Optimal rate of convergence in periodic homogenization of Hamilton-Jacobi equations

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joint works with W. Jing, H. Mitake, and Y. Yu

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Homogenization Theory of Hamilton-Jacobi Equation

Assume $H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is uniformly coercive in $p$, and $\mathbb{Z}^n$-periodic in $x$. Ex: Classical mechanics Hamiltonian $H(p, x) = |p|^2/2 + V(x)$. For each $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^n \times [0, \infty))$ be the viscosity solution to the following Hamilton-Jacobi equation

$$
\begin{align*}
\left\{
\begin{array}{ll}
    u^\varepsilon_t + H(Du^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
    u^\varepsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n.
\end{array}
\right.
\end{align*}
$$

(1)
Homogenization Theory of Hamilton-Jacobi Equation

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$$\begin{cases}
    u_t^\varepsilon + H \left( Du^\varepsilon, \frac{x}{\varepsilon} \right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
    u^\varepsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n.
\end{cases} \tag{1}$$

It was known (Lions-Papanicolaou-Varadhan, 1987), that $u^\varepsilon$, as $\varepsilon \to 0$, converges locally uniformly to $u$, which solves

$$\begin{cases}
    u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
    u(x, 0) = g(x) & \text{on } \mathbb{R}^n.
\end{cases} \tag{2}$$

Here, $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called “effective Hamiltonian” or “$\alpha$ function”, a nonlinear averaging of the original $H$. 
Example – Front propagation $H(p, y) = a(y)|p|$

Typically, $\varepsilon > 0$ is a fixed (very small) length scale. Zero level set \( \Gamma^\varepsilon_t = \{x : u^\varepsilon(x, t) = 0\} \) moves with oscillatory normal velocity \( V(x) = a(\frac{x}{\varepsilon})n \). Mathematically, by letting $\varepsilon \to 0$, we see averaging behavior (zero level set of $u$).

**Figure:** Zero level sets of $u^\varepsilon(\cdot, t)$ and $u(\cdot, t)$, respectively, for some fixed $t > 0$
For any $p \in \mathbb{R}^n$, there exists a **UNIQUE** number $\bar{H}(p) \in \mathbb{R}$ such that

$$H(p + Dv, y) = \bar{H}(p) \quad \text{in} \quad \mathbb{T}^n.$$ 

has a periodic viscosity solution $v$ (often named **corrector**). Write $v = v(y, p)$ to demonstrate clear dependence.
Cell problems and effective Hamiltonian

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Heuristically, the two-scale asymptotic expansion says

$$u^\varepsilon(x, t) \approx u(x, t) + \varepsilon v\left(\frac{x}{\varepsilon}, Du(x, t)\right).$$

**Figure:** Local expansion of $u^\varepsilon$ around $(x, t)$
Main question of interest

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Note: The corrector \( v(y, p) \) for \( p = Du(x, t) \) basically captures (corrects) the oscillation of \( Du^\varepsilon \) around \((x, t)\). Here, \( y = \frac{x}{\varepsilon} \).
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**A natural and fundamental question**

How fast does $u^\varepsilon$ converge to $u$ as $\varepsilon \to 0+$?

According to the above formal expansion, it seems that

$$|u^\varepsilon - u| = O(\varepsilon).$$

However, there is NO way to justify this expansion rigorously!
Why does the expansion not hold generically? $v(y, p)$ lacks enough regularities:

1. The solution of the effective equation $u(x, t)$ is in general not even $C^1$;
2. There does not even exist a continuous selection of

$$p \mapsto v(y, p).$$
Why does the expansion not hold generically? \( v(y, p) \) lacks enough regularities:

(1) The solution of the effective equation \( u(x, t) \) is in general not even \( C^1 \);
(2) There does not even exist a continuous selection of

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p \mapsto v(y, p).
\]

The best known result was due to Capuzzo-Dolcetta and Ishii (2001) based on pure PDE approaches

\[
|u^\varepsilon - u| \leq O \left( \varepsilon^{\frac{1}{3}} \right).
\]

They used an approximation procedure, doubling variables, and the perturbed test function method (Evans (1992)) to obtain the result.
Main Result 1: General Convex Case \((p \to H(p, x))\)

Theorem (Mitake–T.–Yu 2018)

Assume \(H\) is convex in \(p\) and \(g \in \text{Lip}(\mathbb{R}^n)\).

(i) There is \(C > 0\) dependent only on \(H\) and \(\|Dg\|_{L^\infty(\mathbb{R}^n)}\) so that

\[
u^\varepsilon(x, t) \geq u(x, t) - C\varepsilon\quad \text{for all } (x, t) \in \mathbb{R}^n \times [0, \infty).
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\]

(ii) For fixed \((x, t) \in \mathbb{R}^n \times (0, \infty)\), if \(u\) is differentiable at \((x, t)\), and \(\overline{H}\) is twice differentiable at \(p = Du(x, t)\), then

\[
u^\varepsilon(x, t) \leq u(x, t) + \tilde{C}_{x,t}\varepsilon.
\]

If the initial data \(g \in C^2(\mathbb{R}^n)\) with \(\|g\|_{C^2(\mathbb{R}^n)} < \infty\). If \(g\) is merely Lipschitz continuous, then

\[
u^\varepsilon(x, t) \leq u(x, t) + C_{x,t}\sqrt{\varepsilon}.
\]

Here \(\tilde{C}_{x,t}\) (\(C_{x,t}\)) is dependent on \(H, \overline{H}, p\) and \(\|g\|_{C^2(\mathbb{R}^n)}\) \((\|Dg\|_{L^\infty(\mathbb{R}^n)})\).
Theorem (Mitake–T.–Yu 2018)

Assume \( n = 2 \) and \( g \in \text{Lip}(\mathbb{R}^2) \). Assume further that \( H \) is convex and positively homogeneous of degree \( k \) in \( p \) for some \( k \geq 1 \), that is, \( H(\lambda p, x) = \lambda^k H(p, x) \) for all \( (\lambda, x, p) \in [0, \infty) \times \mathbb{T}^2 \times \mathbb{R}^2 \). Then,

\[
|u^\varepsilon(x, t) - u(x, t)| \leq C\varepsilon \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, \infty).
\]

Here \( C > 0 \) is a constant depending only on \( H \) and \( \|Dg\|_{L^\infty(\mathbb{R}^2)} \).
Main result 2: Two dimensions

Theorem (Mitake–T.–Yu 2018)

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$$|u^\varepsilon(x, t) - u(x, t)| \leq C\varepsilon \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, \infty).$$

Here $C > 0$ is a constant depending only on $H$ and $\|Dg\|_{L^\infty(\mathbb{R}^2)}$.

Note that $k = 1$ corresponds to Hamiltonians associated with the front propagation, which is probably one of the most physically relevant situations in the homogenization theory. Recall the example that we presented at the beginning.
Theorem (Mitake–T.–Yu 2018)

Assume that $n = 1$ and $H = H(p,x)$ is convex in $p$. Assume further that $g \in \text{Lip}(\mathbb{R})$. Then,

$$\| u^\varepsilon - u \|_{L^\infty(\mathbb{R} \times [0,\infty))} \leq C\varepsilon.$$

Here $C$ is a constant depending only on $H$ and $\| g' \|_{L^\infty(\mathbb{R})}$. 
Main result 3: One dimension

Theorem (Mitake–T.–Yu 2018)

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$$\| u^\varepsilon - u \|_{L^\infty(\mathbb{R} \times [0, \infty))} \leq C\varepsilon.$$ 

Here $C$ is a constant depending only on $H$ and $\|g'|_{L^\infty(\mathbb{R})}$.

Son Tu (2018) generalized this one dimensional result to $H = H(p, x, \frac{x}{\varepsilon})$. For the one dimension case, the remaining question is to find the optimal rate for general coercive $H$ (i.e. nonconvex $H$). We conjecture that the optimal rate is $O(\sqrt{\varepsilon})$. 
Sketch of Proof of the Lower Bound $u^{\varepsilon} \geq u - C\varepsilon$

$$u^{\varepsilon}(0, 1) = \inf_{\eta(0)=0} \left\{ g(\varepsilon\eta(-\varepsilon^{-1})) + \varepsilon \int_{-\varepsilon^{-1}}^{0} L(\dot{\eta}(t), \eta(t)) \, dt \right\}$$

Here $L(q, x) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(p, x) \}$. Also,

$$u(0, 1) = \inf_{y \in \mathbb{R}^n} \{ g(y) + \bar{L}(-y) \}.$$

For any $p \in \mathbb{R}^n$ and a corrector $\nu_p$:

$$H(p + D\nu_p, y) = \bar{H}(p),$$

$$\int_{-\varepsilon^{-1}}^{0} L(\dot{\eta}(t), \eta(t)) + \bar{H}(p) \, dt \geq p \cdot \eta(0) - p \cdot \eta(-\varepsilon^{-1}) + \nu_p(\eta(0)) - \nu_p(\eta(-\varepsilon^{-1}))$$

Accordingly, since $\bar{L}(q) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - \bar{H}(p) \}$,

$$\varepsilon \int_{-\varepsilon^{-1}}^{0} (L(\dot{\eta}(t), \eta(t)) \, dt \geq \bar{L}(-\varepsilon\eta(-\varepsilon^{-1})) - C\varepsilon.$$
Upper bound

\[
 u^\varepsilon(0, 1) = \inf_{\eta(0)=0} \left\{ g(\varepsilon \eta(-\varepsilon^{-1})) + \varepsilon \int_{-\varepsilon^{-1}}^{0} L(\dot{\eta}(t), \eta(t)) \, dt \right\}
\]

Since \(-\varepsilon^{-1} \to -\infty\) as \(\varepsilon \to 0^+\), we need to have certain good global (or one-sided global) curves to control the upper bound.
The Upper Bound and the Hamiltonian System

For any \( p \in \mathbb{R}^n \), fix a corrector \( v_p \). For \( x_0 \in \mathbb{T}^n \), let \( \xi : (-\infty, 0] \to \mathbb{R}^n \) be a backward characteristic with \( \xi(0) = x_0 \), that is,

\[
p \cdot (\xi(t_2) - \xi(t_1)) + v_p(\xi(t_2)) - v_p(\xi(t_1)) = \int_{t_1}^{t_2} L(\dot{\xi}, \xi) + \overline{H}(p) \, ds.
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for all \( t_1 < t_2 \leq 0 \). This backward characteristic can be found by using the optimal control formula of a corresponding Cauchy problem. Finding upper bound is basically reduced to the following question.
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**Question:** Does the average slope

\[
\frac{\xi(t)}{t}
\]

converge as \( t \to \infty \)? More importantly, what is the convergence rate?
It is known that in weak KAM theory/Aubry-Mather theory (Fathi, E, Evans-Gomes, ...) that if $\overline{H}$ is differentiable at $p$, then
\[
\lim_{t \to \infty} \frac{\xi(t)}{t} = D\overline{H}(p). \tag{3}
\]

**Theorem (Mitake–T.–Yu 2018)**

*Connection with the convergence rate.*

(I) \[
\left| \frac{\xi(t)}{t} - D\overline{H}(p) \right| \leq \frac{C}{t} \Rightarrow |u^\varepsilon - u| \leq O(\varepsilon) \quad \text{for } g \in \text{Lip}(\mathbb{R}^n)
\]

(II) \[
\left| \frac{\xi(t)}{t} - D\overline{H}(p) \right| \leq \frac{C}{\sqrt{t}} \Rightarrow \begin{cases} |u^\varepsilon - u| \leq O(\sqrt{\varepsilon}) & \text{for } g \in \text{Lip}(\mathbb{R}^n) \\ |u^\varepsilon - u| \leq O(\varepsilon) & \text{for } g \in C^2(\mathbb{R}^n). \end{cases}
\]

By modifying the argument of (3), we show that if $\overline{H}$ is twice differentiable at $p$, then (see also the beautiful work of Gomes (2002))
\[
\left| \frac{\xi(t)}{t} - D\overline{H}(p) \right| \leq \frac{C}{\sqrt{t}}.
\]
Two dimensions and the Aubry-Mather Theory

**Key ingredient:** 2d topology + the fact that two absolute minimizers $\xi$ cannot intersect twice lead to good description of the structure of absolute minimizers.
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- In particular, each absolute minimizer can be identified with a **circle map**: $f : \mathbb{R} \to \mathbb{R}$, continuous, increasing and $f(x + 1) = f(x) + 1$.

There exists a rotation number $\alpha$ such that $|f^i(x) - x - \alpha i| \leq 1$ for all $i$. 
Connection with the Convergence Rate

• If $n = 2$ and the Hamiltonian $H(p, x)$ is Tonelli and homogeneous of degree $k$, then $\overline{H}$ is differentiable away from 0 (Carneiro, 1995).

• Combining with the circle map identification and some weak KAM type calculations, we can deduce that for any global characteristic $\xi : \mathbb{R} \rightarrow \mathbb{R}^2$

$$\left| \frac{\xi(t)}{t} - DH(p) \right| \leq \frac{C}{t}.$$ 

Via approximation, this leads to the $O(\varepsilon)$ convergence rate for general convex, coercive and homogeneous $H$. 
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**Conjecture:** For a general convex and coercive $H(p, x)$ when $n = 2$, we are aiming at proving that

$$|u^\varepsilon(x, t) - u(x, t)| \leq C_{x, t} \varepsilon \quad \text{for a.e.} \ (x, t) \in \mathbb{R}^2 \times (0, +\infty).$$
Some Remarks about the Higher Dimension Case $n \geq 3$

When $n \geq 3$, there is **NO** topological obstructions for absolutely minimizing curves. The generalized **Aubry-Mather theory** has very limited applicability in obtaining properties of $\overline{H}$.

For example, for a simple metric Hamiltonian, which corresponds to a normal front propagation problem, $H(p, x) = a(x)|p|$, we know that $\overline{H}$ is convex and homogeneous of degree 1, but not much else is known.

**Jing–T.–Yu (2019)**: study deeper about shapes of $\overline{H}$ and optimal rate of convergence here.
The Hedlund example $H(p, x) = a(x)|p|$ for $n \geq 3$

Here $a \in C^\infty(\mathbb{T}^n)$ is a singular perturbation of 1 such that each minimizing geodesic is confined in a small neighborhood of one of three disjoint lines ($a(x) = C \gg 1$ there). Then, the Aubry-Mather set is very small and $\overline{H}(p) = C \max\{|p_1|, |p_2|, |p_3|\}$. However, for this sort of “bad” example, we still obtain the optimal convergence rate $O(\varepsilon)$. 

\[ y_1 \downarrow \quad y_2 \downarrow \quad y_3 \downarrow \]
Let me just report here two related results.

**Theorem (Jing–T.–Yu (2019))**

Assume that \( H(p, x) = a(x)|p| \) where \( a \in C(\mathbb{T}^n, (0, \infty)) \). Let \( \overline{H} \) be the corresponding effective Hamiltonian. If \( \{ p : \overline{H}(p) \leq 1 \} \) is a centrally symmetric polytope in \( \mathbb{R}^n \), then we have optimal rate of convergence \( O(\varepsilon) \).

And here is a result (inverse problem type) saying that all centrally symmetric polytopes with rational coordinates are admissible.

**Theorem (Jing–T.–Yu (2019)–generalization of Hedlund’s example)**

Assume \( n \geq 3 \). Let \( P \) be a centrally symmetric polytope with nonempty interior in \( \mathbb{R}^n \). Then, there exists \( a \in C^\infty(\mathbb{T}^n, (0, \infty)) \) such that, for \( H(p, x) = a(x)|p| \), the corresponding effective Hamiltonian satisfies \( \{ p : \overline{H}(p) \leq 1 \} = P \).
Some new ideas in the proof of JTY

We consider the reachable set framework. Let $Y = [0, 1]^n$, and

$$\begin{cases} \mathcal{R}_t(x) = \{ y \in \mathbb{R}^n : \gamma(0) = x, \gamma(t) = y, |\gamma'(s)| \leq a(\gamma(s)) \text{ a.e. } s \in [0, t] \}, \\
\mathcal{R}_t(Y) = \bigcup_{x \in Y} \mathcal{R}_t(x). \end{cases}$$

Then, we have the following important observation: For $t, s > 0$,

$$\mathcal{R}_{t+s}(Y) \subset \mathcal{R}_t(Y) + \mathcal{R}_s(Y) + (-Y).$$

Let $X_m = \text{co}\mathcal{R}_m(Y) + (-Y)$ for $m \in \mathbb{N}$. Then, $X_m$ is convex, compact, and

$$X_{m+k} \subset X_m + X_k \quad \text{for all } m, k \in \mathbb{N}.$$ 

This is an important subadditive property for a sequence of convex, compact set $\{X_m\}$. 

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We then use the framework in Jing–Souganidis–T. to get that

\[
\begin{align*}
X_m \rightarrow D & \quad \text{as } m \to \infty \text{ in Hausdorff distance sense,} \\
R_t(Y) \rightarrow D & \quad \text{as } t \to \infty \text{ in Hausdorff distance sense.}
\end{align*}
\]

In fact \( D = \bigcap_{m \in \mathbb{N}} \frac{X_m}{m} \). Let \( \rho \) be the Hausdorff distance.

**Theorem (Jing–T.–Yu (2019))**

**Optimal rate of convergence**

\[ \| u^\varepsilon - u \|_{L^\infty} \leq C\varepsilon \] holds for all initial data \( g \in \text{Lip}(\mathbb{R}^n) \) IFF

\[ \rho \left( \frac{R_t(Y)}{t}, D \right) \leq \frac{C}{t} \quad \text{for all } t > 1. \]
Firstly, we can prove that

\[ \frac{\mathcal{R}_t(Y)}{t} \subset D + B_{\frac{C}{t}}(0). \]

This gives us the optimal lower bound that was proved already, that is, \( u^\varepsilon \geq u - C\varepsilon \).

Secondly, it is clear that \( D \subset \frac{X_m}{m} \) for all \( m \in \mathbb{N} \). However, this does not imply that \( D \subset \frac{\mathcal{R}_m(Y)}{m} + B_{\frac{C}{t}}(0) \). This is quite a subtle point as \( X_m \) might be bigger than \( R_m(Y) \).

Thirdly, if \( D \) is a centrally symmetric polytope in \( \mathbb{R}^n \), then we can show that vertices of \( D \) are in \( \frac{\mathcal{R}_m(Y)}{m} + B_{\frac{C}{t}}(0) \) and hence,

\[ D \subset \frac{\mathcal{R}_m(Y)}{m} + B_{\frac{C}{t}}(0). \]
Concluding remarks

This is just the beginning. There are many more interesting questions to be studied.

1. The case where $H = H(p, x, \frac{x}{\varepsilon})$ or the case where $H = H(p, \frac{x}{\varepsilon}, \frac{t}{\varepsilon})$.

2. Periodic homogenization for static problem $u^\varepsilon + H(Du^\varepsilon, x, \frac{x}{\varepsilon}) = 0$.

3. Quasi/Almost periodic setting.

4. Stationary ergodic setting: Improve the result by Armstrong–Cardaliaguet–Souganidis.

THANK YOU FOR YOUR ATTENTION!