Mixing Problem:

Rate of change of amount in the container = \( \text{(rate at which chemical arrives)} - \text{(rate at which chemical departs)} \)

\[ y(t) \]: amount of chemical in the container at time \( t \)
\[ v(t) \]: total volume of liquid in the container at time \( t \)

Departure rate = \( \frac{y(t)}{v(t)} \cdot \text{(overflow rate)} \)

\[ \frac{dy}{dt} = \text{(chemical's arrival rate)} - \frac{y(t)}{v(t)} \cdot \text{(overflow rate)} \]

Units:

\[ \text{gallons} = \frac{\text{gallons}}{\text{minutes}} = \frac{\text{gallons}}{\text{gallons}} \cdot \frac{\text{gallons}}{\text{minutes}} \]

Exercise Problem:

A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the required time to reach the maximum.
Idea of Taylor Series

Using polynomials to approximate a function near the point \( x = a \) such that they have the same signature.

The signature of a function \( f \) at \( x = a \) is defined by:

\[
( f(a), f'(a), f''(a), \ldots, f^{(k)}(a), \ldots )
\]

where \( f^{(k)}(a) \) is the \( k \)-th derivative of \( f(x) \) at \( x = a \).

Defn. \( T_n f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \)

where \( n! = 1 \cdot 2 \cdot 3 \cdots n \) and \( 0! = 1 \) by definition.

Some Special Taylor Polynomials:

\[
T_n e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]

\[
T_{n+1} \sin x = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]

\[
T_n \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}
\]

\[
T_n \left( \frac{1}{1-x} \right) = 1 + x + x^2 + \cdots + x^n \quad \text{(Geometric Sum)}
\]

\[
T_n \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n}
\]

\[
T_n (1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n.
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).
In general: The coefficient of $x^n$ in $T_\infty^a f(x)$ is:

$$\frac{f^{(n)}(a)}{n!}$$

(Notation: $T_n f(x) := T_0^a f(x)$, $T_\infty f(x) := T_0^\infty f(x)$)

**Example:**

1. $T_\infty (f(x) + g(x)) = T_\infty f(x) + T_\infty g(x)$.
2. $T_\infty (af(x)) = a T_\infty f(x)$, $a \in \mathbb{R}$.
3. $T_\infty (f(ax + b)) = (T_\infty f)(ax + b)$, $a, b$ are real numbers.

The coefficient of $x^n$ in $T_\infty^a f(x) + g(x)$ is:

$$\frac{(f+g)^{(n)}(a)}{n!} = \frac{f^{(n)}(a) + g^{(n)}(a)}{n!}$$ (Sum Rule)

The coefficient of $x^n$ in $T_\infty^a (cf(x))$ is:

$$\frac{(cf)^{(n)}(a)}{n!} = c \cdot \frac{f^{(n)}(a)}{n!} = c \cdot \frac{f^{(n)}(a)}{n!}$$

The coefficient of $x^n$ in $T_\infty^a (f(cx))$ is:

$$\frac{f^{(n)}(a)}{n!} \cdot c^n$$ (Ex: Prove this one) (use chain rule)
Example

Find the coefficient of $x^n$ in $\ln(f(t))$ for the following functions.

1. $\sinh t = \frac{e^t - e^{-t}}{2}$

Using 1 and 2, 3

The coefficient of $x^n$ in $\ln(f(t))$ for $e^{-t}$ is.

$$\frac{(-1)^n}{n!}$$

The coefficient of $x^n$ in $\ln(f(t))$ for $e^t$ is.

$$\frac{1}{n!}$$

The coefficient of $x^n$ in $\ln(f(t))$ for $\frac{e^t - e^{-t}}{2}$ is.

$$\frac{1}{n!} - \frac{(-1)^n}{2n!} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n!} & \text{if } n \text{ is odd} \end{cases} \quad (1) \text{ and (2)}$$

2. $\ln(1+2t)$

The coefficient of $x^n$ in $\ln(f(t))$ for $\ln(1+t)$ is.

$$\frac{(-1)^{n+1}}{n}$$

The coefficient of $x^n$ in $\ln(f(t))$ for $\ln(1+2t)$ is.

$$2^n \cdot \frac{(-1)^{n+1}}{n} \quad (3)$$
Exercise:

3) \( \ln \left( \frac{1+x}{1-x} \right) \)

4) \( \ln \sqrt{\frac{1+x}{1-x}} \)

5) \( 2 \sin t \cos t \)

(Hint: \( 2 \sin t \cos t = \sin 2t \))

6) \( x \cos x \)

The Remainder Term:

\[ f(x) = T_n^a f(x) + R_n^a f(x) \]

\[ \text{approximation} \quad \text{error term} \]

Theorem:

\[ R_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \]

(\( \xi \) is a variable depending on \( x \) and \( \xi \) lies between 0 and \( x \).)

Example. Find the fourth degree Taylor polynomial \( T_4 \cos x \) for the function \( f(x) = \cos x \) and estimate the error \( |\cos x - T_4 \cos x| \) for \( |x| < 1 \).

Thus:

\[ R_4 \cos x = \cos x - T_4 \cos x \]

\[ R_4 \cos x = \frac{f^{(5)}(\xi)}{5!} x^5 = \frac{\sin \xi}{5!} x^5 \]

\[ f^{(5)}(x) = -\sin x \]

Estimate (Find an upper bound of) \( |R_4 \cos x| \) for \( |x| < 1 \).

\[ |R_4 \cos x| = \left| \frac{\sin \xi}{5!} x^5 \right| \leq \frac{1}{5!} |\sin \xi| |x^5| 

| \leq \frac{1}{5!} |x|^5 \]

| \leq \frac{1}{5!} \]
Exercise:

1. The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when $x$ is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

2. Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$.

3. Estimate $\epsilon_{10}$ using Taylor series such that the error is less than or equal to $10^{-5}$.

Advance Exercise:

1. Compute $\ln 2$ for $f(x) = e^{-x^2}$, for $x \neq 0$ and $f(0) = 0$.

2. Estimate $\ln 2$ using Taylor series such that the error is less than $10^{-6}$. (Hint: Use a function other than $\ln(1+x)$.) The answer should not have too much computation involved.