Math 431, Homework 8 Solutions

1. Exercise 4.12

Following the hint that $E[X^2] = \frac{2}{\lambda}$, we have

$$E[X^2] = \int_0^\infty x^3 \lambda e^{-\lambda x} \, dx$$

\[= \left[ x^3 \cdot -\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty - \int_0^\infty 3x^2 \cdot \lambda e^{-\lambda x} \, dx \]

\[= [0 - 0] + 3 \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} \, dx \]

\[= 3E[X^2] \]

\[= \frac{6}{\lambda^2}. \]

2. Exercise 4.14 We start by defining the appropriate random variable.

$T = \text{Length of lightbulbs lifetime.}$

The $T \sim \text{Exp}(\lambda)$ where $\lambda = 1/E[T] = 1/1000$.

(a) We want to find $P(T > 2000)$. Using the CDF for the exponential distribution we get

$$P(T > 2000) = 1 - F_T(2000) = e^{-\frac{1}{1000} \cdot 2000} = e^{-2}. \]

(b) We want to find $P(T > 2000 \mid T > 500)$. We use the memoryless property in this case.

$$P(T > 2000 \mid T > 500) = P(T > 500 + 1500 \mid T > 500) = P(T > 1500) = e^{-3/2}. \]

3. Exercise 4.50

We want to compute $P(T > 7 + 3 \mid T > 7)$. This is easiest using the memoryless property of the exponential distribution.

$$P(T > 7 + 3 \mid T > 7) = P(T > 3) = e^{-\frac{1}{3}}. \]

More generally for an additional $x$ hours, we have

$$P(T > 7 + x \mid T > 7) = P(T > x) = e^{-\frac{1}{3} \cdot x} = e^{-x/3}. \]

4. Exercise 5.2
(a) We need $EX$ and $EX^2$ to find the mean and variance of $X$. So we start by computing the first two derivatives of the MGF.

\[
M'_X(t) = \frac{-4}{3} \cdot e^{-4t} + \frac{5}{6} \cdot e^{5t}
\]
\[
M''_X(t) = \frac{16}{3} \cdot e^{-4t} + \frac{25}{6} \cdot e^{5t}
\]

Now we use the general formula $EX^n = M^{(n)}_X(0)$ to get

\[
EX = \frac{-4}{3} + \frac{5}{6} = \frac{-3}{6} = \frac{-1}{2}
\]
\[
EX^2 = \frac{16}{3} + \frac{25}{6} = \frac{57}{6} = \frac{19}{2}.
\]

Thus,

\[
EX = -\frac{1}{2}
\]
\[
Var(X) = EX^2 - (EX)^2 = \frac{19}{2} - \left(-\frac{1}{2}\right)^2 = \frac{19}{2} - \frac{1}{4} = \frac{37}{4}
\]

(b) We can express the MGF as

\[
M_X(t) = e^{t \cdot 0} \frac{1}{2} + e^{t \cdot -4} \frac{1}{3} + e^{t \cdot 5} \frac{1}{6}.
\]

So the PMF is

\[
\begin{array}{c|c|c|c|}
k & -4 & 0 & 5 \\
\hline
px(k) & \frac{3}{3} & \frac{2}{2} & \frac{1}{6} \\
\end{array}
\]

Computing the first two moments using the PMF gives

\[
EX = -4 \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} + 5 \cdot \frac{1}{6} = -\frac{4}{3} + \frac{5}{6} = \frac{-1}{2}
\]
\[
EX^2 = (-4)^2 \cdot \frac{1}{3} + 0^2 \cdot \frac{1}{2} + 5^2 \cdot \frac{1}{6} = \frac{16}{3} + \frac{25}{6} = \frac{57}{6} = \frac{19}{2}.
\]

The first two moments are the same as our solution to the previous part, so the mean and variance will be equal as well.

5. Exercise 5.3

We start with the definition of the MGF.

\[
M_X(t) = E[e^{tx}] = \int_{0}^{1} e^{tx} \cdot \frac{1}{1-t} \, dx
\]
\[
= \left[ \frac{1}{t} e^{tx} \right]_{x=0}^{1} = \frac{1}{t} e^t - 1.
\]
This computation is valid for all values of $t$ except $t = 0$. We must find $M_X(0)$ through different methods. Fortunately, this is quite direct:

$$M_X(0) = E[e^{0 \cdot X}] = E[1] = 1$$

To summarize,

$$M_X(t) = \begin{cases} \frac{e^t - 1}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

6. **Exercise 5.6**

The possible values $Y$ can take are

$$\text{Range}(Y) = \{(-1 - 1)^2, (0 - 1)^2, (2 - 1)^2, (4 - 1)^2\} = \{4, 1, 1, 9\} = \{1, 4, 9\}.$$  

So we find $p_Y(k) = P(Y = k)$ for $k = 1, 4, 9$.

\[
\begin{align*}
p_Y(1) &= P((X - 1)^2 = 1) = P(X = 0) + P(X = 1) = \frac{4}{14} = \frac{2}{7} \\
p_Y(4) &= P((X - 1)^2 = 4) = P(X = -1) + P(X = 3) = \frac{1}{7} \\
p_Y(9) &= P((X - 1)^2 = 9) = P(X = -2) + P(X = 4) = \frac{4}{7}
\end{align*}
\]

7. **Exercise 5.7**

We begin by computing the cdf of $Y$. First note that because the range of $X$ is $[0, +\infty)$, the range of $Y$ is $(-\infty, +\infty)$ (we can ignore the case in which $Y = -\infty$, as this corresponds to $X = 0$ which occurs with probability 0). For any $t \in \mathbb{R}$, we have

$$F_Y(t) = P(Y \leq t) = P(\ln(X) \leq t) = P(X \leq e^t) = 1 - e^{-\lambda e^t}.$$  

By differentiating, we find that the probability density function of $Y$ is given by

$$f_Y(t) = F_Y'(t) = \lambda e^{-\lambda e^t} e^t = \lambda e^{t-\lambda e^t},$$

for any $t \in \mathbb{R}$. 