Large time behavior of level-set mean curvature flow equations with driving and source terms

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The following is a figure I got from the internet that describes various crystal growth models.
Our main focus in this talk is model (b), a type of birth and spread model, and its properties. It is worth noting that this kind of growth does appear in real experiments. Basically, we look at crystal growth in a supersaturated solution. Here is a sketch of the growth model again.
Main questions of interest

We are interested in the following points

1. How to describe this birth and spread model in PDE terms?
2. Study asymptotic growth speed of the crystal, that is, asymptotic speed of solution to the corresponding PDE.
3. Study finer properties of solutions (e.g., large time behavior).
The crystal grows in both **vertical and horizontal directions.**

- The vertical direction growth is stimulated by a **nucleation** (a source term $f(x)$).
- The horizontal one is given by a surface evolution, which is described by the **mean curvature with unit constant force.**
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When being viewed in discrete time steps (each time step is $\tau > 0$), the growth is simultaneously switched between vertical (V) and horizontal (H) growths.

$$V \rightarrow H \rightarrow V \rightarrow H \rightarrow V \rightarrow H \rightarrow \cdots$$
The source term $f : \mathbb{R}^n \to \mathbb{R}$ is given, which has compact support. Typically, $f$ can be a characteristic function of a compact set $E$ ($f = 1_E$), which is actually quite hard (Giga-Mitake-Tran 2015). Or, a simpler case is when $f \in C^1_c(\mathbb{R}^n)$ (Giga-Mitake-Ohtsuka-Tran 2018), which can be viewed as an approximation of $1_E$.

The vertical growth is quite simple. If we start with initial data $v(x, 0) = \phi(x)$, we can write

\[
\begin{cases}
    v_t = f & \text{for } t \in (0, \tau), \\
    v(x, 0) = \phi(x)
\end{cases}
\]

after time $\tau$, it is clear that $v(x, \tau) = \phi(x) + \tau f(x)$.
Horizontal growth

After a step of vertical growth, we achieve a new function $\psi(x)$ (which, according to the above slide, $\psi(x) = v(x, \tau) = \phi(x) + \tau f(x)$). We now let each of the level set of $\psi(x)$ evolves with normal velocity $V = \kappa + 1$, where $\kappa$ is the mean curvature. The philosophy is simple: each level set can be thought of as a pancake (or a cookie). If the pancake is too small, it will be dissolved into the solution. On the other hand, if it is big enough, it will grow by having adatoms attached to its boundary. By noting that unit normal vector to each level set is $n = \frac{Dw}{|Dw|}$, we write

\[
(H) \quad \begin{cases} 
    w_t = \left( \text{div} \left( \frac{Dw}{|Dw|} \right) + 1 \right) |Dw| & \text{for } t \in (0, \tau), \\
    w(x, 0) = \psi(x)
\end{cases}
\]
As noted above, we switch between (V) and (H), each of which happens in a time interval $\tau$:

$$V \rightarrow H \rightarrow V \rightarrow H \rightarrow V \rightarrow H \rightarrow \cdots$$

We then let $\tau \rightarrow 0$ and use the Trotter-Kato product formula to get our main PDE

(PDE) \[ \begin{aligned}
    & u_t - \left( \text{div}\left( \frac{Du}{|Du|} \right) + 1 \right) |Du| = f(x) \\
    & u(x, 0) = u_0(x)
\end{aligned} \]

in $\mathbb{R}^n \times (0, \infty)$, on $\mathbb{R}^n$.

Here, $u_0(x)$ denotes the initial height of the crystal at location $x$. The unknown is $u(x, t)$, which denotes the height of the crystal at location $x$ at time $t \geq 0$. 
Preliminaries about our PDE

\[
\begin{aligned}
&\left\{ 
  u_t - \left( \text{div} \left( \frac{Du}{|Du|} \right) + 1 \right) |Du| = f(x) \\
  &\quad \text{in } \mathbb{R}^n \times (0, \infty), \\
  &\quad u(x, 0) = u_0(x) \quad \text{on } \mathbb{R}^n.
\end{aligned}
\]

For simplicity, we only consider \( f \in C^1_c(\mathbb{R}^n) \) in this talk. By classical theory (Evans-Spruck, Chen-Giga-Goto), there exists a unique viscosity solution \( u \) to (PDE).

Modern viewpoint: to go further to understand various properties of \( u \) such as asymptotic growth speed, large time behavior, etc.
A question of great interests to both scientists and mathematicians is about the asymptotic growth speed of the crystal, that is,

$$\lim_{t \to \infty} \frac{u(x, t)}{t} = ???$$

**Theorem (GMT 2015, GMOT 2018)**

*For given $f \in C^1_c(\mathbb{R}^n, [0, \infty))$ and $u_0 \in \text{BUC}(\mathbb{R}^n)$, there exists $c = c_f \in \mathbb{R}$ such that*

$$\lim_{t \to \infty} \frac{u(x, t)}{t} = c$$

*Moreover, $c$ is independent of $u_0$.***
Some ideas on proving existence of asymptotic speed

By simple deduction, we only need to consider \( u_0 \equiv 0 \). We plan to

- Keep track with the height of the crystal at each time \( t \geq 0 \), that is, we care about \( t \mapsto M(t) = \sup_{x \in \mathbb{R}^n} u(x, t) \).
- Obtain regularity of \( u \): \( u \) is globally Lipschitz in \( (x, t) \), that is,

\[
\|Du\|_{L^\infty(\mathbb{R}^n \times [0, \infty))} + \|u_t\|_{L^\infty(\mathbb{R}^n \times [0, \infty))} \leq C
\]

I will skip the second claim, whose proof is based on the classical Bernstein method. Basically, the point is that if we are able to obtain that

\[
\lim_{t \to \infty} \frac{M(t)}{t}
\]

exists, then we combine it with the second claim to get the desired result.
$t \mapsto M(t)$ is subadditive

We now show that $t \mapsto M(t)$ is subadditive, that is, for $s, r \geq 0$,

\[ M(s + r) \leq M(s) + M(r). \]

Indeed, it is clear by definition that $u(x, s) \leq M(s)$ for all $x \in \mathbb{R}^n$.

Set $v(x, t) = u(x, s + t) - M(s)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Then $v$ also solves (PDE) with initial condition $v(x, 0) = u(x, s) - M(s) \leq 0 = u_0(x)$.

By the comparison principle for (PDE), $v \leq u$. In particular, $v(x, r) = u(x, r + s) - M(s) \leq u(x, r)$. Hence, subadditivity holds.
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We then have that

\[
\lim_{t \to \infty} \frac{M(t)}{t} = \inf_{s > 0} \frac{M(s)}{s} =: c.
\]
Completion of the proof

Recall that \( f \in C^1_c(\mathbb{R}^n) \), so there is \( R_0 > 0 \) such that \( \text{supp}(f) \subset B(0, R_0) \). There is a step that I skip here, but it is not hard to show that for any \( T > 0 \), \( \max_{\mathbb{R}^n \times [0, T]} u \) is obtained at a point \((x_T, s_T) \in B(0, R_0) \times [0, T]\).

Roughly speaking, max values are always attained in \( B(0, R_0) \). So let’s assume \( M(t) = \sup_{x \in \mathbb{R}^n} u(x, t) = u(x_t, t) \) for some \( x_t \in B(0, R_0) \).

Now, fix \( R > 0 \). Then, for any \( |x| \leq R \),

\[
|u(x, t) - u(x_t, t)| \leq C|x - x_t| \leq C(R + R_0)
\]

\[
\Rightarrow \left| \frac{u(x, t)}{t} - \frac{M(t)}{t} \right| \leq \frac{C(R + R_0)}{t}
\]

\[
\Rightarrow \lim_{t \to \infty} \frac{u(x, t)}{t} = c \quad \text{uniformly for } x \in B(0, R).
\]
Some remarks

(i) Our ideas are quite general, and in fact, we also obtain large time average for solutions to fully nonlinear parabolic PDEs

\[ u_t + F(Du, D^2u) = f(x) \quad \text{in } \mathbb{R}^n \times (0, \infty). \]

(ii) The use of subadditivity is quite natural and simple.

(iii) Although we have the existence of \( c = c_f = \lim_{t \to \infty} \frac{u(x,t)}{t} \), we would like to investigate further this quantity and its dependence on \( f \).

(iv) Further finer asymptotic behavior (or next term in the asymptotic expansion): \( u(x, t) - ct \to ??? \).
More on $c = c_f$

In general, it is quite hard to investigate further properties of $c = c_f$ and its clear dependence on $f$. This is our long term project, and we have obtained some first results. Let us now focus only on the radially symmetric setting.

**Theorem (GMOT 2018)**

Assume that $f(x) = \tilde{f}(|x|)$ for some $\tilde{f} \in C_c^1([0, \infty), [0, \infty))$. Then

$$c = c_f = \max_{|x| \geq n-1} f(x) = \max_{r \geq n-1} \tilde{f}(r).$$

Note that we can assume $u_0 \equiv 0$. 
Some ideas

Search for radially symmetric solution \( u(x, t) = \phi(|x|, t) \). Then \( \phi \) solves

\[
\phi_t - \frac{n-1}{r} \phi_r - |\phi_r| = \tilde{f}(r) \text{ in } (0, \infty) \times (0, \infty), \quad \phi(0, t) \equiv 0.
\]

This is a first-order Hamilton-Jacobi equation with Hamiltonian
\( H(p, r) = -\frac{n-1}{r} p - |p| - \tilde{f}(r) \), which is concave in \( p \), but singular as \( r \to 0 \). We can obtain the result by carefully analyzing the following optimal control formula for \( \phi \)

\[
\phi(r, t) = \sup \left\{ \int_0^t \tilde{f}(\gamma(s)) \, ds : \gamma(t) = r, \left| \gamma'(s) + \frac{n-1}{\gamma(s)} \right| \leq 1 \text{ a.e.} \right\}.
\]
Theorem (GMT 2019)

Assume $f(x) = \tilde{f}(|x|)$, and $u_0(x) = \tilde{u}_0(|x|)$ for $\tilde{f} \in C^1_c([0, \infty), [0, \infty))$, and $\tilde{u}_0 \in \text{BUC}([0, \infty))$. Then

$$u(x, t) - cf t \to u_\infty(x) \quad \text{locally uniformly for } x \in \mathbb{R}^n \text{ as } t \to \infty,$$

where $u_\infty$ is a solution to the stationary (ergodic) equation

$$- \left( \text{div} \left( \frac{Du_\infty}{|Du_\infty|} \right) + 1 \right) |Du_\infty| = f(x) - cf.$$

The proof is a combination of PDE and dynamical system approaches. A main difficulty: the limiting equation might have many solutions. Our key new idea is to show that $\tilde{A} = \{ r \geq 1 : \tilde{f}(r) = cf \}$ is the uniqueness set.
Some ideas

The ergodic problem in the radially symmetric setting is

\[- \frac{n-1}{r} \psi_r - |\psi_r| = \tilde{f}(r) - c_f \quad \text{in } (0, \infty).\]

We show that this equation has bounded from above solutions, and \(\tilde{A} = \{ r \geq 1 : \tilde{f}(r) = c_f \}\) is the uniqueness set. More precisely, if two bounded from above solutions \(\psi_1, \psi_2\) to the ergodic problem satisfy that \(\psi_1 = \psi_2\) on \(\tilde{A}\), then \(\psi_1 = \psi_2\).

Then, define \(\varphi(r, t) = \phi(r, t) - c_f t\). It is worth noting that

\[\varphi_t(r, t) \geq 0 \quad \text{for } r \in \tilde{A}.\]

This fact, together with the uniqueness property of \(\tilde{A}\), helps to deduce the large time behavior result.
Open problems

A lot to be done in the future.

(i) As noted, it is one of our main goals to investigate the dependence of $c_f$ on $f$ for $f \in C^1_c(\mathbb{R}^n)$. We are still far away from satisfactory answers.

(ii) After that, of course, large time behavior is of great interests. So far, we have only got the result for two cases in [GMT 2019].

(iii) Study (i)–(ii) for general $F = F(Du, D^2u)$.

(iv) When $f$ is discontinuous (e.g., $f = 1_E$), there are some first results in [GMT 2015]. Still, many questions to be answered here.
THANK YOU!