Math 431: Homework 9 Solutions

1. Exercise 6.2

(a) To find the PMF of $X$ we compute the row sums. This gives

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_X(k)$</td>
<td>$\frac{1}{15}$</td>
<td>$\frac{2}{15}$</td>
<td>$\frac{3}{30}$</td>
</tr>
</tbody>
</table>

To find the PMF of $Y$ we compute the column sums. This gives

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_Y(k)$</td>
<td>$\frac{6}{30}$</td>
<td>$\frac{6}{30}$</td>
<td>$\frac{5}{15}$</td>
<td>$\frac{8}{30}$</td>
</tr>
</tbody>
</table>

(b) We add the appropriate terms that satisfy the inequality.

$$P(X + Y^2 \leq 2) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 2, Y = 0)$$

$$= \frac{1}{15} + \frac{1}{15} + \frac{1}{10}$$

$$= \frac{7}{30}.$$

2. Exercise 6.5

(a) First, note that

$$xy + y^2 \geq xy \geq 0$$

for all $x, y \geq 0$. So the first criterion for a PDF is satisfied. Next,

$$\int_0^1 \int_0^1 \frac{12}{7} (xy + y^2) \, dx \, dy = \frac{12}{7} \int_0^1 \left( \frac{y}{2} + y^2 \right) \, dy$$

$$= \frac{12}{7} \left[ \frac{1}{4} + \frac{1}{3} \right]$$

$$= \frac{12}{7} \cdot \frac{7}{12} = 1$$

So the second criterion for a PDF is satisfied.

(b) To find $f_X$ we integrate the joint PDF over $y$. For $0 \leq x \leq 1$ we have

$$f_X(x) = \int_0^1 \frac{12}{7} (xy + y^2) \, dy$$

$$= \frac{12}{7} \left( \frac{x}{2} + \frac{1}{3} \right).$$
For all other values, the PDF is 0. To summarize,

\[ f_X(x) = \begin{cases} \frac{12}{7} \left( \frac{x}{2} + \frac{3}{4} \right) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \]

To find \( f_Y \) we integrate the joint PDF over \( x \). For \( 0 \leq y \leq 1 \) we have

\[ f_Y(y) = \int_0^1 \frac{12}{7} (xy + y^2) \, dx \\
= \frac{12}{7} \left( \frac{y^2}{2} + y^2 \right). \]

For all other values, the PDF is 0. To summarize,

\[ f_Y(y) = \begin{cases} \frac{12}{7} \left( \frac{y}{2} + y^2 \right) & : 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases} \]

(c) \[ P(X < Y) = \int_0^1 \int_y^1 \frac{12}{7} (xy + y^2) \, dx \, dy \\
= \frac{12}{7} \int_0^1 \frac{y^3}{2} + y^3 \, dy \\
= \frac{12}{7} \cdot \frac{3}{2} \cdot 1 \\
= \frac{9}{14}. \]

(d) We compute directly.

\[ \int_0^1 \int_y^1 x^2 y \cdot \frac{12}{7} (xy + y^2) \, dx \, dy = \frac{12}{7} \int_0^1 \frac{y^2}{4} + \frac{y^3}{3} \, dy \\
= \frac{12}{7} \left( \frac{1}{12} + \frac{1}{12} \right) \\
= \frac{2}{7}. \]

3. Exercise 6.9

We know that \( X \sim \text{Bin}(3, 1/2) \), because it counts the number of heads in three independent fair coin tosses. So we have the PMF

\[ p_X(j) = \binom{3}{j} \left( \frac{1}{2} \right)^j \left( \frac{1}{2} \right)^{3-j} = \frac{1}{8} \binom{3}{j} \] for \( j = 0, 1, 2, 3 \).

For \( Y \) we have a uniform distribution on all outcomes for a six-sided die roll. This gives the PMF

\[ p_Y(k) = \frac{1}{6} \] for \( k = 1, 2, 3, 4, 5, 6 \).

\( X \) and \( Y \) are independent, so their joint PMF is given by

\[ p_{X,Y}(j, k) = p_X(j)p_Y(k) = \binom{3}{j} \frac{1}{8} \cdot \frac{1}{6} = \frac{1}{48} \binom{3}{j} \] for \( 0 \leq j \leq 3, 1 \leq k \leq 6 \).

For all other values of \( j \) and \( k \), the PMF is 0.
4. Exercise 6.12

First we find the marginal PDF of $X$. For $x > 0$ we have

$$f_X(x) = \int_0^\infty 2e^{-(x+2y)} \, dy$$

$$= e^{-x} [e^{-2y}]_0^\infty$$

$$= e^{-x}.$$ 

If $x \leq 0$ then $f(x, y) = 0$. The PDF of $X$ is

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now we find the marginal PDF of $Y$. For $y > 0$ we have

$$f_Y(y) = \int_0^\infty 2e^{-(x+2y)} \, dx$$

$$= 2e^{-2y} [e^{-x}]_0^\infty$$

$$= 2e^{-2y}.$$ 

If $y \leq 0$ then $f(x, y) = 0$. The PDF of $Y$ is

$$f_Y(y) = \begin{cases} 2e^{-2y} & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We see that $X$ and $Y$ are independent, because

$$f_X(x)f_Y(y) = e^{-x} \cdot 2e^{-2y} = 2e^{-(x+2y)} = f(x, y).$$

5. Exercise 6.22

It is more straightforward to think in terms of the interpretation of the multinomial distribution. $X_1$ gives the number of trials that have result 1. $X_2$ gives the number of trials that have result 2. So $X_1 + X_2$ gives the number of trials out of $n$ that have result 1 or result 2. For any given trial, the probability of result 1 or result 2 is $p_1 + p_2$. As the trials are independent and there are a fixed number of them, we get that $X_1 + X_2 \sim \text{Bin}(n, p_1 + p_2)$. 