Math 431: HW08, Part 2 Solutions

1. Exercise 5.8

The range of $X$ is $[-1, 2]$, so the range of $Y$ is $[0, 4]$. Thus, we know that $f_Y(t) = 0$ for $t \notin [0, 4]$.

For $t \in [0, 4]$ we have

$$F_Y(t) = P(Y \leq t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}),$$

and we have to consider multiple cases. For $t \in [0, 1]$ we have $-\sqrt{t} \geq -1$ and so

$$F_Y(t) = P(Y \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) = \frac{\sqrt{t} - (-\sqrt{t})}{2 - (-1)} = \frac{2\sqrt{t}}{3} \implies f_Y(t) = F_Y'(t) = \frac{1}{3} t^{-1/2}.$$  

For $t \in [1, 4]$ we have

$$F_Y(t) = P(Y \leq t) = P(-1 \leq X \leq \sqrt{t}) = \frac{\sqrt{t} + 1}{3} \implies f_Y(t) = \frac{1}{6} t^{-1/2}.$$  

Putting this together yields:

$$f_Y(t) = \begin{cases} 0 & t \notin [0, 4] \\ \frac{1}{3} t^{-1/2} & 0 \leq t \leq 1 \\ \frac{1}{6} t^{-1/2} & 1 < t < 4 \end{cases}$$

Note that there is another way to proceed from equation (*). We have

$$F_Y(t) = P(Y \leq t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t}).$$

We can get the pdf of $Y$ by differentiating $F_Y$, and since $F_X' = f_X$, we get

$$f_Y(t) = F_Y'(t) = \frac{d}{dt} \left( F_X(\sqrt{t}) - F_X(-\sqrt{t}) \right) = \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} f_X(-\sqrt{t}).$$

The pdf $f_X(x)$ is equal to $\frac{1}{3}$ if $-1 \leq x \leq 2$ and 0 otherwise. Thus if $0 < t < 1$ then

$$f_Y(t) = \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} f_X(-\sqrt{t}) = \frac{1}{2\sqrt{t}} \frac{1}{3} + \frac{1}{2\sqrt{t}} \frac{1}{3} = \frac{1}{3\sqrt{t}}$$

and if $1 < t < 4$ then

$$f_Y(t) = \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} f_X(-\sqrt{t}) = \frac{1}{2\sqrt{t}} \frac{1}{3} + 0 = \frac{1}{6\sqrt{t}}.$$  

This gives the same case-defined function that we have found before.
2. Exercise 5.11

(a) By definition of moment generating function and by applying Fact 3.26, we have

\[ M_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx \]

\[ = \int_{0}^{+\infty} e^{tx} xe^{-x} dx \]

\[ = \int_{0}^{+\infty} xe^{x(t-1)} dx \]

Now, we have two cases:

- If \( t \geq 1 \) then \( x(t-1) \geq 0 \) for any \( x > 0 \), which implies that \( xe^{x(t-1)} \geq xe^0 = x \) for any \( x > 0 \). It follows that

\[ M_X(t) = \int_{0}^{+\infty} xe^{x(t-1)} dx \geq \int_{0}^{+\infty} x dx = \left[ \frac{x^2}{2} \right]_{0}^{+\infty} = +\infty \]

Therefore, if \( t \geq 1 \) then \( M_X(t) = \infty \).

- If \( t < 1 \) then, by integrating by parts (we can use \( u' = e^{x(t-1)} \) and \( v = x \)),

\[ M_X(t) = \int_{0}^{+\infty} xe^{x(t-1)} dx \]

\[ = \left[ \frac{x e^{x(t-1)}}{t-1} \right]_{0}^{+\infty} - \int_{0}^{+\infty} \frac{e^{x(t-1)}}{t-1} dx \]

\[ = \left[ \frac{x e^{x(t-1)}}{t-1} \right]_{0}^{+\infty} - \left[ \frac{e^{x(t-1)}}{(t-1)^2} \right]_{0}^{+\infty} \]

To evaluate the last expressions we note that since \( t-1 < 0 \), both \( \frac{x e^{x(t-1)}}{t-1} \) and \( \frac{e^{x(t-1)}}{(t-1)^2} \) converge to zero as \( x \to \infty \). Thus

\[ M_X(t) = -0 \cdot \frac{e^{0(t-1)}}{t-1} + \frac{e^{0(t-1)}}{(t-1)^2} = \frac{1}{(1-t)^2}. \]

Therefore,

\[ M_X(t) = \begin{cases} \frac{1}{(1-t)^2} & \text{if } t < 1 \\ +\infty & \text{if } t \geq 1 \end{cases} \]

(b) The moment generating function \( M_X(t) \) is finite in an interval around 0, therefore we can apply the fact

\[ E[X^n] = M_X^{(n)}(0). \]

Now, if we try to compute the \( n \)th derivative of \( M_X(t) \), we might recognize a familiar pattern. Indeed, if we consider a random variable \( Y \sim \text{Exp}(1) \), we have for any \( t < 1 \)

\[ M_X(t) = \frac{1}{(1-t)^2} = \frac{d}{dt} \left( \frac{1}{1-t} \right) = M_Y(t) \]
Therefore,

\[ E[X^n] = M_X^{(n)}(0) = M_Y^{(n+1)}(0) = E[Y^{n+1}] = (n + 1)!1^{-n-1} = (n + 1)!, \]

where we made use of the fact (proven in class) that the \( n \)th moment of an exponential random variable with parameter \( \lambda > 0 \) is \( n!\lambda^{-n} \).

Alternatively, we can explicitly compute the \( n \)th derivative of \( M_X(t) \). For \( t < 1 \) we have

\[
\begin{align*}
M_X'(t) &= \frac{2(1-t)}{(1-t)^4} = \frac{2}{(1-t)^3} \\
M_X''(t) &= \frac{2 \cdot 3(1-t)^2}{(1-t)^6} = \frac{2 \cdot 3}{(1-t)^4} \\
M_X'''(t) &= \frac{2 \cdot 3 \cdot 4(1-t)^3}{(1-t)^8} = \frac{2 \cdot 3 \cdot 4}{(1-t)^5}
\end{align*}
\]

So each time to obtain \( M_X^{(j)}(t) \) we multiply \( M_X^{(j-1)}(t) \) by \( j + 1 \) and divide it by \((1 - t)\). This can be readily seen by the fact that

\[
\frac{d}{dt} \left( \frac{1}{(1-t)^{j+1}} \right) = \frac{(j + 1)(1-t)^j}{(1-t)^{2j+2}} = \frac{j + 1}{(1-t)^{j+2}}.
\]

Therefore, for \( t < 1 \) we have

\[ M_X^{(n)}(t) = \frac{(n + 1)!}{(1-t)^{n+2}}. \]

Hence,

\[ E[X^n] = M_X^{(n)}(0) = (n + 1)! \]

3. Exercise 5.16

(a) This is a direct computation.

\[ E[X^n] = \int_0^1 x^n \, dx = \left[ \frac{x^{n+1}}{n + 1} \right]_0^1 = \frac{1}{n + 1} \]

(b) From 5.3 we have

\[ M_X(t) = \begin{cases} 
\frac{e^{t-1}}{t} & \text{if } t \neq 0 \\
1 & \text{otherwise}
\end{cases} \]
Expanding $e^t$ as its Taylor series gives

$$M_X(t) = \frac{1}{t}(e^t - 1)$$

$$= \frac{1}{t} \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} - 1 \right)$$

$$= \frac{1}{t} \left( \sum_{n=1}^{\infty} \frac{t^n}{n!} \right)$$

$$= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(n + 1)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n + 1} \cdot \frac{t^n}{n!}.$$

Note that this computation is for $t \neq 0$. However, the formula is also valid for $t = 0$:

$$M_X(0) = 1 = \sum_{n=0}^{\infty} \frac{0^n}{n!}.$$

So the Taylor series found above is valid for all real values of $t$.

So we have that the $n$-th coefficient of the MGF's Taylor series is

$$\frac{1}{n + 1} = E[X^n],$$

the same as the moments found in (a).

4. **Exercise 5.21**

We approach this directly.

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}]$$

$$= E[e^{(at)X}e^{tb}]$$

$$= e^{tb}M_X(at).$$