Problem 2 (15 points): Mixture problem: A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.

a At what time will the tank be full?

b At the time the tank is full, how many pounds of concentrate will it contain?

1] Solution: a. Since at the beginning, the 200-gal tank is half full, the time to fill this tank is:

$$ T = \frac{100}{5 - 3} = 50 $$

b. Since we can use the mass of concentrate $m(t)$ as our variable, we can also use the density of the concentrate $\rho(t)$ as our variable, there are two ways to build the differential equation. (Actually they are the same)

Method 1: use $m(t)$ as variable.

First we build the differential equation:

Let’s use min as the unit of time.

At time $t$, the volume is: $V(t) = 100 + 2t$.

Let $m(t)$ denote the mass of concentrate in the tank at time $t$. Then we have:

$$ m(t + 1) - m(t) = \{5 \cdot 0.5 - 3 \cdot \frac{m(t)}{V(t)}\} $$

where $\frac{m(t)}{V(t)}$ is the density of the concentrate at time $t$. And $V(t) = 100 + 2t$ as we got above.

Then we have:

$$ \frac{dm}{dt} = \{2.5 - 3 \cdot \frac{m(t)}{100 + 2t}\} $$

Simplify it, we have:

$$ \frac{dm}{dt} + 3 \cdot \frac{m(t)}{100 + 2t} = 2.5 $$

We solve this first-order linear differential equation by integrating factor:

$$ u(t) = e^{\int \frac{3}{100 + 2t} dt} = e^{\frac{3}{2} \ln(100 + 2t)} = (100 + 2t)^{\frac{3}{2}} $$

Then we have:

$$ m(t) \cdot (100 + 2t)^{\frac{3}{2}} = 2.5 \int (100 + 2t)^{\frac{3}{2}} dt $$

which leads to:

$$ m(t) = \frac{1}{2} \cdot (100 + 2t) + C \cdot (100 + 2t)^{-\frac{3}{2}} $$

Use initial condition: $m(0) = 0$, we have: $0 = 50 + \frac{C}{1000}$, $C = -50000$.

Then:

$$ m(t) = 50 + t - 50000(100 + 2t)^{-\frac{3}{2}} $$

Method 2: Use $\rho(t)$ as variable.
First we build the differential equation:
Let’s use min as the unit of time.
At time $t$, the volume is $V(t) = 100 + 2t$.
Let $\rho(t)$ denote the concentration of solution in the tank at time $t$. Then we have:

$$\rho(t + 1) - \rho(t) = \frac{\{5 \cdot 0.5 - 3\rho(t)\}}{V(t)}$$

Then we have:

$$\frac{d\rho}{dt} = \frac{2.5 - 3\rho(t)}{100 + 2t}$$

Simplify this equation, we get:

$$\frac{d\rho}{dt} + \frac{3}{100 + 2t} \rho = \frac{2.5}{100 + 2t}$$

Use integrating factor to solve this first-order linear differential equation, with $m(t) = e^{\frac{3}{2}ln(100 + 2t)} = (100 + 2t)^{\frac{3}{2}}$

Then we have:

$$(100 + 2t)^{\frac{3}{2}} \rho = 2.5 \int (100 + 2t)^{\frac{1}{2}} dt$$

Then we have:

$$\rho(t) = \frac{5}{6} + C (100 + 2t)^{-\frac{3}{2}}$$

With our initial condition: $t = 0, \rho(0) = 0$, we can fix $C = \frac{2500}{3}$

Then we have:

$$\rho = \frac{5}{6} - \frac{2500}{3} (100 + 2t)^{-\frac{3}{2}}$$

Then $m(t) = \rho(t) \cdot V(t)$