How to pass the exam:
Step 1: Study the material on this handout.
Step 2: Don’t fail the exam.

1 Integration

1.1 Remember Your Trigonometry

- Don’t forget the basic identities:
  \[
  \sin(2x) = 2\sin(x)\cos(x) \\
  \cos(2x) = \cos^2(x) - \sin^2(x) \\
  \sin^2(x) + \cos^2(x) = 1
  \]

- You really only need to remember integrals of tangent, and secant; the rest are easy from those:
  \[
  \int \sin(x) \, dx = -\cos(x) + C \\
  \int \cos(x) \, dx = \sin(x) + C \\
  \int \tan(x) \, dx = -\ln(\cos(x)) + C \\
  \int \sec(x) \, dx = \ln(\sec(x) + \tan(x)) + C \\
  \int \csc(x) \, dx = -\ln(\csc(x) + \cot(x)) + C \\
  \int \cot(x) \, dx = \ln(\sin(x)) + C
  \]

- The important inverse-trig functions are arcsine and arctangent:
  \[
  \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C \\
  \int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C \\
  \int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + C
  \]

1.2 Substitution

- The general substitution formula states that \[ \int f'(g(x)) \cdot g'(x) \, dx = f(g(x)) + C \]. It is just the Chain Rule, written in terms of integration via the Fundamental Theorem of Calculus. We generally don’t use the formula written this way. To do a substitution, follow this procedure:
  - Step 1: Choose a substitution \( u = g(x) \).
  - Step 2: Compute the differential \( du = g'(x) \, dx \).
  - Step 3: Rewrite the original integral in terms of \( u \):
    - 3a: Rewrite the integral to peel off what will become the new differential \( du \).
* 3b: Write the remaining portion of the integrand in terms of \( u \).

* 3c: Find the new limits of integration in terms of \( u \), if the integral is a definite integral. [If the old limits are \( x = a \) and \( x = b \), the new ones will be \( u = g(a) \) and \( u = g(b) \). In a very concrete sense, these are “the same” points.]

* 3d: Write down the new integral. If the integral is indefinite, substitute back in for the original variable.

- Substitution is best learned by doing examples; see below.

- Evaluate \( \int_0^3 2x e^{x^2} \, dx \).
  - Step 1: The exponential has a 'complicated' argument \( x^2 \), so we try setting \( u = x^2 \).
  - Step 2: The differential is \( du = 2x \, dx \).
  - Step 3a: We can rearrange the integral as \( \int_0^3 e^u \cdot (2x \, dx) \).
  - Step 3b: The remaining portion of the integrand is \( e^u \), which is just \( e^u \).
  - Step 3c: We see that \( x = 0 \) corresponds to \( u = 0 \) and \( x = 3 \) corresponds to \( u = 3^2 \).
  - Step 3d: Putting it all together gives \( \int_0^3 e^u \, du = e^u \bigg|_{u=0}^{u=9} = e^9 - 1 \)

- Evaluate \( \int_1^e \frac{(\ln(x))^2}{x} \, dx \).
  - Step 1: It might not look like any function has a 'complicated' argument, but if we think carefully we can see that the numerator is what we get if we plug in \( \ln(x) \) to the squaring function. So we try setting \( u = \ln(x) \).
  - Step 2: The differential is \( du = \frac{1}{x} \, dx \).
  - Step 3a: We can rearrange the integral as \( \int_1^e \frac{1}{x} \cdot \left( \frac{1}{x} \, dx \right) \).
  - Step 3b: The “remaining portion” of the integrand is \( \frac{(\ln(x))^2}{x} \), which is just \( u^2 \).
  - Step 3c: We see that \( x = 1 \) corresponds to \( u = \ln(1) = 0 \) and \( x = e \) corresponds to \( u = \ln(e) = 1 \).
  - Step 3d: Putting it all together gives \( \int_1^e u^2 \, du = \frac{1}{3} u^3 \bigg|_{u=0}^{u=1} = \frac{1}{3} \)

- Evaluate \( \int_0^1 x \sqrt{3x^2 + 1} \, dx \).
  - Step 1: Here we see that the square root function has the 'complicated' argument \( 3x^2 + 1 \) so we try \( u = \sqrt{3x^2 + 1} \).
  - Step 2: The differential is \( du = 6x \, dx \).
  - Step 3a: We can rearrange the integral as \( \int_0^1 \sqrt{3x^2 + 1} \cdot \frac{1}{6} \cdot (6x \, dx) \). Note that we introduced a factor of \( \frac{1}{6} \), which is okay since it's just multiplication by 1.
  - Step 3b: The “remaining portion” of the integrand is \( \sqrt{3x^2 + 1} \cdot \frac{1}{6} \), which is just \( \frac{1}{6} u^{1/2} \).
  - Step 3c: We see that \( x = 0 \) corresponds to \( u = 1 \) and \( x = 1 \) corresponds to \( u = 4 \).
  - Step 3d: Putting it all together gives \( \int_0^1 \frac{1}{6} u^{1/2} \, du = \frac{1}{6} \cdot \frac{2}{3} u^{3/2} \bigg|_{u=1}^{u=4} = \frac{7}{9} \)

- Evaluate \( \int_2^3 \frac{2x}{x^2 - 1} \, dx \).
  - Step 1: Try the denominator: \( u = x^2 - 1 \).
  - Step 2: The differential is \( du = 2x \, dx \).
• Step 3a: We can rearrange the integral as \( \int_2^3 \frac{1}{x^2-1} \cdot (2x \, dx) \).

• Step 3b: The “remaining portion” of the integrand is \( \frac{1}{x^2-1} \), which is just \( u^{-1} \).

• Step 3c: We see that \( x = 2 \) corresponds to \( u = 3 \) and \( x = 3 \) corresponds to \( u = 8 \).

• Step 3d: Putting it all together gives \( \int_3^8 u^{-1} \, du = \ln(u)|_3^8 = \ln(8) - \ln(3) \).

• Remark: It’s possible to do this one without substitution, by using partial fractions to see that \( \frac{2x}{x^2-1} = \frac{1}{x+1} + \frac{1}{x-1} \). This gives \( I = [\ln(x+1) + \ln(x-1)]|_2^3 = \ln(4) + \ln(2) - \ln(3) = \ln(8) - \ln(3) \) as before.

1.3 Integration by Parts

- The integration by parts formula states \( \int f' \cdot g \, dx = f \cdot g - \int f \cdot g' \, dx \). It is just the Product Rule, rearranged and rewritten in terms of integrals using the Fundamental Theorem of Calculus. To perform an integration by parts, all that is required is to pick a function \( f(x) \) and a function \( g(x) \) such that the product \( f'(x) \cdot g(x) \) is equal to the original integrand. Examples will make everything clear.

- Evaluate \( \int_0^1 x e^x \, dx \).

  • The integrand \( x \cdot e^x \) is an obvious product, and so we need to decide which of \( x \) and \( e^x \) should be \( f' \). Since \( x \) gets more complicated if we take its antiderivative (since we’d get \( \frac{1}{2}x^2 \)) we try \( f' = e^x \) and \( g = x \), to get \( f = e^x \) and \( g' = 1 \). Plugging into the formula gives \( \int_0^1 x e^x \, dx = x e^x|_0^1 - \int_0^1 1 \cdot e^x \, dx = (x e^x - e^x)|_0^1 = 1 \).

- Evaluate \( \int_1^e [\ln(x)]^2 \, dx \).

  • We can write \( [\ln(x)]^2 = \ln(x) \cdot \ln(x) \), but this doesn’t help unless we remember the antiderivative of \( \ln(x) \). Instead we write the integrand as \( 1 \cdot [\ln(x)]^2 \), so as to take \( f' = 1 \) and \( g = [\ln(x)]^2 \). Then we get \( f = x \) and \( g' = \frac{1}{x} \) by the Chain Rule. So plugging in will yields

\[
\int_1^e [\ln(x)]^2 \, dx = x \cdot [\ln(x)]^2|_1^e - \int_1^e x \cdot 2 \ln(x) \cdot \frac{1}{x} \, dx
\]

\[
= e - \int_1^e 2 \ln(x) \, dx
\]

\[
= e - \left[ 2x \cdot \ln x|^1_e - \int_1^e 2x \cdot \frac{1}{x} \, dx \right] \quad \text{(IBP again)}
\]

\[
= e - [2e - (2e - 2)]
\]

\[
= e - 2
\]

- Evaluate \( \int_0^1 (x^2 - 2x + 2)e^{3x} \, dx \).

  • We want \( g \) to be the thing which gets simpler when we differentiate. The polynomial \( x^2 - 2x + 2 \) gets much simpler if we differentiate it, while the exponential \( e^{3x} \) stays basically the same. So we should take \( g = x^2 - 2x + 2 \) and \( f' = e^{3x} \), so that \( g' = 2x - 2 \) and \( f = \frac{1}{3}e^{3x} \). Then integrating by parts yields
expression which we can’t evaluate directly - we have to integrate by parts again:

\[
\int_0^1 (x^2 - 2x + 2)e^{3x} \, dx = (x^2 - 2x + 2) \cdot \frac{1}{3}e^{3x} \bigg|_0^1 - \int_0^1 (2x - 2) \cdot \frac{1}{3}e^{3x} \, dx \quad \text{(IBP once)}
\]

\[
= \left( \frac{2}{3}e^3 - \frac{2}{3} \right) - \int_0^1 (2x - 2) \cdot \frac{1}{3}e^{3x} \, dx
\]

\[
= \left( \frac{2}{3}e^3 - \frac{2}{3} \right) - \left[ \left( x - 1 \right) \left( \frac{1}{9}e^{3x} \right) \bigg|_0^1 \right]
\]

\[
= \left( \frac{2}{3}e^3 - \frac{2}{3} \right) - \left[ \frac{2}{9} - \left( \frac{2}{27}e^3 - \frac{2}{27} \right) \right]
\]

\[
= \frac{20}{27}e^3 - \frac{26}{27}.
\]

- Evaluate \( \int x^3 \sin(x) \, dx \).
  - Here we just need to integrate by parts repeatedly. We get

\[
\int x^3 \sin(x) \, dx = -x^3 \cos(x) + \int 3x^2 \cos(x) \, dx \quad \text{(IBP once)}
\]

\[
= -x^3 \cos(x) + \left[ 3x^2 \sin(x) - \int 6x \sin(x) \, dx \right] \quad \text{(IBP again)}
\]

\[
= -x^3 \cos(x) + 3x^2 \sin(x) - \left[ -6x \cos(x) + \int 6 \cos(x) \, dx \right] \quad \text{(IBP again)}
\]

\[
= -x^3 \cos(x) + 3x^2 \sin(x) + 6x \cos(x) + 6 \sin(x) + C
\]

- Find \( \int x^3 \sin(x^2) \, dx \).
  - First we look for a substitution. We try the argument of the sine: \( u = x^2 \). The differential is \( du = 2x \, dx \), and we can rearrange the integral as \( \int x^2 \sin(x^2) \cdot \frac{1}{2} \cdot (2x \, dx) = \int u \sin(u) \cdot \frac{1}{2} \, du \).
  - Now we integrate by parts, to get \( -\frac{1}{2} u \cos(u) + \int \frac{1}{2} \cos(u) \, du = \frac{1}{2} [-u \cos(u) + \sin(u)] + C \).
  - Finally substitute back for \( x \) to get \( \frac{1}{2} \left[ -x^2 \cos(x^2) + \sin(x^2) \right] + C \).

1.4 Trig Sub

- Some kinds of integrals require a more clever sort of substitution to evaluate, that’s sort of ‘backwards’ from the usual way we try to do substitutions: instead of \( u = f(x) \) we try \( x = f(u) \) for some appropriate (trig function) \( f \). The idea is to use one of the basic trig relations (e.g., \( \sin^2(x) + \cos^2(x) = 1 \)) to simplify something more complicated. As always, examples make everything clear.

- Evaluate \( \int \sqrt{1 - x^2} \, dx \).
  - Traditional substitution - along the lines of \( u = 1 - x^2 \) - doesn’t work like we’d hope.
  - Instead we try \( x = \sin(u) \); then \( dx = \cos(u) \, du \).
  - So we get \( \int \sqrt{1 - \sin^2(u)} \cdot \cos(u) \, du = \int \sqrt{\cos^2(u)} \cdot \cos(u) \, du = \int \cos^2(u) \, du \).
  - Remembering \( \cos^2(u) = \frac{1 + \cos(2u)}{2} \) we can evaluate the integral to get \( \frac{u}{2} + \frac{\sin(2u)}{4} + C \).
  - Finally substitute back for \( u = \sin^{-1}(x) \) to obtain \( \frac{\sin^{-1}(x)}{2} + \frac{\sin(2\sin^{-1}(x))}{4} + C \). If desired, we can simplify this to the equivalent form \( \frac{\sin^{-1}(x)}{2} + \frac{x\sqrt{1 - x^2}}{2} + C \).
• Evaluate $\int \frac{1}{(1+x^2)^2} \, dx$.
  
  ○ This time we think of the arctangent antiderivative and try $x = \tan(u)$. Then $dx = \sec^2(u) \, du$.
  
  ○ We obtain $\int \frac{1}{(1+\tan^2(u))^2} \cdot \sec^2(u) \, du = \int \frac{1}{\sec^4(u)} \cdot \sec^2(u) \, du = \int \cos^2(u) \, du$.
  
  ○ Remembering the identity $\cos^2(u) = \frac{1+\cos(2u)}{2}$ we can evaluate the integral to get $\frac{u}{2} + \frac{\sin(2u)}{4} + C$.
  
  ○ Finally substitute back for $u = \tan^{-1}(x)$ to obtain $\tan^{-1}(x) + \frac{\sin(2\tan^{-1}(x))}{4} + C$. If desired, we can simplify this to the equivalent form $\tan^{-1}(x) + \frac{x}{2(1+x^2)} + C$.

• Evaluate $\int \frac{2-x}{\sqrt{4-x^2}} \, dx$.
  
  ○ We’d like to do the $x = \sin(u)$ substitution again but it doesn’t quite work, since then $\sqrt{4-x^2}$ isn’t nice.
  
  ○ Instead we have to try $x = 2\sin(u)$, since then $\sqrt{4-x^2} = \sqrt{4-4\sin^2(u)} = 2\cos(u)$, and we’re much happier.
  
  ○ Then we have $dx = 2\cos(u) \, du$, so we get $\int \frac{2-2\sin(u)}{2\cos(u)} \cdot 2\cos(u) \, du = \int [2-2\sin(u)] \, du = 2u + 2\cos(u) + C$.
  
  ○ Substituting back yields $2\sin^{-1}(x) + 2\cos(\sin^{-1}(x)) + C$, or equivalently, $2\sin^{-1}(x) + 2\sqrt{1-x^2} + C$.

1.5 Partial Fractions

• Generally, integrating rational functions is difficult, unless we simplify them. We do this via the partial fraction decomposition, which breaks down rational functions into simpler parts which we know how to integrate. To find the PFD, follow these steps:
  
  ○ Step 1: Factor the denominator.
  
  ○ Step 2: Find the form of the PFD:
    
    * For each term $(x+a)^n$ the PFD has terms $\frac{C_1}{x+a} + \frac{C_2}{(x+a)^2} + \cdots + \frac{C_n}{(x+a)^n}$.
    * For each non-factorable term $(x^2+ax+b)^n$, the PFD has terms $\frac{C_1x+D_1}{x^2+ax+b} + \frac{C_2x+D_2}{(x^2+ax+b)^2} + \cdots + \frac{C_nx+D_n}{(x^2+ax+b)^n}$.
  
  ○ Step 3: Solve for the coefficients $C_1, C_2, \ldots, C_n$.
    
    * The best way is to clear all denominators and then substitute ‘intelligent’ values for $x$ (e.g., the roots of the linear factors).
    * Sometimes plugging in other values of $x$ is necessary, in order to find the coefficients of the higher terms.
  
  ○ Step 4: Evaluate the integral.

• As ever, examples make everything easier.

• Evaluate $\int \frac{1}{x^2 + 3x} \, dx$.
  
  ○ We factor and get $\frac{1}{x(x+3)}$ so we want $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$.
Clear denominators so that $1 = A(x + 3) + B(x)$.
Set $x = 0$ and $x = -3$ to see that $1 = 3A$ and $1 = -3B$.
Then $\int \frac{1}{x^2 + 3x} \, dx = \int \left[ \frac{1/3}{x} - \frac{1/3}{x + 3} \right] \, dx = \frac{1}{3} \ln(x) - \frac{1}{3} \ln(x + 3) + C$.

- Evaluate $\int \frac{1}{x^3 + x^2} \, dx$.
- We factor and get $\frac{1}{x^2(x + 1)}$ so we want $\int \frac{1}{x^2(x + 1)} = A \frac{x}{x} + B \frac{x}{x + 1} + D \frac{x}{x + 1}$.
- Clear denominators so that $1 = Ax(x + 1) + B(x + 1) + Dx^2$.
- Set $x = 0$ to get $1 = B$.
- Set $x = -1$ to get $1 = D$.
- Set $x = 1$ to get $1 = 2A + 2B + D$ so that $A = -1$.
- Then $\int \frac{1}{x^3 + x^2} \, dx = \int \left[ \frac{-1}{x} + \frac{1}{x^2 + 1} + \frac{1}{x + 1} \right] \, dx = -\ln(x) - \frac{1}{x} + \ln(x + 1) + C$.

- Evaluate $\int \frac{2x}{(x + 1)(x^2 + 1)} \, dx$.
- We want $\frac{2x}{(x + 1)(x^2 + 1)} = A \frac{x}{x + 1} + Bx + D \frac{x}{x + 1}$.
- Clear denominators so that $2x = A(x^2 + 1) + (Bx + D)(x + 1)$.
- Set $x = -1$ to get $-2 = 2A$ so that $A = -1$.
- So we want $x^2 + 2x + 1 = (Bx + D)(x + 1)$ so we see that we need $B = D = 1$.
- Then $\int \frac{2x}{(x + 1)(x^2 + 1)} \, dx = \int \left[ \frac{-1}{x + 1} + \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} \right] \, dx = -\ln(x + 1) + \frac{1}{2} \ln(x^2 + 1) + \tan^{-1}(x) + C$.

1.6 Improper Integrals

- Sometimes we like to integrate to infinity, by which we mean, take the limit as a limit of integration becomes arbitrarily large. In other words, we write $\int_a^\infty f(x) \, dx$ as shorthand for $\lim_{q \to \infty} \int_q^a f(x) \, dx$.
- Other times, we like to integrate through a 'singularity' of a function - that is, through a point where a function is undefined because it blows up. [E.g., $x = 0$ on $y = 1/x$.] Again, we will write something like $\int_0^1 \frac{1}{x} \, dx$ as shorthand for $\lim_{y \to 0^+} \int_y^1 \frac{1}{x} \, dx$.
- Integrals with either of these two kinds of 'bad behaviors' are called improper integrals. (I can only assume that the terminology stems from the fact that trying to evaluate such integrals is not something done in polite company.)
- Typically we will be interested in asking (i) if the integral actually converges to a finite value, and (ii) if it does, what the value is.
- There are approximately two ways to solve problems like this:
  - Method 1: Find the indefinite integral, and then evaluate the limit. If the limit is hard, try rewriting the function to make the limit doable.
  - Method 2: Compare the integrand to another function whose integral is easier to evaluate. We are aided in the following theorem:

* Theorem (Comparison Test for Integrals): If $0 < f(x) < g(x)$ for all $x$ in the interval $[a, b]$ where one or both of $a$ and $b$ can be infinite, then if $\int_a^b f(x) \, dx$ diverges, so does $\int_a^b g(x) \, dx$. If $\int_a^b g(x) \, dx$ converges, then so does $\int_a^b f(x) \, dx$. 
* The idea behind the theorem is to say that if $0 < f(x) < g(x)$ then $\int_a^b f(x) \, dx < \int_a^b g(x) \, dx$, and so if the $f(x)$-integral goes to $\infty$ then so must the $g(x)$-integral, and conversely if the $g(x)$-integral stays finite then so must the $f(x)$-integral.

Example time!

• Evaluate $\int_0^\infty e^{-x} \, dx$.
  
  - We just evaluate to see $\int_0^\infty e^{-x} \, dx = \lim_{q \to \infty} (-e^{-x})|_{x=0}^{q} = \lim_{q \to \infty} [-e^{-q} + 1] = 1$

• Evaluate $\int_1^\infty \frac{1}{x^2 + x} \, dx$.
  
  - We do partial fractions to see $\int \frac{1}{x^2 + x} \, dx = \ln(x) - \ln(x + 1) + C$.
  - Now to take the limit as $x \to \infty$ is not so easy, unless we notice that we can rewrite the indefinite integral as $\ln\left(\frac{x}{x+1}\right)$.
  - Then we have $\int_1^\infty \frac{1}{x^2 + x} \, dx = \lim_{q \to \infty} \left[ \ln\left(\frac{x}{x+1}\right) \right]_{x=1}^{q} = \lim_{q \to \infty} \left[ \ln\left(\frac{x}{x+1}\right) - \ln\left(\frac{1}{2}\right) \right] = \ln(2)$
  - If the problem had only asked about convergence, we could have observed that $x^2 + x > x^2$ for positive $x$, so $\frac{1}{x^2 + x} < \frac{1}{x^2}$. Then the Comparison Test would have said that, because $\int_1^\infty \frac{1}{x^2} \, dx = -\frac{1}{x}|_{x=1}^{\infty} = 1$ converges, then so does $\int_1^\infty \frac{1}{x^2 + x} \, dx$. However, the Comparison Test doesn’t help us in computing the actual value.

• Determine whether the integral $\int_0^\infty \frac{1}{(x+1)(x+2)(x+3)(x+4)} \, dx$ converges.
  
  - Doing partial fractions is too much work (although it does give the correct answer, if the trick in the previous example is used repeatedly).
  - But it’s easier to notice that $\frac{1}{x+4} < \frac{1}{x+3} < \frac{1}{x+2} < \frac{1}{x+1}$.
  - Then $\int_0^q \frac{1}{(x+1)(x+2)(x+3)(x+4)} \, dx < \int_0^q \frac{1}{(x+1)^4} \, dx = \frac{1}{4}(x+1)^{-3}|_{x=0}^{q}$. As $q \to \infty$ this remains finite (in fact, it is $\frac{1}{4}$).
  - Therefore this integral [converges]

• Determine whether the integral $\int_0^1 \frac{1}{x} \, dx$ converges.
  
  - We integrate to get $\int_0^1 \frac{1}{x} \, dx = -\ln(q)$.
  - So as $q \to 0^+$ we see that this diverges to $-\infty$.
  - Therefore this integral [diverges to $-\infty$]

2 Sequences and Series

2.1 Sequences, Series, Convergence

• We say a sequence $a_1, a_2, \ldots, a_k, \ldots$ converges to 0 if the terms of the sequence eventually become arbitrarily small.