1. Exercise 2.21

(a) We start by defining some appropriate random variables.

\[ X = \text{Number of questions on exam Jane gets correct} \]

and

\[ X_i = \begin{cases} 1 & \text{if Jane gets question } i \text{ correct} \\ 0 & \text{otherwise} \end{cases} \]

for \( i = 1, 2, 3, 4 \). We are given that \( X_1, X_2, X_3, X_4 \) are independent, because it is assumed the results on different problems are independent. We know that \( X_i \sim \text{Bernoulli}(0.8) \). \( X \sim \text{Binomial}(4, 0.8) \), because the results on each problem are independent and all have the same success probability.

With these random variables, we only need to compute \( P(X \geq 3) \). This is straightforward using the Binomial(4, 0.8) PMF.

\[
P(X \geq 3) = P(X = 3) + P(X = 4) \\
= \binom{4}{3} 0.8^3 \cdot 0.2^1 + \binom{4}{4} 0.8^4 \cdot 0.2^0 \\
\approx 0.8192.
\]

(b) Now we have assumed that the event \( X_1 = 1 \).

\[
P(X \geq 3|X_1 = 1) = \frac{P(X_1 + X_2 + X_3 + X_4 \geq 3|X_1 = 1)}{P(X_1 = 1)} \\
= \frac{P(X_2 + X_3 + X_4 \geq 2, X_1 = 1)}{P(X_1 = 1)} \\
= \frac{P(X_2 + X_3 + X_4 \geq 2)}{P(X_1 = 1)} \\
= P(X_2 + X_3 + X_4 \geq 2)
\]

The sum of 3 independent Bernoulli(0.8) random variables has a Binomial(3, 0.8) distribution. So

\[
P(X \geq 3|X_1 = 1) = P(X_2 + X_3 + X_4 \geq 2) \\
= P(X_2 + X_3 + X_4 = 2) + P(X_2 + X_3 + X_4 = 3) \\
= \binom{3}{2} 0.8^2 \cdot 0.2^1 + \binom{3}{3} 0.8^3 \cdot 0.2^0 \\
\approx 0.896.
\]
2. Exercise 2.23

We begin by defining the appropriate random variables.

\[ X_i = \begin{cases} 
1 & \text{if there is an accident on day } i \\
0 & \text{if there is not an accident on day } i 
\end{cases} \]

So \( X_i \sim \text{Bernoulli}(0.05) \). We assume accidents occur independently from day to day, so this means that we are assuming \( X_1, X_2, X_3, \ldots \) are independent random variables.

(a) In terms of our random variables, this is

\[
P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 = 0, X_7 = 0) \\
= P(X_1 = 0)P(X_2 = 0)P(X_3 = 0)P(X_4 = 0)P(X_5 = 0)P(X_6 = 0)P(X_7 = 0) \\
= 0.95^7
\]

(b) If we set \( X = X_1 + X_2 + \ldots + X_{30} \) then

\[ X = \text{Number of days with accidents over the next 30 days} \]

This is what we want, because September has 30 days. We must compute \( P(X = 2) \).

\( X \) is the sum of independent Bernoulli(0.05) random variables, so \( X \sim \text{Binomial}(30, 0.05) \).

We can just use the PMF for this distribution to get

\[
P(X = 2) = \binom{30}{2} 0.05^2 \cdot 0.95^{28} \approx 0.259.
\]

(c) We are given \( X_1 = 0 \). We must compute

\[
P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \ldots + X_{11} \geq 1 | X_1 = 0) \\
= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \ldots + X_{11} \geq 1) \\
= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0)P(X_6 + \ldots + X_{11} \geq 1) \\
= 0.95^4 \cdot (1 - P(X_6 + \ldots + X_{11} = 0)) \\
= 0.95^4 \cdot (1 - 0.95^6) \\
\approx 0.2158.
\]

It is also reasonable to have interpreted the problem as meaning the first day had no accidents, and we only consider 9 more days. In that case the solution is the following.

\[
P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \ldots + X_{10} \geq 1 | X_1 = 0) \\
= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 + \ldots + X_{10} \geq 1) \\
= P(X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0)P(X_6 + \ldots + X_{10} \geq 1) \\
= 0.95^4 \cdot (1 - P(X_6 + \ldots + X_{10} = 0)) \\
= 0.95^4 \cdot (1 - 0.95^5) \\
\approx 0.184.
\]

Both answers are acceptable.
3. Exercise 2.25

We start by defining events.

\[ A = \text{The first roll is 3} \]
\[ B = \text{The second roll is 4} \]
\[ D_k = \text{The die used is } k\text{-sided, for } k = 4, 6, 12 \]

The events \( A \) and \( B \) are conditionally independent given \( D_k \). Also, \( D_4, D_6, \) and \( D_{12} \) form a partition of the sample space. In terms of these events, we want to find

\[ P(D_6|AB) = \frac{P(AB|D_6)P(AB)}{P(AB)}. \]

We can compute the denominator separately, as it is the largest part of the computation. So we can use the law of total probability to get

\[ P(AB) = P(AB|D_4)P(D_4) + P(AB|D_6)P(D_6) + P(AB|D_{12})P(D_{12}) \]
\[ = P(A|D_4)P(B|D_4)P(D_4) + P(A|D_4)P(B|D_6)P(D_6) + P(A|D_4)P(B|D_{12})P(D_{12}) \]
\[ = \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{12} \cdot \frac{1}{3} \]
\[ = \left( \left( \frac{1}{4} \right)^2 + \left( \frac{1}{6} \right)^2 + \left( \frac{1}{12} \right)^2 \right) \cdot \frac{1}{3} \]

From here, we use the Bayes formula to finish the problem.

\[ P(D_6|AB) = \frac{P(AB|D_6)P(AB)}{P(AB)} \]
\[ = \frac{\left( \frac{1}{4} \right)^2 \cdot \frac{1}{3}}{\left( \left( \frac{1}{4} \right)^2 + \left( \frac{1}{6} \right)^2 + \left( \frac{1}{12} \right)^2 \right) \cdot \frac{1}{3}} \]
\[ = \frac{2}{7} \]

4. Exercise 2.28

(b) We define

\[ X_b = \text{Number of games with at least one ace} \]

The number of aces received in each game are independent. So

\[ X_b \sim \text{Binomial}(50, p) \]

where

\[ p = \text{Probability of receiving at least one ace in a game}. \]

Using part (a), we have

\[ p = P(X_a \geq 1) = 1 - P(X_a = 0) = 1 - \left( \frac{48}{52} \right)^{50}. \]
(c) We define
\[ X_c = \text{Number of games with cards of all the same suit}. \]

The cards received in each game are independent. So \[ X_c \sim \text{Binomial}(50, p) \]
where \[ p = \text{Probability of receiving cards of all the same suit}. \]

This is a more direct probability calculation:
\[ p = \frac{4}{\binom{52}{13}}. \]

5. **Exercise 2.47**

We start by defining appropriate random variables.
\[ X = \text{Number of patients who have a successful trial} \]
The trials between patients are independent and all have success probability \( p \), so \( X \sim \text{Binomial}(80, p) \).

We also define random variables for the outcomes of individual patients.
\[ X_j = \begin{cases} 1 & \text{if the } i\text{-th person has a successful trial} \\ 0 & \text{if the } i\text{-th person does not have a successful trial} \end{cases} \]
This is for \( j = 1, 2, \ldots, 80 \). Note that \( X_j \sim \text{Bernoulli}(p) \), and \( X_1, \ldots, X_{80} \) are independent. We also have
\[ X = X_1 + X_2 + \cdots + X_{79} + X_{80}. \]

For simplicity, we can assume that our two friends are the 79th and 80th persons in the medical trial. So we want to compute the probability below.
\[
P(X_{79} = 1, X_{80} = 1 | X = 55) = \frac{P(X_{79} = 1, X_{80} = 1, X = 55)}{P(X = 55)} \\
= \frac{P(X_{79} = 1, X_{80} = 1, X_1 + \cdots + X_{78} = 53)}{P(X = 55)} \\
= \frac{P(X_{79} = 1, X_{80} = 1)P(X_1 + \cdots + X_{78} = 53)}{P(X = 55)} \\
= \frac{p^2 \cdot \binom{78}{53} p^{53} (1-p)^{25}}{\binom{80}{55} p^{55} (1-p)^{25}} \\
= \frac{\binom{78}{53}}{\binom{80}{55}}
\]

6. **Exercise 2.57**
(a) We start by defining appropriate events and random variables.

\[ C_1 = \text{The first component is working} \]
\[ C_2 = \text{The second component is working} \]
\[ X_1 = \text{The number of working elements in the first component} \]
\[ X_2 = \text{The number of working elements in the second component} \]

Because elements work independently of one another and with the same probability, we know that \( X_1 \sim \text{Binomial}(8, 0.95) \) and \( X_2 \sim \text{Binomial}(4, 0.90) \). This also tells us that \( C_1 \) and \( C_2 \) are independent.

The connection between these events and random variables is

\[ C_1 = \{ X_1 \geq 6 \} \]
\[ C_2 = \{ X_2 \geq 3 \} \]

The system functions if both components are working. So we want that probability

\[ P(C_1 C_2) = P(C_1) P(C_2), \]

because the elements operate independently.

\[ P(C_1) = P(X_1 \geq 6) = \binom{8}{6} 0.95^6 \cdot 0.05^2 + \binom{8}{7} 0.95^7 \cdot 0.05^1 + \binom{8}{8} 0.95^8 \cdot 0.05^0 \approx 0.9942 \]

\[ P(C_2) = P(X_2 \geq 3) = \binom{4}{3} 0.90^3 \cdot 0.10^1 + \binom{4}{4} 0.90^4 \cdot 0.10^0 \approx 0.9477 \]

Now we just take this product to get our final answer.

\[ P(C_1 C_2) = P(C_1) P(C_2) = P(X_1 \geq 6) P(X_2 \geq 3) \approx 0.9942 \cdot 0.9477 \approx 0.9422. \]

(b) This is a direct computation in which we can use Bayes’ formula.

\[ P(C_2^c | (C_1 C_2)^c) = \frac{P((C_1 C_2)^c | C_2^c) P(C_2^c)}{P((C_1 C_2)^c)} = \frac{1 \cdot (1 - 0.9477)}{1 - 0.9422} \approx 0.9048. \]

7. Exercise 2.67

This is a direct computation.

\[ P(X = n + k | X > n) = \frac{P(X = n + k, X > n)}{P(X > n)} = \frac{P(X = n + k)}{P(X > n)} = \frac{(1 - p)^{n+k-1}p}{(1 - p)^n} = (1 - p)^{k-1}p = P(X = k). \]
8. Exercise 2.74

We start by defining events.

\[ D = \text{Steve is a drug user} \]
\[ T_1 = \text{Steve fails the first test} \]
\[ T_2 = \text{Steve fails the second test}. \]

We are given that

\[ P(T_k|D) = 0.99 \]
\[ P(T_k|D^c) = 0.02 \]
\[ P(D) = 0.01. \]

Note that \( T_1, T_2 \) are assumed to be conditionally independent given \( D \) (or \( D^c \)).

(a) First we find \( P(D|T_1) \) using the Bayes formula.

\[
P(D|T_1) = \frac{P(T_1|D)P(D)}{P(T_1|D)P(D) + P(T_1|D^c)P(D^c)}
\]
\[
= \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.02 \cdot 0.99}
\]
\[
= \frac{1}{3}.
\]

(b) Now we find \( P(T_2|T_1) \).

\[
P(T_2|T_1) = \frac{P(T_1T_2)}{P(T_1)}
\]
\[
= \frac{P(T_1T_2|D)P(D) + P(T_1T_2|D^c)P(D^c)}{P(T_1|D)P(D) + P(T_1|D^c)P(D^c)}
\]
\[
= \frac{0.99^2 \cdot 0.01 + 0.02^2 \cdot 0.99}{0.99 \cdot 0.01 + 0.02 \cdot 0.99}
\]
\[
\approx 0.3433
\]

(c) Finally, we find \( P(D|T_1T_2) \) using the Bayes formula.

\[
P(D|T_1T_2) = \frac{P(T_1T_2|D)P(D)}{P(T_1T_2|D)P(D) + P(T_1T_2|D^c)P(D^c)}
\]
\[
= \frac{0.99^2 \cdot 0.01}{0.99^2 \cdot 0.01 + 0.02^2 \cdot 0.99}
\]
\[
= \frac{99}{103} \approx 0.9612.
\]