \tau(\varphi) \sum_{mn>0 \mod (M\phi)} \psi(n)\varphi(m)m^{k-1}q^{mn}
$$

$$
= 2\frac{\Omega_k}{M^k\phi} \tau(\varphi) \sum_{m,n>0} \psi(n)\varphi(m)m^{k-1}q^{mn}
$$

$$
= 2\frac{\Omega_k}{M^k\phi} \tau(\varphi) \sum_{n=1}^{\infty} \left( \sum_{m|n \mod (M\phi)} \psi\left(\frac{n}{m}\right) \varphi(m)m^{k-1} \right)q^n
$$

$$
= 2\frac{\Omega_k}{M^k\phi} \tau(\varphi) \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n
$$

where

$$
\sigma_{k-1}(n) := \psi \ast (\varphi m^{k-1}) = \sum_{m|n \mod (M\phi)} \psi\left(\frac{n}{m}\right) \varphi(m)m^{k-1}.
$$

The constant term is

$$
\sum_c \sum_d \sum_e \psi(c)\varphi(d)\delta(cM\phi)\zeta_{d+eM\phi}(k) = \psi(0) \sum_d \sum_e \varphi(d)\zeta_{d+eM\phi}(k).
$$

Recall that $\psi(0) \neq 0$ iff $\psi = 1$. Take $\psi(0) \neq 0$, so $\psi(0) = 1$, $M\psi = 1$, and $M\phi = N$, whence

$$
= \psi(0) \sum_{d \mod N} \varphi(d) \sum_{m \equiv d \mod N} \frac{1}{m^k}
$$

$$
= 2\psi(0)L(k, \varphi)
$$

by the functional equation for $L(s, \chi)$. Let

$$
E_{k,\psi,\phi}^\psi(\tau) = \frac{M^k\phi}{2\Omega_k\tau(\varphi)} \tilde{E}_{k,\psi,\phi}^\psi(\tau).
$$

Then:

**Theorem III.8.3.** $E_{k,\psi,\phi}^\psi(\tau) = \psi(0) \frac{L(1-k, \varphi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in \mathcal{E}_k(N, \psi\varphi)$. 
Definition III.8.4. Let $D_{N,k}$ be the set of triples $(\psi, \varphi, t)$ such that $\psi, \varphi$ are primitive mod $M_\psi, M_\varphi$, respectively, $(\psi \varphi)(-1) = (-1)^k$, and $t \in \mathbb{Z}_{\geq 1}$ such that $t M_\psi M_\varphi | N$.

Let $e_k^{\psi,\varphi,t}(\tau) := E_k^{\psi,\varphi}(t \tau) \in M_k(\Gamma_1(t M_\psi M_\varphi)) \subseteq M_k(\Gamma_1(N))$.

Theorem III.8.5. Let $\chi$ be a character mod $N$. Then

\[ \left\{ E_k^{\psi,\varphi,t} : (\psi, \varphi, \tau) \in D_{N,k} \text{ and } \psi \varphi = \chi \right\} \]

is a basis of $\mathcal{E}_k(N, \chi)$.

III.9 2014-03-24: Families of Eisenstein series [incomplete]

III.9.1 Weight 2 Eisenstein series

Let

\[ \tilde{E}_2(\tau) := \sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} \frac{1}{(c \tau + d)^2} \]

This converges conditionally. Then

\[ \left( \tilde{E}_2|_2 \gamma \right)(\tau) = \tilde{E}_2(\tau) - \frac{2\pi i c}{c \tau + d} \]

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, but $\tilde{E}_2(\tau) - \frac{\pi}{y}$ (where $\tau = x + iy$) is weight 2 invariant under $\text{SL}_2(\mathbb{Z})$, but not holomorphic.

Note III.9.1. $M_2(\text{SL}_2(\mathbb{Z})) = 0$.

Recall: $\dim \mathcal{E}_2(\Gamma) = \nu_\infty - 1$.

For $\nu_\infty \geq 2$, here’s the idea: Say you find $E^{(1)}$ and $E^{(2)}$ weight 2 invariant, but not holomorphic. Then we hope that $E^{(1)} - E^{(2)}$ is holomorphic.

Fix $\bar{v} \in (\mathbb{Z}/N)^2$ of order $N$. Let

\[ \tilde{E}_{2,\bar{v}}^\varphi(\tau) := \delta(\tau_\varphi) \zeta^\pm_{d_\varphi}(2) + \frac{\Omega_2}{N^2} \sum_{n=1}^{\infty} \sigma_1^n(n) q_{N}^n, \]

\[ g_2^\varphi(\tau) := \tilde{E}_{2,\bar{v}}^\varphi(\tau) - \frac{\pi}{N^2 y} \]

weight 2 invariant under $\Gamma(N)$.

Theorem III.9.2. $\mathcal{E}_2(\Gamma(N)) = \left\{ \sum_{\varphi} a_\varphi \tilde{E}_{2,\bar{v}}^\varphi : \sum_{\varphi} a_{\varphi} = 0, a_{\varphi} \in \mathbb{C} \right\}$.

...
III.9.2 Families of Eisenstein series

Hecke introduced a parameter $s \in \mathbb{C}$. Let $k \in \mathbb{Z}$, and let $\psi, \varphi$ be primitive characters mod $M_\psi, M_\varphi$, respectively. Then

$$\tilde{E}_k(\tau, s; \psi, \varphi) := \sum_{m,n \in \mathbb{Z}} \frac{\psi(m) \varphi(n)}{(m \tau + n)^k |m \tau + n|^{2s}}$$

converges absolutely and uniformly on $\text{Re}(k+2s) \geq 2+\varepsilon$ for all $\varepsilon > 0$, and so it’s holomorphic. Let

$$\Gamma_0(M_\psi, M_\varphi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : b \equiv 0 \pmod{M_\varphi}, c \equiv 0 \pmod{M_\psi} \right\}.$$ Then for all $\gamma \in \Gamma_0(M_\psi, M_\varphi)$,

$$\tilde{E}_k(\gamma \cdot z, s; \psi, \varphi) = (\psi \varphi)(d) j(cz + d)^d |cz + d|^{2s} \tilde{E}_k(z, s; \psi, \varphi).$$

So if $k \geq 3$, we can plug in $s = 0$ to obtain

$$\tilde{E}_k(z, 0; \psi \varphi)\big|_{k \gamma} = (\psi \varphi)(d) \tilde{E}_k(z, 0; \psi, \varphi)$$

for all $\gamma \in \Gamma_0(M_\psi, M_\varphi)$. In fact, for $k \geq 3$,

$$\tilde{E}_k(M_\varphi z, 0; \psi, \varphi) = \tilde{E}^{\psi \varphi}_k(z).$$

The idea is to analytically continued $\tilde{E}_k(z, s; \psi, \varphi)$ to $\text{Re}(s) \geq -\varepsilon$ for some $\varepsilon > 0$ and $k = 1, 2$, and plug in $s = 0$. 

Chapter IV
Hecke operators

IV.1 2014-03-26: Hecke operators

IV.1.1 Motivation

Let \( \Gamma = \text{SL}_2(\mathbb{Z}) \). Think of \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \) as the space of 2-dimensional lattices (up to homothety, i.e., scaling and rotating). Let

\[
\mathcal{L} := \{ \text{full rank lattices } \Lambda \subseteq \mathbb{C} \},
\]

\[
B := \left\{ \text{pairs } (\omega_1, \omega_2) \in \mathbb{C}^\times \text{ such that } \text{Im} \left( \frac{\omega_1}{\omega_2} \right) > 0 \right\}.
\]

There is a natural surjection

\[
\Phi : B \twoheadrightarrow \mathcal{L},
\]

\[
(\omega_1, \omega_2) \mapsto \Lambda(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2.
\]

Two pairs \( \omega, \omega' \in B \) give the same lattice iff there exists \( \gamma \in \text{SL}_2(\mathbb{Z}) \) such that \( \gamma \omega = \omega' \).

Thus, \( \mathcal{L} \cong \text{SL}_2(\mathbb{Z}) \backslash B \).

There is also a map

\[
\Psi : B \rightarrow \mathfrak{h},
\]

\[
(\omega_1, \omega_2) \mapsto \frac{\omega_1}{\omega_2}.
\]

This commutes with the action of \( \text{SL}_2(\mathbb{Z}) \).

Note that \( \mathbb{C}^\times \) acts on \( \mathcal{L} \) and \( B \) by scalar multiplication, and \( \Psi \) induces an \( \text{SL}_2(\mathbb{Z}) \)-equivariant isomorphism

\[
\Psi : B / \mathbb{C}^\times \xrightarrow{\cong} \mathfrak{h}.
\]

So we have an isomorphism

\[
\text{SL}_2(\mathbb{Z}) \backslash \mathcal{B} / \mathbb{C}^\times \cong \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}
\]

between the space of lattices modulo homothety and the modular curve.

Remark IV.1.1. An elliptic curve \( E / \mathbb{C} \) is a complex torus, i.e., there is a lattice \( \Lambda \subseteq \mathbb{C} \) such that \( E \cong \mathbb{C} / \Lambda \). Moreover, \( \mathbb{C} / \Lambda \cong \mathbb{C} / \Lambda' \) as elliptic curves iff \( \Lambda \sim \Lambda' \) (homothety). So we can think of \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \) as the moduli space of complex elliptic curves.
**IV.1.2 Connection to modular forms**

Since $\mathbb{C}^\times$ acts on $\mathcal{L}$ and $\mathbb{C}$, it acts on the set of maps $\text{Hom}_{\text{set}}(\mathcal{L}, \mathbb{C})$.

The characters of $\mathbb{C}^\times$ are of the form $z \mapsto z^s$. For $k \in \mathbb{Z}$, say $F$ is of weight $k$ if for all $\lambda \in \mathbb{C}^\times$ and $\Lambda \in \mathcal{L}$,

$$F(\lambda \Lambda) = \lambda^{-k} F(\Lambda).$$

Denote $F(\omega_1, \omega_2) := F(\Lambda(\omega_1, \omega_2))$. Weight $k$ means:

$$F(\lambda \omega_1, \lambda \omega_2) = \lambda^{-k} F(\omega_1, \omega_2).$$

We can dehomogenize in $F$: let

$$\tilde{F}(\omega_2) := \omega_2^k F(\omega_1, \omega_2).$$

If $F$ has weight $k$, then $\tilde{F}(\omega_1, \omega_2) = F(\omega_1, \omega_2)$. There is a function $f : \mathfrak{h} \to \mathbb{C}$ such that $\tilde{F}(\omega_1, \omega_2) = f(z)$.

For $F$ to be $\text{SL}_2(\mathbb{Z})$-invariant means that

$$(c \omega_1 + d \omega_2)^k F(a \omega_1 + b \omega_2, c \omega_1 + d \omega_2) = (c \omega_1 + d \omega_2)^k F(\omega_1, \omega_2) = \left( \frac{c \omega_1}{\omega_2} + d \right)^k \omega_2^k F(\omega_1, \omega_2),$$

i.e., $f|_k \gamma = f$.

**IV.1.3 Hecke operators**

Let $\mathcal{D}$ be the free abelian group on $\mathcal{L}$. For $n \in \mathbb{Z}_{\geq 1}$, we have $\mathbb{Z}$-linear operators

$$T(n) : \mathcal{D} \to \mathcal{D}$$

$$[\Lambda] \mapsto \sum_{\substack{\Lambda' \subseteq \Lambda \\ \text{index } n}} [\Lambda']$$

and

$$S(n) : \mathcal{D} \to \mathcal{D}$$

$$[\Lambda] \mapsto [n \Lambda].$$

Now any map $F : \mathcal{L} \to \mathbb{C}$ yields $F : \mathcal{D} \to \mathbb{C}$, and for any linear $T : \mathcal{D} \to \mathcal{D}$, we can define

$$(T \cdot F)([\Lambda]) = F(T \cdot [\Lambda]).$$

So we can define $T(n)$ on $f : \mathfrak{h} \to \mathbb{C}$ by

$$(T(n) \cdot f)(z) = \text{function on } \mathfrak{h} \text{ corresponding to } n^{k-1} T(n) \cdot F.$$ 

This is a Hecke operator.

Let’s rephrase this definition in terms of group theory. For $n \geq 1$, let

$$M_2(n) := \left\{ \alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = n \right\}.$$ 

Let $\alpha \in M_2(n)$ and $\Lambda \in \mathcal{L}$. Then $\Lambda' := \alpha \Lambda \subseteq \Lambda$ has index $n$. Conversely, if $\Lambda' \subseteq \Lambda$ has index $n$, then there exists $\alpha \in M_2(n)$ such that $\Lambda' = \alpha \Lambda$. Moreover, $\alpha \Lambda = \beta \Lambda$ iff there exists $\gamma \in \text{SL}_2(\mathbb{Z})$ such that $\beta = \gamma \alpha$. 
Lemma IV.1.2. \( M_2(n) = \bigsqcup_{\substack{a \geq 1 \\ 0 \leq b < d \\ ad = n}} \SL_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \).

So sublattices of \( \Lambda(\omega_1, \omega_2) \) of index \( n \) are exactly \( \Lambda(a\omega_1 + b\omega_2, d\omega_2) \), where \( a, b, d \) are as above.

Fact IV.1.3 (Structure theorem of finitely-generated abelian groups). Let \( \alpha \in M_2(n) \) and \( \Lambda' = \alpha\Lambda \). We can choose bases \( \omega \) and \( \omega' \) of \( \Lambda \) and \( \Lambda' \) such that \( \omega_1' = a\omega_1 \) and \( \omega_2' = d\omega_2 \), where \( a \geq 1 \), \( a \mid d \), and \( ad = n \). That is,

\[ M_2(n) = \bigsqcup_{\substack{a \geq 1 \\ a|d \\ ad = n}} \SL_2(\mathbb{Z}) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \SL_2(\mathbb{Z}). \]

Remark IV.1.4. Next time, we’ll discuss “abstract Hecke algebras”, first introduced by Shimura. The idea is, given \( \Gamma \leq \SL_2(\mathbb{Z}) \), and \( T(n) \) acting on \( M_k(\Gamma) \), we need to think of \( T(n) \) as a formal sum over double cosets.

IV.2 2014-03-28: Abstract Hecke algebras

Let \( G \) be a group (e.g., \( \GL_2^+(\mathbb{R}) \)) and \( \Gamma, \Gamma' \leq G \) subgroup (e.g., congruence subgroups).

Definition IV.2.1. We say that \( \Gamma \) and \( \Gamma' \) are commensurable, denoted \( \Gamma \approx \Gamma' \), if \( [\Gamma : \Gamma \cap \Gamma'] \) and \( [\Gamma' : \Gamma \cap \Gamma'] \) are both finite. The commensurator of \( \Gamma \) is \( \tilde{\Gamma} := \{ g \in G \mid g\Gamma g^{-1} \approx \Gamma \} \leq G \).

(If \( \Gamma \leq \GL_2^+(\mathbb{R}) \) is a congruence subgroup, then \( \tilde{\Gamma} = \mathbb{R}^\times \cdot \GL_2^+(\mathbb{Q}) \).)

Lemma IV.2.2. Suppose \( \Gamma \approx \Gamma' \). Then:

(i) \( \tilde{\Gamma} = \tilde{\Gamma}' \).

(ii) For all \( \alpha \in \tilde{\Gamma} \),

\[ \Gamma \alpha \Gamma' = \bigsqcup_{i=1}^{d} \Gamma \alpha \gamma_i = \bigsqcup_{j=1}^{e} \delta_j \alpha \Gamma', \]

where the \( \gamma_i \) and \( \delta_j \) are coset representatives of \((\Gamma' \cap \alpha^{-1} \Gamma \alpha) \backslash \Gamma' \) and \( \Gamma / (\Gamma \cap \alpha \Gamma' \alpha^{-1}) \), respectively.

Definition IV.2.3 (Abstract Hecke algebras). Let \( \Gamma \approx \Gamma' \), and let \( \Delta \subseteq \tilde{\Gamma} \) be a submonoid such that \( \Delta \supseteq \Gamma, \Gamma' \). Let \( R \) be a commutative ring. Then we define \( \mathcal{H}_R(\Gamma, \Gamma'; \Delta) \) to be the free \( R \)-module generated by double cosets \( \Gamma \alpha \Gamma' \) for \( \alpha \in \Delta \). (We also write \( \mathcal{H}_R(\Gamma; \Delta) \) when \( \Gamma = \Gamma' \), and we drop the subscript when \( R = \mathbb{Z} \).)
Let $\Gamma \approx \Gamma' \approx \Gamma''$, and let $\Delta \supseteq \Gamma''$. Define a multiplication map

$$\mathcal{H}_R(\Gamma, \Gamma'; \Delta) \times \mathcal{H}_R(\Gamma', \Gamma''; \Delta) \to \mathcal{H}_R(\Gamma, \Gamma''; \Delta)$$

as follows: for $\alpha, \beta \in \Delta$, write

$$\Gamma \alpha \Gamma' = \bigsqcup_i \Gamma \alpha_i,$$

$$\Gamma' \beta \Gamma'' = \bigsqcup_j \Gamma' \beta_j.$$

Then

$$\Gamma \alpha \Gamma' \beta \Gamma'' = \bigsqcup_j \Gamma \alpha \Gamma' \beta_j = \bigsqcup_i \Gamma \alpha_i \beta_j = \bigsqcup_k \Gamma \gamma_k \Gamma'',$$

so we define

$$(\Gamma \alpha \Gamma') \cdot (\Gamma' \beta \Gamma'') \overset{\text{def}}{=} \sum_{\Gamma \gamma \Gamma'' \subseteq \Gamma \alpha \Gamma' \beta \Gamma''} c_\gamma \Gamma \gamma \Gamma''$$

where $c_\gamma := \# \{(i, j) : \Gamma \alpha_i \beta_j = \Gamma \gamma\}$. This is well-defined and associative.

**Remark IV.2.5.** The multiplication defined above makes $\mathcal{H}_R(\Gamma; \Delta)$ into a unital associative algebra.

**Theorem IV.2.6.** Suppose there is an involution $c : \Delta \to \Delta$ such that:

- $(\alpha \beta)^c = \beta^c \alpha^c$
- $(a^c)^c = a$
- $\Gamma^c = \Gamma$
- $\Gamma \alpha^c \Gamma = \Gamma \alpha \Gamma$ for all $\alpha \in \Delta$

Then for all $\alpha \in \Delta$, $\Gamma \setminus \Gamma \alpha \Gamma$ and $\Gamma \alpha \Gamma / \Gamma$ have a common set of representatives, and $\mathcal{H}_R(\Gamma; \Delta)$ is commutative.

Let $M$ be an $R$-linear right $\Delta$-module. Then $M^\Gamma$ has a natural $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ “action”. Write $\Gamma \alpha \Gamma' = \bigsqcup_{i=1}^d \Gamma \alpha_i$, and define

$$m |\Gamma \alpha \Gamma' \overset{\text{def}}{=} \sum_{i=1}^d m |\alpha_i \in M^\Gamma.$$
IV.2.1 Concrete example

Now let $\Gamma = \Gamma_0(N)$, and consider

$$\Delta_0(N) := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{N}, \gcd(a, N) = 1, \det A > 0 \right\},$$

$$\Delta^*_0(N) := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{N}, \gcd(d, N) = 1, \det A > 0 \right\}.$$ 

Let

$$\mathcal{H}(N) := \mathcal{H}(\Gamma_0(N), \Delta_0(N)),$$

$$\mathcal{H}^*(N) := \mathcal{H}(\Gamma_0(N), \Delta^*_0(N)).$$

Fact IV.2.7. If $\alpha \in \Delta_0(N)$ (resp. $\Delta^*_0(N)$), then there are unique $a, d \in \mathbb{Z}_{\geq 1}$ such that $a \mid d$, $\gcd(a, N) = 1$, and $\Gamma \alpha \Gamma = \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ (resp. $\Gamma \alpha \Gamma = \Gamma \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$).

Theorem IV.2.8. $\mathcal{H}(N)$ and $\mathcal{H}^*(N)$ are commutative.

Proof. Apply Theorem [IV.2.6] using the following involution:

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix}^c = \begin{pmatrix} a & c \\ bN & d \end{pmatrix}.$$

For $\gcd(a, N) = 1$, let

$$\Gamma(a, d) := \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \Gamma \in \mathcal{H}(N),$$

$$\Gamma^*(a, d) := \Gamma \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}, \Gamma \in \mathcal{H}^*(N).$$

For $\chi$ a Dirichlet character mod $N$, and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(N)$, extend $\chi$ to a map $\Gamma_0(N) \to \mathbb{C}^\times$ by $\chi(\alpha) := \overline{\chi}(a)$. If $\alpha \in \Delta^*_0(N)$, extend to $\chi^*(\alpha) := \chi(d)$. If $\alpha \in \Gamma_0(N)$, then $\chi(\alpha) = \chi^*(\alpha)$.

For $G$ any of $A, M, S$, and $\lambda$ either nothing or $\ast$, we have the Hecke action

$$\mathcal{H}^\lambda(N) \subset G_k(N, \chi),$$

defined by

$$f|_K \Gamma \alpha \Gamma := \det(\alpha)^{\frac{k-1}{2}} \sum_i \overline{\chi^\lambda(\alpha_i)} f|_{k \alpha_i},$$

where $\Gamma \alpha \Gamma = \bigcup_i \Gamma \alpha_i$. 
IV.3 2014-03-31: Hecke actions

Continuing with the notation from last time: For \( n \geq 1 \), let

\[
T^2(n) := \sum_{\Gamma \alpha \Gamma \in \mathcal{H}(n) \atop \det(\alpha) = n} \alpha \Gamma \alpha^{-1}.
\]

Then:

\[
T(n) = \sum_{\substack{a \mid n \atop aN = 1}} T(a, d),
\]

\[
T^*(n) = \sum_{\substack{d \mid n \atop aN = 1}} T^*(d, a).
\]

So, \( T(p) = T(1, p) \) and \( T^*(p) = T^*(p, 1) \) for \( p \) prime.

**Theorem IV.3.1.**

- If \( (m, n) = 1 \), then \( T(m)T(n) = T(mn) \).
- If \( p \mid N \), then \( T(p^e) = T(p)T(p^{e-1}) \).
- Otherwise, \( T(p^e) = T(p)T(p^{e-1}) - pT(p, p)T(p^{e-2}) \).
- Generally, \( T(m)T(n) = \cdots \).

IV.3.1 Petersson inner product

This is an inner product \( \langle \cdot, \cdot \rangle \) on \( S_k(\Gamma) \).

Recall that \( \mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}) \). Since \( \mathbb{P}^1(\mathbb{Q}) \) is countable, it has measure zero. We have

\[
\mathfrak{h} \cong \text{GL}_2^+ (\mathbb{R}) / \mathbb{R}^*_+ O_2(\mathbb{R}),
\]

which is a locally compact group, and thus has a Haar measure \( \mu_G \) such that \( \mu_G(\alpha X) = \mu_G(X) \) for all \( \alpha \in \text{GL}_2^+ (\mathbb{R}) \) and all Borel \( X \subseteq \text{GL}_2^+ (\mathbb{R}) \). This induces a \( \text{GL}_2^+ (\mathbb{R}) \)-invariant measure on \( \mathfrak{h} \). If \( \tau = x + iy \), then \( d\mu(\tau) = \frac{dx dy}{y^2} \).

Recall that \( \mathcal{F}^* = \{ |\text{Re}(\tau)| \leq \frac{1}{2}, \ |\tau| \geq 1 \} \cup \{ \infty \} \).

**Lemma IV.3.2.** If \( \varphi : \mathfrak{h} \to \mathbb{C} \) is continuous and bounded, then for all \( \alpha \in \text{SL}_2(\mathbb{Z}) \),

\[
\int_{\mathcal{F}^*} \varphi(\alpha \tau) d\mu(\tau)
\]

converges.

**Example IV.3.3.** For \( \varphi = 1 \), we have

\[
\int_{\mathcal{F}^*} d\mu(\tau) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^\infty \frac{dy}{y^2} dx = 2 \int_{0}^{1/2} \frac{1}{\sqrt{1-x^2}} dx = 2 \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{3} = \zeta(2) \cdot \frac{2}{\pi}.
\]

Thus, \( \text{Vol}(X(\text{SL}_2(\mathbb{Z}))) = \frac{\pi}{3} \).
IV.4 2014-04-09: Newforms

(1) We’ve seen that $S_k(N, \chi)$ has basis of eigenvectors for $T_{\text{cusp}}(N, \chi)$. Why not all of $T_{\text{cusp}}(N, \chi)$?

(2) Let $f$ be a newform eigenvector for all Hecke operators. Then $L(s, f)$ has an Euler product.

(3) Theory of conductors for modular forms.

Hecke showed that, if $\chi = 1$ and $N$ is prime and $k < 12$ or $k = 14$, or if $\chi$ is primitive, then $S_k(N, \chi)$ does have a basis for eigenvectors for the full Hecke algebra.

However: suppose, for example, that $f \in S_k(N, \chi)$, $p \nmid N$, $f|_k T(p) = \lambda_p f$, and $f \in S_k(Np^r, \chi)$ for all $r \geq 0$. Moreover, write $f_j(z) := f(p^j z) \in S_k(Np^r, \chi)$ for $0 \leq j \leq r$ and $\delta_j = \begin{pmatrix} p^j & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(f|_k \delta_j)(z) = p^{jk/2} f(p^j z),$$

and $\delta_j \Gamma_0(N) \delta_j^{-1} \subseteq \Gamma_0(N)$.

Consider the actions $T(p) \subset S_k(N, \chi)$ and $U(p) \subset S_k(Np^r, \chi)$ for $r \geq 1$. Then

$$T(p) = U(p) \sqcup \Gamma_0(Np^r) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

$$f_0|_k U(p) = f|_k T(p) - p^{k-1} f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \lambda_p f - p^{k-1} \chi(p) f.$$

If $j \geq 1$, then

$$f_j|_k U(p) = p^{k-1} \sum_{m=0}^{p-1} \chi \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix} f_j|_k \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}$$

$$= p^{-1} \sum_{m=0}^{p-1} f_{j-1}|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}$$

$$= p^{-1} \sum_{m=0}^{p-1} f_{j-1}|_k \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = f_{j-1}.$$ 

So the action $U(p) \subset V := \langle f_0, f_1, \ldots, f_r \rangle$ is given by the matrix

$$\begin{pmatrix} \lambda_p & 1 & 0 & 0 & \ldots \\ -p^{k-1} \chi(p) & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that $W := \langle f_0, f_1 \rangle \subseteq V$ is $U(p)$-stable, with characteristic polynomial $x^2 - \lambda_p x + p^{k-1} \chi(p)$. 
Lemma IV.4.1. If $T : V \to V$ is a diagonalizable linear operator and $W \leq V$ is $T$-stable, then $\bar{T} : V/W \to V/W$ is diagonalizable.

Hence, if $r \geq 3$, then $U(p)$ is not diagonalizable on $\langle f_0, \ldots, f_r \rangle$.

We want to develop a theory of “primitive” modular forms and “conductors”. For $t \in \mathbb{Z}_{\geq 1}$, $\delta_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ and $f(tz) = t^{-k/2}f|_{k}\delta_t$, so if $f \in M_k(N, \chi)$, then $f(tz) \in M_k(N_t, \chi)$.

Let $M, N$ be positive integers with $M \mid N$, so $S_k(M, \chi) \subseteq S_k(N, \chi)$. For all $t \mid \frac{N}{M}$, we have a “degeneracy map”

$$\iota_{M,N,t} : S_k(M, \chi) \to S_k(N, \chi),$$

$$f \mapsto f|_{k}\delta_t \in S_k(Mt, \chi).$$

Fact IV.4.2. Let $T = T(n)$ or $T(n, n)$, where $(n, T) = 1$. (For $T(n, n)$, also require $(n, N) = 1$.) Then the following diagram commutes:

$$\begin{CD}
S_k(M, \chi) @> T >> S_k(M, \chi) \\
\downarrow \iota_{M,N,t} @. \downarrow \iota_{M,N,t} \\
S_k(N, \chi) @> T >> S_k(N, \chi)
\end{CD}$$

Define the old subspace by

$$S_k^{old}(N, \chi) \overset{\text{def}}{=} \bigcup_{M|N, M \neq N} \bigcup_{t \mid \frac{N}{M}} \iota_{M,N,t}(S_k(M, \chi)) \subseteq S_k(N, \chi).$$

Proposition IV.4.3. $S_k^{old}(N, \chi)$ is Hecke stable.

Proof. The fact for $p \mid N$ is similar to computations of $U(p) \subseteq f$.

The new subspace is

$$S_k^{new}(N, \chi) \overset{\text{def}}{=} S_k^{old}(N, \chi) \perp \{ f \in S_k(N, \chi) \mid \langle f, g \rangle = 0 \forall g \in S_k^{old}(N, \chi) \}.$$  

A form $f \in S_k(N, \chi)$ is called new or primitive if $f \in S_k^{new}(N, \chi)$.

Proposition IV.4.4. $S_k^{new}(N, \chi)$ is Hecke stable.

Proof. Observe that $\langle f|_{k}T(n), g \rangle = \langle f, g|_{k}T^{*}(n) \rangle$, where $T^{*}(n) = \overline{\chi(n)}T(n)$.

Corollary IV.4.5. Both $S_k^{new}$ and $S_k^{old}$ have a basis of eigenvectors for the anemic Hecke algebra.

Our goal is to show $S_k^{new}(N, \chi)$ has a basis of eigenvectors for the full Hecke algebra.
IV.4.1 Atkin–Lehner–Li theory

Note IV.4.6. If \( f = \sum_{n \geq 1} a_n q^n \in S_k(M, \chi) \) and \( d \mid \frac{N}{M} \), then
\[
\iota_{M,N,d}(f) = d^{\frac{k}{2}} \sum_{n \geq 1} a_n q^{dn},
\]
so \( \iota_{M,N,d}(f) \) has \( a_n = 0 \) whenever \( (n, d) = 1 \).

Lemma IV.4.7 (Main lemma of Atkin–Lehner–Li theory). If \( f \in S_k(\Gamma, (N)) \) and \( a_n(f) = 0 \) for all \( (n, N) = 1 \), then for all \( p \mid N \), there is a modular form \( f(p) \in S_k(\Gamma, \frac{N}{p}) \) such that
\[
f = \sum_{p\mid N} \iota_{N/p,N,p}(f(p)).
\]

Proof. See §5.7 of Diamond–Shurman or §4.6 of Miyake. \( \Box \)

IV.5 2014-04-11: More about newforms

If \( f \in S_k(\Gamma_1(N)) \) with \( a_n(f) = 0 \) for all \( (n, N) = 1 \), then \( f \) is old.

Definition IV.5.1. If \( f \not\equiv 0 \) in \( M_k(\Gamma_1(N)) \) and \( f \) is an eigenvector for all \( T \in \mathcal{H}(N) \), then \( f \) is called an eigenform. An eigenform \( f = \sum_{n \geq 0} a_n q^n \) is normalized if \( a_1(f) = 1 \). A newform is a normalized eigenform in \( S_k^{\text{new}}(\Gamma_1(N)) \).

Let \( f \in S_k(N, \chi) \) be an eigenvector for \( \mathcal{H}^{(N)}(N) \). Then for all \( n \) with \( (n, N) = 1 \), there are \( \lambda_n, d_n \in \mathbb{C} \) such that \( f|_k T(n) = \lambda_n f \) and \( f|_k T(n, n) = n^{k-1}d_n f \). One can check that \( \chi : n \mapsto d_n \) is a Dirichlet character of level \( N \), so \( f \in S_k(N, \chi) \).

For all \( n \) with \( (n, N) = 1 \), we know \( a_n(f) = \lambda_n a_1(f) \). So if \( a_1(f) = 0 \), then \( f \) is old by the main lemma. Hence, if \( f \in S_k^{\text{new}}(N, \chi) \), then \( a_1(f) \neq 0 \), so we can normalize to \( a_1(f) = 1 \) without loss of generality. Then \( a_n(f) = \lambda_n \).

For \( m \geq 1 \), let \( g_m = f|_k T(m) - a_m(f) f \in S_k^{\text{new}}(N, \chi) \). Then \( g_m \) is also an eigenvector for \( T(n) \) and \( T(n, n) \) whenever \( (n, N) = 1 \). We compute \( a_1(g_m) \):
\[
a_1(g_m) = a_1(f|_k T(m)) - a_1(a_m(f) f) = a_m(f) - a_m(f) = 0.
\]
So \( g_m \) is both old and new, hence \( g_m = 0 \). Thus, \( f|_k T(m) = \lambda_m f \) (where \( \lambda_m = a_m(f) \)) for all \( m \geq 1 \).

Theorem IV.5.2. If \( f \in S_k^{\text{new}}(\Gamma_1(N)) \) with \( f \not\equiv 0 \) is an eigenvector for \( T(n) \) and \( T(n, n) \) whenever \( (n, N) = 1 \), then:

(i) \( f \) is an eigenform and a multiple of a newform.

(ii) (Multiplicity one): If \( \tilde{f} \in S_k^{\text{new}}(\Gamma_1(N)) \) with the same Hecke eigenvalues for the full Hecke algebra, then \( \tilde{f} = cf \) for some \( c \in \mathbb{C}^\times \).

(iii) The set of newforms of level \( N \) and character \( \chi \) are an orthogonal basis of \( S_k(N, \chi) \).

We can prove a stronger form of the above “multiplicity one” result:

**Theorem IV.5.3.** Suppose $f, g \in S^\text{new}_k(\Gamma_1(N))$ and $D \in \mathbb{Z}_{\geq 1}$ such that $f, g$ are eigenvectors for $T_{cusp}^{(ND)}(N, \chi)$ with $\lambda_p(f) = \lambda_p(g)$ for all prime $p \nmid ND$. Then $f = cg$ for some $c \in \mathbb{C}^\times$.

**Theorem IV.5.4.** The set $\{f(tz) : f \in S^\text{new}_k(M, \chi), tM \mid N\}$ is a basis of $S_k(N, \chi)$.

**Theorem IV.5.5.** If $g \in S_k(\Gamma_1(N))$ is a normalized eigenform, then there exists $M \mid N$ and $f \in S^\text{new}_k(\Gamma_1(M))$ such that $a_p(f) = a_p(g)$ for all $p \nmid N$. We define $\text{cond}(g) := M$.

### IV.5.1 Euler products

Next time, we’ll talk about $L$-functions properly. Today, we’ll just briefly remark on Euler products.

Given $f = \sum_{n \geq 0} a_n q^n$, define

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}.$$ 

When is there an Euler product?

**Theorem IV.5.6.** Let $R$ be a commutative ring and $t(n), d(n) \in R$ such that $t(1) = d(1) = 1$ and $d(mn) = d(m)d(n)$ for all $m, n \in \mathbb{Z}_{\geq 1}$. Then the following are equivalent:

1. $t(mn) = t(m)t(n)$ whenever $(m, n) = 1$, and $t(p^e)t(p^f) = t(p^{e+1})pd(p)t(p^{e-1})$ for all primes $p$ and $e \geq 1$.
2. $\sum_{n \geq 1} \frac{t(n)}{n^s} = \prod_p \left(1 - t(p)p^{-s} + pd(p)p^{-2s}\right)^{-1}$.

### IV.6 2014-04-14: L-functions

Let $f = \sum_{n \geq 0} a_n q^n \in M_k(N, \chi)$. Define

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}.$$ 

The Dirichlet series converges for $\text{Re}(s) > C_f$. Indeed, if $|a_n| = O(n^\nu)$, then

$$\left|\frac{a_n}{n^s}\right| \leq Cn^{\nu-s},$$ 

so it converges absolutely and uniformly on compact subsets of $\text{Re}(s) > \nu + 1$.

**Lemma IV.6.1.** If $f$ is a cusp form of weight $k$, then $|a_n| = O(n^{k/2})$. Otherwise, $|a_n| = O(n^{k-1+\varepsilon})$ for all $\varepsilon > 0$.

In fact, Deligne proved the following theorem, conjectured by Ramanujan and Petersson:
Theorem IV.6.2. If $f$ is a newform of level $N$, then for all $n$ with $(n, N) = 1$,

$$|a_n| = O(n^{\frac{k-1}{2}}).$$

The idea is to consider $x^2 - a_p x - p^{k-1} \chi(p) = (x - \alpha_p)(x - \beta_p)$. The modular form is attached to a 2-dimensional Galois representation coming from the $\ell$-adic cohomology of a variety; $a_p$ and $p^{k-1} \chi(p)$ are the trace and determinant, respectively, of this Galois representation. In particular, $L(s, f)$ is a factor in the zeta function of some proper smooth variety, occurring in $H^{k-1}$, so the Weil conjectures say that $|\alpha_p| = |\beta_p| = p^{\frac{k-1}{2}}$.

There is a converse:

Proposition IV.6.3. If $f : \mathfrak{h} \rightarrow \mathbb{C}$ is a holomorphic function such that $f|_{k}\gamma = f$ for all $\gamma \in \Gamma$, then the following are equivalent:

(i) $f|_{k}\alpha$ is holomorphic at $\infty$ for all $\alpha \in \text{SL}_2(\mathbb{Z})$ (i.e., $f \in M_k(\Gamma)$).

(ii) There exists $\nu$ such that $|a_n| = O(n^\nu)$.

IV.6.1 Euler products

Does $L(s, f)$ have an Euler product?

Recall from last time:

Theorem IV.6.4. Let $R$ be a commutative ring and $t(n), d(n) \in R$ such that $t(1) = d(1) = 1$ and $d(mn) = d(m)d(n)$ for all $m, n \in \mathbb{Z}_{\geq 1}$. Then the following are equivalent:

(i) $t(mn) = t(m)t(n)$ whenever $(m, n) = 1$, and $t(p)t(p^e) = t(p^{e+1})pd(p)t(p^{e-1})$ for all primes $p$ and $e \geq 1$.

(ii) $\sum_{n \geq 1} \frac{t(n)}{n^s} = \prod_p \left( \frac{1 - t(p)p^{-s} + pd(p)p^{-2s}}{1 - t(p)p^{-s}} \right)^{-1}$.

Proof. Suppose (i). Then

$$\sum_{n \geq 1} \frac{t(n)}{n^s} = \prod_p \left( \sum_{e \geq 0} \frac{t(p^e)}{p^{es}} \right)^{-1}.$$

We know $t(p^{e+1}) - t(p)t(p^e) + pd(p)t(p^{e-1}) = 0$. So

$$(1 - t(p)p^{-s} + pd(p)p^{-2s}) \sum_{e \geq 0} \frac{t(p^e)}{p^{es}} = t(1) + \frac{t(p)}{p^s} + \frac{t(p^2)}{p^{2s}} + \frac{t(p^3)}{p^{3s}} + \ldots$$

$$- \left( \frac{t(p)}{p^s} + \frac{t(p^2)}{p^{2s}} + \frac{t(p)t(p^2)}{p^{3s}} + \ldots \right)$$

$$+ \frac{pd(p)}{p^{2s}} + \frac{pd(p)t(p)}{p^{3s}} + \ldots,$$
and adding diagonally the terms with the same \( p^e \) term in the denominator, everything but the initial \( t(1) = 1 \) cancels, yielding

\[
\sum_{n \geq 1} \frac{t(n)}{n^s} = \prod_p \left( \sum_{e \geq 0} \frac{t(p^e)}{p^{es}} \right) = \prod_p \left( 1 - t(p)p^{-s} + pd(p)p^{-2s} \right)^{-1}.
\]

For the converse, see Diamond–Shurman.

For our application, let \( R = \mathcal{H}(N) \), and let

\[
t(n) = T(n),
\]

\[
d(n) = \begin{cases} 
T(n, n) & \text{if } (n, N) = 1, \\
0 & \text{if } (n, N) > 1.
\end{cases}
\]

If \( f \in M_k(N, \chi) \) and \( \mathcal{P} \) is a set of primes such that \( f|_kT(p) = \lambda(p)f \) for all \( p \in \mathcal{P} \), then

\[
L(s, f) = \left( \prod_{p \in \mathcal{P}} (1 - \lambda(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1} \right) \cdot \sum_{(n, \mathcal{P}) = 1} \frac{a_n}{n^s}.
\]

So if \( f \) is a newform, then

\[
L(s, f) = \prod_p \left( 1 - a_p p^{-s} + \chi(p)p^{k-1-2s} \right)^{-1}.
\]

### IV.6.2 Analytic continuation and functional equation

Our next goal is to show \( L(s, f) \) has analytic continuation, a functional equation, and other analytic properties.

The basic tool is the Mellin transform: Given \( f = \sum_{n \geq 0} a_n q^n \in M_k(N, \chi) \), let

\[
\Lambda(s, f) := \mathcal{M}(f(it))(s) \overset{\text{def}}{=} \int_0^\infty (f(it) - a_0)t^s \frac{dt}{t}.
\]

Recall that \( a_n = O(n^\nu) \), so if \( \text{Re}(s) > \nu + 1 \), then

\[
\int_0^\infty (f(it) - a_0)t^s \frac{dt}{t} = \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n t}t^s \frac{dt}{t} = \sum_{n=1}^\infty \int_0^\infty a_n e^{-t(2\pi n)^s}t^{s-1} \frac{dt}{t} = \frac{1}{(2\pi)^s} \int_0^\infty e^{-t(2\pi)^s}t^{s-1} \frac{dt}{t} = \frac{1}{(2\pi)^s} \Gamma(s)L(s, f).
\]
Remark IV.6.5. There is an inverse Mellin transform: if $L(s) = \mathcal{M}(F)(s) = \int_0^\infty (F(t) - F(\infty)) t^s dt/t$, then (under analytic assumptions),
\[
F(s) - F(\infty) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s) t^{-s} ds \overset{\text{def}}{=} \mathcal{M}^{-1}(L)(s).
\]
So if we have $L(s)$, we can ask when
\[
\mathcal{M}^{-1}\left(\frac{(2\pi)^s}{\Gamma(s)} L(s)\right)
\]
is a modular form.

\section*{IV.7 2014-04-16}

The contents of today’s lecture is given in more detail in Miyake, §4.3 and §4.7.

Let $\{a_n\}_{n=0}^\infty$ be a sequence in $\mathbb{C}$, and let
\[
f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}
\]
for $z \in \mathbf{h}$.

\textbf{Fact IV.7.1 (†).} If $a_n = O(n^{\nu})$ for some $\nu > 0$, then the series defining $f$ converges absolutely and uniformly on compact subsets (of $\mathbf{h}$), so $f(z)$ is holomorphic on $\mathbf{h}$. Moreover, writing $z = x + iy$, we have $|f(z)| = O(y^{-\nu-1})$ as $y \to 0$ and $|f(z) - a_0| = O(e^{-2\pi y})$ as $y \to \infty$, both uniformly in $x$.

Conversely, let $f(z)$ be holomorphic on $\mathbf{h}$ such that there exist $a_n \in \mathbb{C}$ with $f(z) = \sum_{n \geq 0} a_n q^n$ converging absolutely and uniformly on compact subsets, and such that there exists $\nu > 0$ with $|f(z)| = O(y^{-\nu-1})$ as $y \to 0$. Then $|a_n| = O(n^{\nu})$ (and so $|f(z) - a_0| = O(e^{-2\pi y})$ as $y \to \infty$).

Suppose $f$ satisfies the above conditions (†). Then
\[
L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}
\]
converges absolutely and uniformly on compact subsets for $\text{Re}(s) > \nu + 1$.

Let $N \in \mathbb{Z}_{\geq 1}$, and define
\[
\Lambda_N(s, f) := \frac{N^{s/2}}{(2\pi)^s} \Gamma(s) L(s, f) = \int_0^\infty \left( f\left(\frac{it}{\sqrt{N}}\right) - a_0\right) t^s dt/t.
\]

We now define the \emph{Atkin–Lehner operator}: let $W_n = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Then $W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N)$. 


Fact IV.7.2. The map \( f \mapsto f|_{kW_N} \) gives an isomorphism
\[
G_k(N, \chi) \xrightarrow{\simeq} G_k(N, \bar{\chi}),
\]
where \( G \) is any of \( M, S, \) or \( E \).

Theorem IV.7.3 (Hecke). Let \( k, N \in \mathbb{Z}_{\geq 1} \), and let \( f = \sum_{n \geq 0} a_n q^n \) and \( g = \sum_{n \geq 0} b_n q^n \) be functions satisfying (†). The following are equivalent:

1. \( g(z) = (f|_{kW_N})(z) \).
2. Both \( \tilde{\Lambda}_N(s, f) \) and \( \tilde{\Lambda}_N(s, g) \) can be analytically continued to all \( s \in \mathbb{C} \),

\[
\tilde{\Lambda}_N(s, f) = i^k \tilde{\Lambda}_N(k - s, g),
\]

and the function
\[
\tilde{\Lambda}_N(s, f) + \frac{a_0}{s} + \frac{i^k b_0}{k - s}
\]

is entire and bounded in vertical strips.

Remark IV.7.4. The proof (which we won’t present here) uses the Phragmén–Lindelöf principle: Let \( F(s) \) be holomorphic on the vertical strip \( a \leq \text{Re}(s) \leq b \), and suppose \( |F(s)| = O(e^{\sigma|s|^\delta}) \) for some \( \delta \geq 0 \). If there are constants \( M_a \) and \( M_b \) such that, for all \( t \in \mathbb{R} \),
\[
|F(a + it)| \leq M_a \cdot (1 + |t|)^\alpha,
\]
\[
|F(b + it)| \leq M_b \cdot (1 + |t|)^\beta,
\]
then for all \( a \leq \sigma \leq b \),
\[
|F(\sigma + it)| \leq M_a^{\ell(\sigma)} M_b^{1-\ell(\sigma)} (1 + |t|)^{\alpha(\sigma)+\beta(1-\ell(\sigma))},
\]
where \( \ell(\sigma) = 1 - \frac{\sigma-a}{b-a} \).

Corollary IV.7.5. If \( f \in S_k(N, \chi) \), then \( \tilde{\Lambda}_N(s, f) \) is entire and satisfies
\[
\tilde{\Lambda}_N(s, f) = i^k \tilde{\Lambda}_N(k - s, f|_{kW_N}).
\]

Corollary IV.7.6 (\( N = 1 \)). Recall that \( \text{SL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = W_1 \). Let \( k \in \mathbb{Z}_{\geq 2} \) be even, and suppose \( f \) satisfies (†). Then \( f \in M_k(\text{SL}_2(\mathbb{Z})) \) if and only if:

- \( \Lambda(s, f) \) can be analytically continued to all \( s \in \mathbb{C} \).
- \( \Lambda(s, f) = (-1)^{k/2} \Lambda(k - s, f) \).
- \( \Lambda(s, f) + \frac{a_0}{s} + \frac{(-1)^{k/2} a_0}{k - s} \) is entire and bounded in vertical strips.

Furthermore, if \( a_0 = 0 \), then \( f \in S_k(\text{SL}_2(\mathbb{Z})) \).
Proof. By definition, \( f|_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = f \). Then \( f \in M_k(\text{SL}_2(\mathbb{Z})) \) iff \( f|_kW_1 = f \). Apply Hecke’s theorem with \( N = 1 \) and \( f = g \).

To generalize to \( M_k(N, \chi) \), let \( f \) satisfy (†), and let \( \psi \) be a Dirichlet character. Let \( f_\psi = \sum_{n \geq 0} \psi(n)a_nq^n \) (which also satisfies (†)) and

\[
L(s, f, \psi) := L(s, f_\psi) = \sum_{n \geq 1} \frac{\psi(n)a_n}{n^s}.
\]

Let \( m_\psi \) be the conductor of \( \psi \), and let

\[
\tilde{\Lambda}_N(s, f, \psi) := \tilde{\Lambda}_{Nm_\psi}(s, f_\psi) = \frac{(m_\psi \sqrt{N})^s}{(2\pi)^s} \Gamma(s)L(s, f, \psi).
\]

**Lemma IV.7.7.** If \( f, g \) satisfy (†) and \( \psi \) is primitive of conductor \( m > 1 \), then the following are equivalent:

(A) \( \psi \) There is a constant \( C_\psi \) such that \( f_\psi|_kW_{Nm_\psi^2} = C_\psi g_\psi \).

(B) \( \tilde{\Lambda}_N(s, f, \psi) \) has analytic continuation and is bounded in vertical strips, and

\[
\tilde{\Lambda}_N(s, f, \psi) = i^k C_\psi \tilde{\Lambda}_N(k - s, g, \overline{\psi}).
\]

**Lemma IV.7.8.** Let \( f \in M_k(N, \chi) \), \( \psi \) primitive of conductor \( m \), and \( M = \text{lcm}(N, m^2, mm_\chi) \). Then \( f_\psi \in M_k(M, \chi\psi^2) \).

Let \( \mathcal{P} \) be any set of odd primes and 4 such that, for all \( p \in \mathcal{P} \), we have \( (p, N) = 1 \) and

\[
\mathcal{P} \cap \{a + nb : n \in \mathbb{Z}\} \neq \emptyset
\]

for all \( a, b \geq 1 \) with \( (a, b) = 1 \). (For example, we could take \( \mathcal{P} \) to be all odd primes \( p \nmid N \).)

**Theorem IV.7.9** (Weil’s converse theorem). Let \( k, N \in \mathbb{Z}_{\geq 1}, \chi \) a Dirichlet character mod \( N \) such that \( \chi(-1) = (-1)^k \), and \( a_n, b_n \in \mathbb{C} \) both \( O(n^\nu) \) (for some \( \nu > 0 \)). Let \( f = \sum_{n \geq 0} a_nq^n \) and \( g = \sum_{n \geq 0} b_nq^n \). Then \( f \in M_k(N, \chi) \), \( g \in M_k(N, \overline{\chi}) \), and \( f|_kW_N = g \) if and only if:

1. \( \tilde{\Lambda}_N(s, f) \) and \( \tilde{\Lambda}_N(s, g) \) satisfy (B).

2. For all primitive Dirichlet characters \( \psi \) of conductor \( m \in \mathcal{P}, \tilde{\Lambda}_N(s, f, \psi) \) and \( \tilde{\Lambda}_N(s, g, \psi) \) satisfy (B)\( \psi \) with

\[
C_\psi = \chi(m)\psi(-N) \frac{\tau(\psi)}{\tau(\psi)},
\]

where \( \tau \) denotes the Gauss sum.

Furthermore, if \( L(s, f) \) is absolutely convergent at \( s = k - \delta \) for any \( \delta > 0 \), then \( f, g \) are cusp forms.
Theorem IV.8.1. Let \( f \in M_k(N, \chi) \) be a newform. Then there is a number field \( K_f/\mathbb{Q} \) such that \( a_n(f) \in \mathcal{O}_{K_f} \) for all \( n \geq 0 \). In fact, we can take \( K_f = \mathbb{Q}(a_n : n \geq 1) \), and \( \mathbb{Q}(\chi) \subseteq K_f \) and \( K_f \) is totally real or CM.

Let \( \mathbb{T} := \mathbb{T}(N, \chi) = \text{im}(\mathcal{H}(N) \hookrightarrow \text{End}_\mathbb{C}(M_k(N, \chi))) \). Note that \( M_k(N, \chi, \mathbb{Z}[\chi]) \) is Hecke stable (since \( a_n(f|_kT(m)) = \sum_{0 < d | (m, n)} \chi(d) a_{mn/d^2}(f) \)), and it contains a basis of \( M_k(N, \chi) \). Hence, \( \mathbb{T} \subseteq \text{End}_{\mathbb{Z}[\chi]}(M_k(N, \chi, \mathbb{Z}[\chi])) \), recalling that \( \mathcal{H}(N) = \mathbb{Z}[T(n), T(n, n)] \). We have proved the following:

Proposition IV.8.2. \( \mathbb{T} \) is a finitely-generated free \( \mathbb{Z}[\chi] \)-module.

Let \( f \) be an eigenvector for all \( T(m) \). We can define a homomorphism \( \theta_f : \mathbb{T} \to \mathbb{C} \) by \( f|_kT = \theta_f(T)f \) for all \( T \in \mathbb{T} \). The image of \( \theta_f \) is a finite module over \( \mathbb{Z}[\chi] \), so it’s in the ring of integers of some \( K_f \).

Let \( K_f' := \mathbb{Q}(a_n : n \geq 1) \). We want to show that we can take \( K_f = K_f' \), for which it suffices to show \( \chi(d) \in K_f' \) whenever \( (d, N) = 1 \).

Lemma IV.8.3. Let \( K/\mathbb{Q} \) be a finite extension. If \( a_n \in K \) for all \( n \geq 1 \), then so is \( a_0 \).

Fact IV.8.4. A number field \( K/\mathbb{Q} \) has a well-defined complex conjugation iff \( K \) is totally real or CM.

Lemma IV.8.5. For all \( n \) with \( (n, N) = 1 \), \( a_n = \overline{a_n} \chi(n) \).

Proof. We have \( T^*(n) = \overline{\chi}(n)T(n) \), so
\[
a_n \langle f, f \rangle = \langle f|_kT(n), f \rangle = \langle f, f|_kT^*(n) \rangle = \langle f, \overline{\chi}(n)a_n f \rangle = \chi(n) \overline{a_n} \langle f, f \rangle. \]
\[
\]
\[
\]

Corollary IV.8.6. For all \( \sigma \in \text{Aut}_\mathbb{Q}(\mathbb{C}) \) and all \( (n, N) = 1 \),
\[
\sigma(a_n) = \overline{\sigma(a_n)} \chi^\sigma(n).
\]

Proof. Apply the above lemma to \( f^\sigma \).

So for all \( (n, N) = 1 \),
\[
\sigma(\overline{a_n}) \chi^\sigma(n) = \sigma(a_n) = \overline{\sigma(a_n)} \chi^\sigma(n),
\]
so \( \overline{\sigma(a_n)} = \sigma(a_n) \). Thus, \( a_n \) is well-defined for \( (n, N) = 1 \). The existence of a well-defined complex conjugation on \( K_f \) shows that it is totally real or CM.

When is \( K_f \) totally real?

Theorem IV.8.7. \( K_f \) is totally real iff \( a_p = \chi(p)a_p \) for all \( p \nmid N \). In this case, \( \chi^2 = 1 \).
IV.8.1 CM forms

Let $f \in S_k(N, \chi)$ be a newform, and $\varphi$ a Dirichlet character mod $M$. Let

$$f_\varphi = \sum_{n \geq 1} \varphi(n) a_n q^n \in S_k(NM^2, \chi \varphi^2),$$

$$f_\varphi|_k T(p) = (\varphi(p) a_p) f_\varphi \quad \forall p \nmid NM.$$ 

**Definition IV.8.8.** Let $\varphi$ be a non-trivial Dirichlet character. Say $f$ has CM by $\varphi$ if $\varphi(p) a_p = a_p$ for all $p \nmid NM$.

**Note IV.8.9.**

- $\varphi^2 = \mathbf{1}$

  - $\varphi$ corresponds to an imaginary quadratic $F/\mathbb{Q}$, i.e.,

    $$\varphi(p) = \begin{cases} 
    1 & \text{if } p \text{ split in } F, \\
    -1 & \text{if } p \text{ inert in } F.
    \end{cases}$$

  - If $f$ is a newform, $\chi \neq \mathbf{1}$, and $K_f$ is totally real, then $f$ has CM by its Nebentypus $\chi$. 
Chapter V

Automorphic forms

V.1 2014-04-23: Automorphic forms and representations

Today, we will put modular forms into the larger context of automorphic forms and automorphic representations of $\text{SL}_2(\mathbb{R})$ and $\text{GL}_2(\mathbb{A})$. The first step is to go from $S_k(\Gamma)$ to automorphic forms on $\text{SL}_2(\mathbb{R})$ (Gelbart, chapter 2).

Recall: $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}$ by the map $g \mapsto g \cdot i$.

The idea: given $f : \mathfrak{h} \to \mathbb{C}$, “lift” this to a map $\varphi_f : \text{SL}_2(\mathbb{R}) \to \mathbb{C}$. Specifically, for $f \in S_k(\Gamma)$ (where $\Gamma$ is a congruence subgroup and $k$ is even), define

$$\varphi_f(g) := (g \cdot i) j(g, i)^k.$$

Then for all $\gamma \in \Gamma$, let $z = g \cdot i$, whence

$$\varphi_f(\gamma g) = f(\gamma \cdot (g \cdot i)) j(\gamma g, i)^{-k} = f(\gamma \cdot z) j(\gamma, z)^{-k} j(g, i)^{-k} = (f|k\gamma)(z) j(g, i)^{-k} = f(g \cdot i) j(g, i)^{-k} = \varphi_f(g).$$

Thus, $\varphi_f$ is left-$\Gamma$-invariant.

What properties characterize the image of $f \mapsto \varphi_f$?

There are nice coordinates on $G := \text{SL}_2(\mathbb{R})$: let

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_{>0} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{R} \right\},$$

$$K = \text{SO}_2(\mathbb{R}) = \left\{ \kappa_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

Note that $\text{SO}_2(\mathbb{R}) \subseteq \text{SL}_2(\mathbb{R})$ is a maximal compact subgroup.

The Iwasawa decomposition of $\text{SL}_2(\mathbb{R})$ is:

$$G = NAK.$$
Let $B = NA$ be the upper-triangular subgroup, called the Borel subgroup. Under the action $\SL_2(\mathbb{R}) \circlearrowleft \mathfrak{h}$, $B$ acts transitively: for $z = x + iy \in \mathfrak{h}$, let

$$b_z := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}.$$ 

Then $b_z \cdot i = x + iy = z$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = b_\kappa \theta$, then $z = g \cdot i$ and $\theta = \arg(ci + d)$. Then:

**Proposition V.1.1.** Let $f \in S_k(\Gamma)$.

(i) $\varphi_f$ is left-$\Gamma$-invariant.

(ii) $\varphi_f(g\kappa\theta) = e^{-ik\theta}\varphi_f(g)$.

(iii) $\varphi_f(g)$ is bounded and in $L^2(\Gamma \setminus G)$.

(iv) $\varphi_f$ is cuspidal, i.e., for all $g \in \SL_2(\mathbb{R})$ and $\alpha \in \SL_2(\mathbb{Z})$,

$$\int_0^1 \varphi_f \left( \alpha \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g \right) dx = 0,$$

where $h$ is the width of the cusp $\alpha \cdot \infty$.

**Proof.** (i) Already proven.

(ii) Observe that

$$\varphi_f(g\kappa\theta) = f(g\kappa\theta \cdot i)j(g\kappa\theta, i)^{-k} = f(g \cdot i)j(g, i)^{-k}j(\kappa\theta, i)^{-k} = \varphi_f(g)((\sin \theta)i + \cos \theta)^{-k}.$$

(iii) Note that $f \in S_k(\Gamma)$ iff $y^{k/2} |f(z)|$ is bounded. For $\varphi: \SL_2(\mathbb{R}) \to \mathbb{C}$, define

$$\int_G \varphi(g) \, dg := \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y, \theta) \frac{dx \, dy \, d\theta}{y^2}.$$ 

Then

$$\int_{\Gamma \setminus \mathbb{C}} |\varphi_f(g)|^2 \, dg = \int_{\Gamma \setminus \mathbb{C}} |f(z)|^2 \frac{y^k \, dx \, dy}{y^2} = \langle f, f \rangle_{\Gamma \setminus \mathbb{C}} \Vol(X(\Gamma)).$$

(iv) By definition, $f$ is a cusp form $\iff a_0(f|_k \alpha) = 0$ for all $\alpha \in \SL_2(\mathbb{Z})$. Thus, for all
\[ z \in \mathfrak{h}, \]
\[
0 = a_0(f|_{k\alpha}) = \int_0^1 (f|_{k\alpha})(x + z) \, dx
\]
\[
= h \int_0^1 (f|_{k\alpha})(hx + z) \, dx
\]
\[
= h \int_0^1 f(\alpha \cdot (hx + z)) j(\alpha, hx + z)^{-k} \, dx
\]
\[
= h \int_0^1 f \left( \alpha \begin{pmatrix} 1 & hx \\ 0 & 1 \end{pmatrix} g \cdot i \right) j \left( \alpha, \frac{1}{hx} g \cdot i \right)^{-k} \, dx
\]
\[
= hj(g, i)^{-k} \int_0^1 \varphi_f \left( \alpha \begin{pmatrix} 1 & hx \\ 0 & 1 \end{pmatrix} gi \right) \, dx.
\]

So we have a map
\[
S_k(\gamma) \rightarrow L^2(\Gamma \backslash G)
\]
\[
f \mapsto \varphi_f.
\]
A function \( f \) is holomorphic \( \iff \partial_z f = 0 \), where
\[
\partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

To characterize the image of the above map, we now define the Laplace–Beltrami operator on \( SL_2(\mathbb{R}) \): in terms of \((x, y, \theta)\),
\[
\Delta = -y^2 (\partial_x^2 + \partial_y^2) - y \partial_x \partial_\theta.
\]

Facts:

- \( \Delta \) is self-adjoint.
- \( \Delta \) is non-negative.
- \( \Delta \) commutes with \( R(g) \) for all \( g \in G \), where \( R \) denotes the right regular representation of \( G \) on \( L^2(\Gamma \backslash G) \), i.e., for all \( g \in G \) and \( \varphi : G \rightarrow \mathbb{C} \) in \( L^2 \),
\[
(R(g)\varphi)(hx) = \varphi(hgx).
\]

This is a unitary representation of \( G \).

**Theorem V.1.2.** Let
\[
A_k^2(\Gamma) := \left\{ \varphi \in L^2(\Gamma \backslash G) : \varphi \text{ satisfies (i) to (iv) above, and } \Delta \varphi = -\frac{k}{2} \left( \frac{k}{2} - 1 \right) \varphi \right\}.
\]
Then the map \( f \mapsto \varphi_f \) is an isomorphism \( S_k(\Gamma) \cong A_k^2(\Gamma) \).

**Proof sketch.** If \( g = b_2 \kappa_0 \), then \( \varphi_f(g) = y^{k/2} f(z) e^{-ik\theta} \), so
\[
(\Delta \varphi_f)(g) = (\ldots) (\partial_x^2 + \partial_y^2) f + (\ldots) \partial_z f - \frac{k}{2} \left( \frac{k}{2} - 1 \right) \varphi_f.
\]
V.1.1 Automorphic forms

Let us generalize the above situation.

**Definition V.1.3.** An automorphic form on \( G = \text{SL}_2(\mathbb{R}) \) of level \( \Gamma \) is a map \( \varphi : G \to \mathbb{C} \) such that:

1. \( \varphi \in C^\infty(G, \mathbb{C}) \).
2. \( \varphi(\gamma g) = \varphi(g) \) for all \( \gamma \in \Gamma \).
3. \( \varphi \) is right \( K \)-finite, i.e., \( g \mapsto \varphi(gk) \) for \( k \in K \) span a finite-dimensional vector space.
4. \( \varphi \) is slowly increasing, i.e., \( |\varphi(z, \theta)| = O(y^N) \) for some \( N \) as \( y \to \infty \).
5. \( \varphi \) is an eigenfunction for \( \Delta \).

We say \( \varphi \) is cuspidal if

\[
\int_0^1 \varphi \left( \alpha \begin{pmatrix} 1 & hx \\ 0 & 1 \end{pmatrix} g \right) dx = 0 \quad \text{for all} \ g \in G \quad \text{and} \ \alpha \in \text{SL}_2(\mathbb{Z}), \quad \text{where} \ h \ \text{is the width of} \ \alpha \cdot \infty. 
\]

V.2 2014-04-25: Adelic stuff

Define the ring of adeles of \( \mathbb{Q} \) by

\[
\mathbb{A} \overset{\text{def}}{=} \mathbb{R} \times \prod_p \mathbb{Q}_p,
\]

where the product is the restricted direct product

\[
\mathbb{A}_f \overset{\text{def}}{=} \prod_p \mathbb{Q}_p \overset{\text{def}}{=} \left\{ (b_p) \in \prod_p \mathbb{Q}_p : b_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\}.
\]

We also define the ideles \( \text{GL}_1(\mathbb{A}) = \mathbb{A}^\times \).

A Hecke character of \( \mathbb{Q} \) is a continuous map \( \psi : \mathbb{A}^\times \to \mathbb{C}^\times \) such that \( \psi(\mathbb{Q}^\times) = 1 \). The idele class group is \( \mathbb{Q}^\times \backslash \mathbb{A}^\times \).

We have a strong approximation theorem

\[
\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_>^\times \widehat{\mathbb{Z}}^\times,
\]

so \( \chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times \) induces a Hecke character \( \chi_\Lambda \), and every \( \psi \) is equal to \( \chi_\Lambda |_{\mathbb{A}} \) deles for some \( \chi \) and some \( s \in \mathbb{C} \), where

\[
|z|_\mathbb{A} = |z_\infty|_\infty \cdot \prod_p |z_p|_p
\]

for \( z \in \mathbb{A}^\times \).
Let $G = \text{GL}_2$. We will study
\[
G_A = \text{GL}_2(\mathbb{A}) = \text{GL}_2(\mathbb{R}) \times \prod_p \text{GL}_2(\mathbb{Q}_p).
\]

Let $N \in \mathbb{Z}_{\geq 1}$, and let $K_{p,0}(N) \subseteq G_p := \text{GL}_2(\mathbb{Q}_p)$ be the compact open subgroup
\[
K_{p,0}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod{N} \right\}.
\]

Let $K_{f,0}(N) = \prod_p K_{p,0}(N) \subseteq G_f := \text{GL}_2(\mathbb{A}_f)$. Then we have a strong approximation theorem:
\[
G_A = G_Q G_+ \mathcal{K}_f, 0(N).
\]

Let $\Gamma_0(N) = G_Q \cap G_+ K_{f,0}(N)$.

Let $f \in S_k(N, \chi)$, and let $\varphi_f : G_A \to \mathbb{C}$ be given by
\[
\varphi_f(g) = f(g_\infty \cdot i) j(g_\infty, i)^{-k} \chi(k_0),
\]
where $g = \gamma g_\infty k_0$.

Note V.2.1. Let $Z_\mathbb{A} := \text{center of } G_A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{A}^\times \right\}$. Then
\[
G_Q \backslash G_A / (Z_\mathbb{A} K_{f,0}(N)) \cong \Gamma_0(N) \backslash \text{SL}_2(\mathbb{R}) / \text{SO}_2(\mathbb{R}) \cong Y_0(N).
\]

**Definition V.2.2.** An *automorphic form* on $\text{GL}_2(\mathbb{Q})$ is a function $\varphi : G_A \to \mathbb{C}$ such that:

(i) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G_Q$.

(ii) $\varphi$ is right $K = K_\infty K_f$-finite.

(iii) $\varphi$ is slowly-increasing, i.e., for all $c > 0$ and compact $C \subseteq G_A$, there is a constant $M$ such that
\[
\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = O(|a|^M)
\]
for all $a \in \mathbb{A}^\times$ with $|a| > c$ and all $g \in C$.

(iv) $\varphi$ is $\mathcal{Z}$-finite as a function of $g_\infty$.

(v) There is a Hecke character $\psi$ such that $\varphi(zg) = \psi(z) \varphi(g)$ for all $z \in Z_\mathbb{A}$ and $g \in G_A$.

**Remark V.2.3.** To explain (iv), let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ be the Lie algebra. Any $X \in \mathfrak{g}$ acts as a first-order differential operator on $C^\infty(\text{GL}_2(\mathbb{R}))$. Given a Lie algebra $\mathfrak{g}$, we have the “universal enveloping algebra” $U(\mathfrak{g})$, a unital associative algebra.

The representation theory of $\mathfrak{g}$ is given by the representation theory of $U(\mathfrak{g})$. Elements of $U(\mathfrak{g})$ are higher-order differential operators on $C^\infty(\text{GL}_2(\mathbb{R}))$, and
\[
\mathcal{Z} := \text{center of } U(\mathfrak{g}).
\]
Let $\mathcal{A}(G)$ be the set of automorphic forms on $\text{GL}_2(\mathbb{Q})$.

**Definition V.2.4.** An automorphic form $\varphi \in \mathcal{A}(G)$ is *cuspidal* if

$$\int_{\mathbb{Q} \setminus \mathcal{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$$

for all but finitely many $g$.

**Definition V.2.5.** An *automorphic representation* of $\text{GL}_2(\mathbb{Q})$ is (roughly speaking) an irreducible constituent of $\text{GL}_2(\mathbb{A}) \subset \mathcal{A}(\text{GL}_2)$.

**Definition V.2.6 (Admissible representations).** Say $G$ is a locally profinite group (like $G_f$ or $G_p$), and $\pi : G \to \text{GL}(V)$ is a representation of $G$ on some $\mathbb{C}$-vector space $V$. We say $\pi$ is *smooth* if it is locally constant, i.e., for all $v \in V$, there is a compact open subgroup $K \leq G$ such that $Kv = v$.

We say $\pi$ is *admissible* if $\pi$ is smooth and $\text{dim}_{\mathbb{C}} V^K < \infty$ for all compact open $K \leq G$.

**Fact V.2.7.** $\mathcal{A}(G)|_{G_f}$ is admissible.

The Hecke algebra is the algebra $\mathcal{H}(G) = C_c^\infty(G, \mathbb{Z})$, consisting of smooth, compactly supported functions $G \to \mathbb{Z}$ with the operation given by convolution:

$$(\varphi_1 \ast \varphi_2)(g) = \int_G \varphi_1(x)\varphi_2(x^{-1}g) dx.$$ 

Inside $\mathcal{H}(G_p)$, the characteristic function of the double coset $K_{p,0}(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_{p,0}(N)$ corresponds to $T(p)$, and that of $K_{p,0}(N) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K_{p,0}(N)$ corresponds to $T(p, p)$.

If $\pi$ is an admissible representation of $G$, and $\varphi \in \mathcal{H}(G)$, then $\pi$ induces an action of $\mathcal{H}(G)$ on $\pi$ given by

$$\pi(\varphi) \cdot v = \int_G \varphi(g)\pi(g) \cdot v \, dg.$$ 

There’s also a Hecke algebra at $\infty$, basically $U(\mathfrak{g})$.

**Remark V.2.8.** (1) Admissible representations of $G_p$ or $G_f$ correspond to admissible representations of $\mathcal{H}(G_p)$ or $\mathcal{H}(G_f)$. (The Hecke algebra is like a group ring construction.)

(2) $\mathcal{H}(G_\mathfrak{A}) = (\bigotimes \mathcal{H}(G_p)) \otimes \mathcal{H}(G_\infty)$.

(3) Fact: for any irreducible admissible representation $\pi$ of $(\mathfrak{g}, K_\infty) \times G_f$, there exists $\pi_\infty$ admissible of $(\mathfrak{g}, K_\infty)$ and $\pi_p$ admissible of $G_p$ such that

$$\pi \cong \pi_\infty \otimes \bigotimes_p \pi_p.$$
Chapter VI

Elliptic curves

VI.1 2014-04-28: Algebro-geometric perspective

Recall that $\mathcal{A}_k(\Gamma)$ are sections of $\Omega^{k/2}_{X(\Gamma)}$ over $X(\Gamma)$.

To view modular forms over $\mathbb{Z}$, we want to define $X(\Gamma)$ over $\mathbb{Z}$. The idea is that $X(1)$ parametrizes elliptic curves over $\mathbb{C}$, so to define $X(1)/\mathbb{Z}$, we write down a “moduli problem” of elliptic curves over $\mathbb{Z}$ and show it is “represented” by some curve.

Let $S$ be a scheme, and let $\textbf{Sch}_S$ be the category of schemes over $S$. By the Yoneda Lemma, $\textbf{Sch}_S$ embeds into $\textbf{Fun}^\text{op}_{/S}$, i.e., we can identify $X \in \textbf{Sch}_S$ with its functor of points $X$.

Example VI.1.1. Let $S = \text{Spec} \, \mathbb{Z}$, $X = \text{Spec} \, \mathbb{Z}[t, t^{-1}]$, $R$ any ring, and $T = \text{Spec} \, R$. Then

$$X(T) = \text{Hom}_{\textbf{Sch}}(T, X) = \text{Hom}_{\text{Rings}}(\mathbb{Z}[t, t^{-1}], R) = R^\times.$$ 

We could ask if there is $X \in \textbf{Sch}_\mathbb{Z}$ such that $X(\text{Spec} \, R) = R^\times$. Indeed, there is; $X$ is usually denoted $\mathbb{G}_m$ and called the “multiplicative group”. In other words, the functor $\text{Spec} \, R \mapsto R^\times$ is representable by $X = \text{Spec} \, \mathbb{Z}[t, t^{-1}]$.

Over $\mathbb{C}$, an elliptic curve is a genus 1 compact Riemann surface, i.e., a compact smooth algebraic curve of genus 1. Recall:

$$\text{genus}(X) = \dim_{\mathbb{C}}(\text{differentials}).$$

Definition VI.1.2 (Dimension and relative dimension). Given an $S$-scheme $X \to S$, the relative dimension of $X$ at $s \in S$ is $\dim X_s$, where $X_s := X \times_S s$ is the fiber over $s$. If this is independent of $s$, we say $X$ has relative dimension $\dim X_s$.

Definition VI.1.3. An $S$-scheme $f : X \to S$ is smooth of relative dimension $n$ if

(i) $f$ is locally of finite presentation.

(ii) $f$ has relative dimension $n$.

(iii) The sheaf of relative differentials $\Omega_{X/S}$ is locally free of rank $n$.

If $n = 1$, we call $X$ a smooth curve.
CHAPTER VI. ELLIPTIC CURVES

Definition VI.1.4. An elliptic curve $\pi : E \to S$ is a proper smooth curve such that:

(i) For any $\text{Spec}(\bar{k}) \to S$ (where $\bar{k}$ is an algebraically closed field), $E_{\bar{k}} := E \times_S \text{Spec}(\bar{k})$ is connected, and $\dim_{\bar{k}} \pi_* \Omega_{E_{\bar{k}}/\text{Spec}(\bar{k})} = 1$.

(ii) There is a section $0 : S \to E$.

VI.2 2014-04-30: Elliptic curves

Let $\pi : E \to S$ be an elliptic curve with identity $0 : S \to E$, i.e., for all $\text{Spec}(k) \to S$, $E_k := E \times_S \text{Spec}(k)$ is connected and $\dim_k \pi_* \Omega_{E_k/S} = 1$. Hence, $E$ is a commutative group scheme over $S$.

Given $\pi : E \to S$, let $\omega := \omega_{E/S} := \pi_* \Omega_{E/S}$; this is an invertible $\mathcal{O}_S$-module. By Serre duality, $\pi_* \Omega_{E/S} \cong R^1 \pi_* \mathcal{O}_E$.

VI.2.1 Geometric definition of modular forms of level 1

This definition is due to Katz.

Definition VI.2.1. Let $k \in \mathbb{Z}$. A (meromorphic) modular form of weight $k$ (and level 1) is a rule $f$ which assigns to any $E/S$ an $f(E/S) \in H^0(S, \omega^{\otimes k}) = \Gamma(S, \omega^{\otimes k})$ satisfying the following properties:

(i) isomorphism independence: if $E \xrightarrow{\sim} E'$ (as $S$-schemes), then $f(E/S) = f(E'/S)$.

(ii) independence of base change: if $g : S' \to S$, then $f(E/S) = f(E_{S'}/S')$.

Let $A_k(\text{SL}_2(\mathbb{Z}), \mathbb{Z})$ be the $\mathbb{Z}$-module of such rules.

Remark VI.2.2. • If we fix $S_0 = \text{Spec} R_0$ and consider $S$ over $S_0$, then we get $A_k(\text{SL}_2(\mathbb{Z}), R_0)$.

• If $A$ is a commutative ring that is flat over $\mathbb{Z}$, then

$$A_k(\text{SL}_2(\mathbb{Z}), A) = A_k(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$ 

• $A_k(\text{SL}_2(\mathbb{Z}), \mathbb{C}) = A_k(\text{SL}_2(\mathbb{Z}))$.

Here is an equivalent definition:

Definition VI.2.3. Consider pairs $(E/R, \omega)$ with $R$ a ring, $E$ an elliptic curve over $R$, and $\omega$ a nowhere vanishing section of $\omega$. Say $\varphi : (E, \omega) \xrightarrow{\sim} (E', \omega')$ if $\varphi : E \xrightarrow{\sim} E$ and $\omega = \varphi^* \omega'$.

Then we define $A_k(\text{SL}_2([\mathbb{Z}], \mathbb{Z})$ to be the set of rules $(E/R, \omega) \mapsto f(E, \omega)$ which are:

(i) isomorphism independent;

(ii) invariant under base change;

(iii) for all $\lambda \in R^\times$, $f(E, \lambda \omega) = \lambda^{-k} f(E, \omega)$. 


VI.2.2 Tate curves

The behavior at infinity is described by Tate curves. Let

$$E_{\text{Tate}/\mathbb{Z}(q)} : y^2 + xy = x^3 - b_2(q)x - b_3(q),$$

where $$b_2, b_3 \in q\mathbb{Z}[q]$$ are defined by

$$b_2(q) = 5 \sum_{n \geq 1} \frac{n^3 q}{1 - q^n},$$

$$b_3(q) = \sum_{n \geq 1} \frac{7n^5 + 5n^2}{12} \cdot \frac{q^n}{1 - q^n}.$$ 

The discriminant and $$j$$-invariant of the Tate curve are

$$\Delta(E_{\text{Tate}}) = q \prod_{n \geq 1} (1 - q^n)^{24},$$ 

$$j(E_{\text{Tate}}) = \frac{1}{q} + 744 + 196884q + \ldots$$

Geometrically, we can think of $$\mathbb{Z}(q)$$ as a formal neighborhood of the cusp of $$\text{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}$$, and $$E_{\text{Tate}}$$ is an elliptic curve over that neighborhood. If $$f \in A_k(\text{SL}_2(\mathbb{Z}), \mathbb{R}_0)$$, then its $$q$$-expansion is $$f(E_{\text{Tate}}, \omega_{\text{can}}) \in \mathbb{Z}(q) \otimes_\mathbb{Z} \mathbb{R}_0$$, where $$\omega_{\text{can}} = \frac{dx}{y}$$ is the “canonical differential”.

Aside VI.2.4. Say $$E$$ is an elliptic curve over $$\mathbb{Z}_p$$ with multiplicative reduction. Then Tate’s uniformization theorem says that $$E \cong \mathbb{Q}_p^\times / q^\mathbb{Z}$$ for some $$q$$. (This is a $$p$$-adic, multiplicative analogue of a complex torus $$\mathbb{C}/\Lambda$$.)

VI.2.3 Moduli

Let

$$\mathcal{B} = \left\{ (\omega_1, \omega_2) \in \mathbb{C}^\times \times \mathbb{C}^\times : \frac{\omega_2}{\omega_1} \in \mathfrak{h} \right\}.$$ 

If $$\omega = (\omega_1, \omega_2)$$, and if $$\gamma \in \text{SL}_2(\mathbb{Z})$$, then

$$\omega' = \left( \begin{array}{c} \omega_1' \\ \omega_2' \end{array} \right) = \gamma \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right).$$

We have

$$\mathcal{L}/\mathbb{C}^\times \cong \text{SL}_2(\mathbb{Z}) \setminus \mathcal{B}/\mathbb{C}^\times \cong \text{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}.$$ 

Take $$\Gamma = \Gamma_0(N)$$. What does $$\Gamma \setminus \mathfrak{h}$$ parametrize?

Let $$\Lambda = \Lambda(\omega_1, \omega_2)$$, $$\gamma = \left( \begin{array}{cc} a & b \\ Nc & d \end{array} \right)$$, and $$\omega' = \gamma \omega = \left( \begin{array}{c} a\omega_1 + b\omega_2 \\ Nc\omega_1 + d\omega_2 \end{array} \right)$$, so

$$\frac{\omega_2'}{N} \equiv \frac{d\omega_2}{N} \pmod{\Lambda}.$$ 

We have $$(d, N) = 1$$, so $$\frac{\omega_2}{N}$$ and $$\frac{\omega_2'}{N}$$ generate the same cyclic subgroup of order $$N$$ in $$\mathbb{C}/\Lambda$$. 
**Lemma VI.2.5.** $\Gamma_0(N) \setminus B \cong \mathcal{L}_0(N) := \{(\Lambda, C_N) \mid C_N \subseteq \Lambda \text{ cyclic of order } N\}$.

Call the choice of $C_N \leq E$ (cyclic of order $N$) a **level $\Gamma_0(N)$-structure**.

The following is due to work of Deligne–Rapoport, Drinfeld, and Katz–Mazur. Given $E \rightarrow S$, we can define $C_N \leq E$. To formalize this, let $(\mathbf{Ell})$ be the category whose objects are elliptic curves $E \rightarrow S$, and whose maps are commutative squares

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

such that $E' \cong E \times_S S'$. A "moduli problem" is a contravariant functor

$$
P : (\mathbf{Ell}) \rightarrow (\mathbf{Sets}).
$$

An element of $P(E/S)$ is called a level $P$-structure on $E/S$.

**Definition VI.2.6.** $A_k(P, \mathbb{Z})$ is a rule that assigns to $(E/S, \alpha \in P(E/S))$ some $f(E, \alpha) \in H^0(S, \omega^{\otimes k})$ independent under isomorphism and base change.

Say $P$ is representable, i.e., $\mathcal{E}_P \rightarrow \mathcal{M}_P$ and $\alpha_P \in P(\mathcal{E}_P, \mathcal{M}_P)$ such that any $(E/S, \alpha)$ is a "pullback" of $(\mathcal{E}_P/\mathcal{M}_P, \alpha_P)$, i.e., $E \cong \mathcal{E}_P \times_{\mathcal{M}_P} S$. Then there is an isomorphism

$$
\varphi : A_k(P, \mathbb{Z}) \xrightarrow{\sim} H^0(\mathcal{M}_P, (\omega_{\mathcal{E}_P/\mathcal{M}_P})^{\otimes k}),
$$

$$
\varphi(f) = f(\mathcal{E}_P/\mathcal{M}_P, \alpha_P) \quad \forall f \in A_k(P, \mathbb{Z}).
$$

**VI.3 2014-05-05: Galois representations attached to modular forms**

Say $f$ is a newform in $S_k(N, \chi)$. Then

$$
L(s, f) = \prod_p \left(1 - a_p p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}.
$$

This looks like the $L$-function of a 2-dimensional Galois representation.

If $A$ is an abelian variety of dimension $g$ (e.g., if $g = 1$, then $E$ is an elliptic curve). If $\overline{k}$ is algebraically closed and $(n, \text{char } k) = 1$, then

$$
A(\overline{k})[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.
$$

If $A$ is defined over $\mathbb{Q}$, and $G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $A(\overline{\mathbb{Q}})$ and $A(\overline{\mathbb{Q}})[n]$ for all $n$, then for $\ell$ prime, we define the $\ell$-adic Tate module of $A$ to be

$$
T_\ell(A) \overset{\text{def}}{=} \lim_n A(\overline{\mathbb{Q}})[\ell^n].
$$
VI.3. 2014-05-05: GALOIS REPRESENTATIONS ATTACHED TO MODULAR FORMS

Note that $T_\ell(A) \cong \mathbb{Z}_\ell^{2g}$ as modules. So we get a $2g$-dimensional Galois representation

$$\rho_{A,\ell}: G_\mathbb{Q} \to \text{GL}(T_\ell(A)) \cong \text{GL}_{2g}(\mathbb{Z}_\ell).$$

Let $p \neq \ell$ be prime. Define

$$P_{A,p}(x) \overset{\text{def}}{=} \det \left(1 - x \rho_{A,\ell}(\text{Frob}_p)|V_\ell(A)^{I_p}\right).$$

Here, $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and an embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_p$ induces an inclusion $G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$. There is an exact sequence

$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1,$$

and $G_{\mathbb{F}_p} \cong \hat{\mathbb{Z}} = \langle a \mapsto a^p \rangle$, the profinite completion of the free cyclic group generated by the Frobenius map $\text{Frob}_p: a \mapsto a^p$. Hence, $\text{Frob}_p$ is well-defined in $G_{\mathbb{Q}_p}$ up to the inertia group $I_p$.

Moreover, the inclusion $G_\mathbb{Q} \hookrightarrow G_{\mathbb{Q}_p}$ is well-defined up to conjugation, so $\text{Frob}_p$ is well-defined in $G_\mathbb{Q}$ up to inertia and conjugation. Since $I_p$ acts trivially on $V_\ell(A)^{I_p}$ (by definition), and characteristic polynomials are invariant under conjugation, $P_{A,p}(x)$ is well-defined.

Theorem VI.3.1. $P_{A,p}(x)$ is independent of $\ell$ (assuming $\ell \neq p$).

Example VI.3.2. Let $E$ be an elliptic curve, and assume $\rho_{A,\ell}$ is unramified at $p$, i.e., $V_\ell(A)^{I_p} = V_\ell(A)$. Then

$$\det(1 - x \cdot \rho_{A,\ell}(\text{Frob}_p)) = 1 - a_p x + px^2,$$

where $a_p = p + 1 - \#E(\mathbb{F}_p)$. Furthermore, if $N$ is the conductor of $E$ and $p | N$, then

$$P_{E,p}(x) = 1 - a_p x,$$

where $a_p = 0$ if $p^2 | N$ (additive reduction), and $a_p = \pm 1$ otherwise (multiplicative reduction); $a_p = 1$ corresponds to split multiplicative reduction, and $a_p = -1$ nonsplit.

Returning to the general case, define

$$L(s, A) \overset{\text{def}}{=} \prod_p P_{A,p}(p^{-s})^{-1}.$$

For example, if $A = E$, then

$$L(s, E) = \prod_p \left(1 - a_p p^{-s} + 1_N(p)p^{1-2s}\right)^{-1},$$

where $1_N$ is the trivial Dirichlet character mod $N$.

For an elliptic curve $E/\mathbb{F}_p$, the numbers $\#E(\mathbb{F}_{p^n})$ can be put together into a generating series $Z(x, E/\mathbb{F}_p)$. Weil proved that

$$Z(x, E/\mathbb{F}_p) = \frac{1 - a_p x + x^2}{(1 - x)(1 - px)}.$$
Weil also proved the Riemann hypothesis for elliptic curves, i.e., \( | \alpha_p | = | \beta_p | = p^{1/2} \), where
\[
1 - \alpha_p x + x^2 = (1 - \alpha_p x)(1 - \beta_p x).
\]
This generating function is closely related to the L-function:
\[
L(s, E/\mathbb{F}_p) = Z(p^{-s}, E/\mathbb{F}_p).
\]
We can attach Galois representations to newforms:

**Theorem VI.3.3.** If \( f = \sum_{n \geq 1} a_n q^n \in S_k(N, \chi) \) is a newform, then for all primes \( \ell \), there is a continuous, irreducible Galois representation \( \rho_{f, \ell} : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_\ell) \), unramified at \( p \) for all \( p \nmid \ell N \), such that
\[
P_{\rho_{f, \ell}, p}(x) = 1 - a_p x + \chi(p)p^{k-1}x^2.
\]

**Remark VI.3.4.** The Galois representation \( \rho_{f, \ell} \) is uniquely defined by the collection of polynomials \( \{ P_{\rho_{f, \ell}, p} : p \nmid \ell N \} \). This is because \( \{ \text{Frob}_p : p \nmid \ell N \} \) is dense in \( G_{\mathbb{Q}} \) (by Čebotarev density), and we can recover an irreducible representation from characteristic polynomials.

This was proved for \( k = 2 \) by Eichler, Shimura, and Igusa: the Hecke operators have a geometric interpretation as well. Let \( J_1(N) = \text{Jac}(X_1(N)/\mathbb{Q}) \) over \( \mathbb{Q} \). The Hecke algebra \( T \) acts on \( J_1(N) \). For all \( p \nmid \ell N \), by the Eichler–Shimura congruence, \( \rho_{J_1(N), \ell}(\text{Frob}_p) \) satisfies
\[
x^2 - T(p)x + T(p, p).
\]
Under this correspondence, \( f \leftrightarrow \lambda_f : T \to K_f = \mathbb{Q}(a_n) \). Let \( I_f := \ker \lambda_f \). Then
\[
\mathbb{T}/I_f \simeq \mathbb{Z}[a_n] := \mathcal{O}_f,
\]
and \( A_f := J_1(N)/I_f J_1(N) \) is an abelian variety of dimension \( d_f := [K_f : \mathbb{Q}] \). Then \( \rho_{f, \ell} := T_\ell(A_f) \) is a \( 2d_f \)-dimensional \( \mathbb{Z}_\ell \)-module, and hence a 2-dimensional \( \mathcal{O}_f \)-module.

**VI.4 2014-05-07: Galois representations of higher weight**

The following construction of Galois representations for \( k \geq 2 \) is due to Deligne and Shimura (unpublished).

Let \( \mathcal{E} \xrightarrow{\pi} X_1(N) \) be the universal elliptic curve over \( \mathbb{Q} \), and consider the Kuga–Sato variety
\[
\mathcal{E}^{(n)} := \mathcal{E} \times_{X_1(N)} \cdots \times_{X_1(N)} \mathcal{E}.
\]

Deligne defines a “canonical desingularization” \( \mathcal{E}^{(n)} \). Consider the Galois representation
\[
H_{et}^{k-1}(\overline{\mathcal{E}^{(k-2)}} \times_{\mathbb{Q}} \overline{\mathbb{Q}_\ell}),
\]
with actions of \( G_{\mathbb{Q}} \) and \( T_1(N) \). Let \( \rho_{f, \ell} \) be the piece on which \( T_1(N) \) acts like it acts on \( f \).

For \( k = 2 \), we have \( \mathcal{E}^{(0)} = X_1(N) \) and
\[
H_{et}^1(X_1(N), \overline{\mathbb{Q}_\ell}) \cong T_\ell(J_1(N)).
\]
Note VI.4.1. This attaches a motive $M(f)$ to $f$. (Deligne looked at

$$H^1_{\mathrm{ét}}(X_1(N), \mathrm{Sym}^{k-2} R^1 \pi_* (\overline{\mathbb{Q}}_\ell)),$$

which is isomorphic.)

For $k = 1$, this is a theorem of Deligne–Serre by looking at congruences with higher weight forms.
Chapter VII

Generalizations of modular forms

VII.1 2014-05-07, continued

For the rest of today, we’ll talk about some classical generalizations of modular forms.

VII.1.1 Hilbert modular forms

Let $F$ be a totally real number field of degree $d$, let $\sigma_1, \ldots, \sigma_d : F \hookrightarrow \mathbb{R}$ be the $d$ distinct embeddings into $\mathbb{R}$, and let $\mathcal{O}_F$ be its ring of integers. This yields

\[ \text{GL}_2(\mathcal{O}_F) \subseteq \text{GL}_2(F) \overset{\prod \sigma_i}{\rightarrow} \text{GL}_2(\mathbb{R})^d. \]

If $a \in F$, we say $a$ is totally positive (denoted $a \gg 0$) if $\sigma_i(a) > 0$ for all $i$. Let

\[ \text{GL}_2^+(F) := \{ \alpha \in \text{GL}_2(F) : \det(\alpha) \gg 0 \}. \]

If $\alpha \in \text{GL}_2^+(F)$ and $z = (z_1, \ldots, z_d) \in \mathfrak{h}^d$, let

\[ \alpha \cdot z := (\sigma_i(\alpha) \cdot z_i)_{i=1,\ldots,d}, \]

\[ j(\alpha, z) := \prod_{i=1}^d j(\sigma_i(\alpha), z_i). \]

For $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $a \in F$, let

\[ a^k := \prod_{i=1}^d \sigma_i(a)^{k_i}. \]

For $f : \mathfrak{h}^d \rightarrow \mathbb{C}$, let

\[ (f|_k \alpha)(z) := \det(\alpha)^{-k/2} j(\alpha, z) f(\alpha \cdot z) \]

for $\alpha \in \text{GL}_2^+(F)$.

Let $\Gamma \leq \text{SL}_2(\mathcal{O}_F)$ be a subgroup of finite index.

Definition VII.1.1. A Hilbert modular form of weight $\underline{k}$ and level $\Gamma$ is a function $f : \mathfrak{h}^d \rightarrow \mathbb{C}$ such that:

85
(i) $f$ is holomorphic.

(ii) $f|_k \gamma = f$ for all $\gamma \in \Gamma$.

(iii) $f|_k \alpha$ is holomorphic at $\infty$ for all $\alpha \in \text{GL}_2^+(F)$, in the following sense: Let

$$M_\Gamma := \left\{ m \in \mathcal{O}_F : \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \Gamma \right\},$$

$$M_\Gamma^\vee := \left\{ \nu \in F : \text{Tr}_{F/Q}(\nu m) \in \mathbb{Z} \ \forall m \in M_\Gamma \right\}.$$

Then $f$ satisfying (i) and (ii) can be written as

$$f(z) = \sum_{\nu \in M_\Gamma^\vee} a_\nu e^{2\pi i \text{Tr}(\nu z)},$$

where $\text{Tr}(\nu z) := \sum_{i=1}^{d} \sigma_i(\nu) z_i$.

We say $f$ is holomorphic at $\infty$ if $a_\nu = 0$ unless $\nu = 0$ or $\nu \gg 0$.

Theorem VII.1.2 (Köcher’s principle). If $d \geq 2$, if $f$ satisfies (i) and (ii), then it satisfies (iii).

Definition VII.1.3. As before, $f$ is a cusp form if $a_0(f|_k \alpha) = 0$ for all $\alpha$.

If $k = (k, k, \ldots, k)$, we say $f$ has parallel weight $k$.

Theorem VII.1.4. Some facts about Hilbert modular forms:

- $\dim_{\mathbb{C}} M_k^0(\Gamma) < \infty$.
- $M_k^0(\Gamma) = \mathbb{C}$.
- $S_k(\Gamma) = 0$.
- If $k_i \leq 0$ for some $i$ and $k \neq 0$, then $M_k^0(\Gamma) = 0$.
- If $k$ is not parallel, then $M_k(\Gamma) = S_k(\Gamma)$.

Remark VII.1.5. In the representation theory of $\text{GL}_2(\mathbb{Q})$, every irreducible representation is of the form $\text{Sym}^{k_0} \mathbb{Q}^2 \otimes \text{det}^{k_1}$ for some $k_0, k_1$. The weight of the representation is $(k_0, k_1)$, corresponding to characters of the maximal torus $\begin{pmatrix} \mathbb{Q}^\times & 0 \\ 0 & \mathbb{Q}^\times \end{pmatrix}$. In our context, only $k_0 - k_1 =: k$ matters; this is the weight $k$ of a modular form.

For Hilbert modular forms, the weights of $\text{Res}_{F/Q} \text{GL}_2(F)$ correspond to $(k_0, \ldots, k_d)$, and $k = (k_1 - k_0, k_2 - k_0, \ldots, k_d - k_0)$.

Let $Y(\Gamma) := \Gamma \backslash \mathfrak{h}^d$; this is a $d$-dimensional complex manifold and a $d$-dimensional algebraic variety over $\mathbb{C}$. Moreover, $Y(1)$ parametrizes principally polarized $d$-dimensional abelian varieties with RM (real multiplication) by $\mathcal{O}_F$, i.e., a triple $(A, \lambda, \iota)$ with $A$ a $d$-dimensional abelian variety, $\lambda : A \to A^\vee$, and $\iota : \mathcal{O}_F \hookrightarrow \text{End}(A)$.

If $k$ is paritious (i.e., $k_i = k_j$ mod 2), then we get a Galois representation

$$\rho_{f, \ell} : G_F \to \text{GL}_2(\mathbb{Q}_\ell)$$

(Wiles, Taylor, Carayol, Blasius–Rogowski, Ohta, Janis, Tunnell–?).

Next time, we'll talk about Siegel modular forms and other generalizations.
VII.2 2014-05-09: Siegel modular forms, CFT, and Langlands

VII.2.1 Siegel modular forms

Let $\text{Sp}_{2g}(\mathbb{R}) = \{ \alpha \in \text{GL}_{2g}(\mathbb{R}) : \alpha^T J \alpha = J \}$, where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. (Note that $\text{Sp}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})$, so the $g = 1$ case recovers classical modular forms.) This has a maximal compact subgroup $K_{2g} = \text{O}_{2g}(\mathbb{R}) \cap \text{Sp}_{2g}(\mathbb{R})$.

The Siegel upper-half space of degree $g$ is $	ext{Sp}_{2g}(\mathbb{R})/K_{2g} \cong \mathfrak{h}_g := \{ Z = X + iY \in M_g(\mathbb{C}) : Z^T = Z \text{ and } Y > 0 \}$. This is a $\frac{1}{2} g(g + 1)$-dimensional complex manifold with an action of $\text{Sp}_{2g}(\mathbb{R})$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$ Congruence subgroups are defined by

$$\Gamma(N) := \{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) : \gamma \equiv I_{2g} \mod N \}.$$ 

**Definition VII.2.1.** A *Siegel modular form* is a function $f : \mathfrak{h}_g \to \mathbb{C}$ such that:

(i) $f$ is holomorphic.

(ii) $f|_{k \gamma} = f$ for all $\gamma \in \Gamma$, where

$$\left( f|_{k \gamma} \right)(z) := j(\gamma, z)^{-k} f(\gamma \cdot z)$$

and

$$j \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) := \det(CZ + D).$$

(iii) If $g = 1$, then $f$ is holomorphic at $\infty$. (If $g \geq 2$, this is automatic.)

We get a Fourier expansion

$$f(Z) = a_0 + \sum_{Q>0} a_Q e^{2\pi i \text{Tr}(QZ)},$$

where $Q$ ranges over positive-definite, symmetric $g \times g$ matrices such that the associated quadratic form $(x_1, \ldots, x_g)^T Q (x_1, \ldots, x_g)^T$ has integer coefficients.

Let $Y(\Gamma) := \Gamma \backslash \mathfrak{h}_g$. Then, in particular, $Y(1)$ parametrizes all principally polarized abelian varieties of dimension $g$.

We also have a notion of Eisenstein series. For $\text{SL}_2(\mathbb{R})$, Dirichlet characters $\varphi, \psi$ yield $E_{k, \varphi, \psi}$. For $g = 2$, given a Siegel modular form $f$, we have an attached Galois representation $\rho_f : G_{\mathbb{Q}} \to \text{GSp}_4(\overline{\mathbb{Q}_\ell})$. 
CHAPTER VII. GENERALIZATIONS OF MODULAR FORMS

VII.2.2 Class field theory

Let $F$ be a number field (or a global field). Let $\mathbb{A}_F$ be the ring of adeles of $F$, and let $C_F := F^\times \backslash \mathbb{A}_F^\times$ be the idele class group.

Class field theory consists of an Artin reciprocity map
\[
\text{rec}_F : C_F \to G_F^{\text{ab}} = \text{maximal abelian quotient of } \text{Gal}(\overline{F}/F)
\]
\[
p \mapsto \text{Frob}_p
\]
with dense image. We can think of this as a bijection
\[
\left\{\text{continuous characters } \chi : G_F \to \mathbb{C}^\times \right\} \leftrightarrow \left\{\text{finite order continuous characters } C_F \to \mathbb{C}^\times \right\}.
\]
(Similarly, if $F$ is a local field, an analogous statement is true with $C_F := F^\times$.)

What if we consider all continuous characters, rather than only those of finite order? Class field theory provides the Weil group of $F$, a group $W_F$ with a map $\varphi_F : W_F \to G_F$ with dense image.

Let $E/F$ be a finite extension. Then $W_E = \varphi_F^{-1}(G_E)$. Define the relative Weil group
\[
W_{E/F} := W_F/[W_E,W_E].
\]
If $E/F$ is Galois, we obtain a short exact sequence
\[
1 \to C_E \to W_{E/F} \to \text{Gal}(E/F) \to 1.
\]
Now the reciprocity map provides a topological isomorphism
\[
\text{rec}_F : C_F \xrightarrow{\sim} W_F^{\text{ab}},
\]
giving a bijection
\[
\left\{\text{1-dimensional irreducible representations of } C_F \right\} \leftrightarrow \left\{\text{1-dimensional irreducible representations of } W_F \right\}.
\]

VII.2.3 Langlands reciprocity

What if we look at higher-dimensional irreducible representations of $W_F$? Langlands conjectured a bijection
\[
\left\{\text{automorphic irreducible representations of } \text{GL}_n(\mathbb{A}_F) \right\} \leftrightarrow \left\{\text{n-dimensional irreducible representations of } W_F \right\}.
\]

Langlands proved this conjecture for $F = \mathbb{R}$ and $G = \text{GL}_1$ or any reductive group over $\mathbb{R}$, giving a natural correspondence
\[
\left\{\text{admissible representation of } G(\mathbb{R}) \right\} \leftrightarrow \left\{\rho : W_{\mathbb{R}} \to {}^L G(\mathbb{C}) \right\},
\]
where ${}^L G$ denotes the Langlands dual of $G$. Langlands predicted that generalizing this correspondence should provide insight into global and local fields.

Here’s the general idea: The collection of automorphic representations of $G(\mathbb{A}_F)$ should be a neutral Tannakian category. As such, there should exist a group $L_F$ (with a map $L_F \to W_F \to G_F$), the Langlands group, such that this category is the category of representations of $L_F$. 
Index

additive character, 42
adeles, 74
admissible representation, 76
Atkin–Lehner operator, 65
automorphic form, 10, 74, 75
cuspidal, 74, 76
automorphic representation, 76
automorphy factor, 7
Borel subgroup, 72
canonical divisor, 27
Cayley transformation, 14
CM, 69
commensurable, 59
commensurator, 55
conductor, 42
congruence subgroup, 8
cusp form, 8
cuspidal, 7, 72
degeneracy map, 60
degree, 18
differential form, 22
Dirichlet L-function, 43
Dirichlet character, 41
primitive, 42
divisor group, 26

Eichler–Shimura congruence, 82
eigenform, 61
normalized, 61
Eisenstein series, 9
elliptic, 14
elliptic point, 13
Euler characteristic, 19

Fourier transform, 43
fundamental domain, 11

Hecke action, 57
Hecke character, 74
Hilbert modular form, 85
holomorphic at \( \infty \), 7
idele class group, 74
ideles, 74
inverse Mellin transform, 65
irregular cusp, 29
Iwasawa decomposition, 71
Jacobi sum, 43
Kuga–Sato variety, 82
\( \ell \)-adic Tate module, 80
Langlands dual, 88
Langlands group, 88
Laplace–Beltrami operator, 73
level structure, 80
linear space, 27
Mellin transform, 45, 64
meromorphic modular form, 22
modular curve, 8, 11
modular form, 8
geometric definition, 78
moduli problem, 80
motive, 83
Nebentypus, 69
neutral Tannakian category, 88
new subspace, 60
newform, 60, 61
old subspace, 60
order of vanishing, 26
parabolic, 15
parallel weight, 86
Phragmén–Lindelöf principle, 66
primitive form, 60
principal congruence subgroup, 8
principal divisor, 26
ramification degree, 18
relative dimension, 77
Riemann–Hurwitz formula, 18
root number, 47

Siegel modular form, 10, 87
Siegel upper-half space, 87
slowly increasing, 74
smooth, 77
smooth curve, 77
smooth representation, 76

Tate’s uniformization theorem, 79
totally positive, 85

weight $k$, 54
weight $k$ action, 8
Weil group, 88
relative, 88