A PARA-DIFFERENTIAL RENORMALIZATION TECHNIQUE FOR NONLINEAR DISPERSIVE EQUATIONS

SEBASTIAN HERR, ALEXANDRU D. IONESCU, CARLOS E. KENIG,
AND HERBERT KOCH

Abstract. For $\alpha \in (1,2)$ we prove that the initial-value problem

$$\begin{cases} \partial_t u + D^\alpha \partial_x u + \partial_x (u^2/2) = 0 \text{ on } \mathbb{R}_x \times \mathbb{R}_t; \\ u(0) = \phi, \end{cases}$$

is globally well-posed in the space of real-valued L^2-functions. We use a frequency dependent renormalization method to control the strong low-high frequency interactions.

Contents

1. Introduction

In this paper we consider the initial-value problem for the Benjamin-Ono equation with generalized dispersion

$$\begin{cases} \partial_t u + D^\alpha \partial_x u + \partial_x (u^2/2) = 0 \text{ on } \mathbb{R}_x \times \mathbb{R}_t; \\ u(0) = \phi, \end{cases} \tag{1.1}$$

where $\alpha \in (1,2)$, and D^α denotes the operator defined by the Fourier multiplier $\xi \to |\xi|^\alpha$. Let $H_r^\sigma = H_r^\sigma(\mathbb{R})$, $\sigma \in [0,\infty)$, denote the space of *real-valued* functions ϕ

2000 *Mathematics Subject Classification.* Primary: 35Q53; Secondary: 35Q35, 35S50.

S.H. was supported in part by the DFG, Sonderforschungsbereich 611. A. I. was supported in part by a Packard Fellowship. C. K. was supported in part by NSF grant DMS0456583. H.K. was supported in part by the DFG, Sonderforschungsbereich 611.

with the usual Sobolev norm

$$\|\phi\|_{H^\sigma_r} = \|\phi\|_{H^\sigma} = \|(1+|\xi|^2)^{\sigma/2}\widehat{\phi}(\xi)\|_{L^2_\xi}.$$

Let $H^\infty_r = \cap_{\sigma\in\mathbb{Z}_+} H^\sigma_r$ with the induced metric. Suitable solutions of (1.1) satisfy the L^2 conservation law: if $T_1 < T_2 \in \mathbb{R}$ and $u \in C((T_1,T_2) : H^\infty_r)$ is a solution of the equation $\partial_t u + D^\alpha \partial_x u + \partial_x(u^2/2) = 0$ on $\mathbb{R}\times(T_1,T_2)$ then

$$\|u(t_1)\|_{H^0_r} = \|u(t_2)\|_{H^0_r} \text{ for any } t_1, t_2 \in (T_1,T_2). \tag{1.2}$$

Our main theorem concerns the global well-posedness in H^0_r of the initial-value problem (1.1).

Theorem 1.1. *(a) Assume $\phi \in H^\infty_r$. Then there is a unique global solution*

$$u = S^\infty(\phi) \in C(\mathbb{R} : H^\infty_r)$$

of the initial-value problem (1.1).

(b) Assume $T \in \mathbb{R}_+$. Then the mapping

$$S^\infty_T = \mathbf{1}_{(-T,T)}(t)\cdot S^\infty :\ H^\infty_r \to C((-T,T) : H^\infty_r)$$

extends uniquely to a continuous mapping

$$S^0_T : H^0_r \to C((-T,T) : H^0_r),$$

and

$$\|S^0_T(\phi)(t)\|_{H^0_r} = \|\phi\|_{H^0_r} \textit{ for any } t \in (-T,T).$$

One-dimensional models such as (1.1) have been studied extensively. The case $\alpha = 2$ corresponds to the KdV equation, while the case $\alpha = 1$ corresponds to the Benjamin–Ono equation. Global well-posedness in H^0_r is known in both of these cases, see [1] and [10] respectively. Other local and global well-posedness results for (1.1) in Sobolev spaces H^s_r have been obtained by several authors, see [15, 16, 17, 4, 3, 7] for the KdV case $\alpha = 2$, and [13, 22, 19, 14, 26] for the Benjamin–Ono case $\alpha = 1$.

The dispersion generalized model (1.1) has also been analyzed in the literature, see for example [15, 5, 20, 9, 8]. For example, local well-posedness in Sobolev spaces $H^s(\mathbb{R})$ for $s > -3/4(\alpha-1)$, and global well-posedness in $H^s_r(\mathbb{R})$ in the range $s \geq 0$, has been shown by the first author [9] under an additional *low frequency constraint* on the initial data. Without this low frequency constraint, the Sobolev index for local well-posedness has been pushed down to $s > 2-\alpha$ by Z. Guo in [8], using the method of [12].

The nonlinearity of the dispersion generalized Benjamin-Ono equation (1.1) is too strong to allow a direct perturbative argument (without a low frequency constraint) since the flow map is not locally uniformly continuous in $H^s_r(\mathbb{R})$, $s > 0$. Problems with this feature have attracted considerable interest in recent years. It is not difficult to see the reason for this failure at the hand of the model problem

$$v_t + vv_x = 0,$$

see [19, p.2]. Given a solution v we obtain the family of solutions

$$v_c(t,x) := v(t,x-ct) + c\ , \quad c \in \mathbb{R}. \tag{1.3}$$

If $v(0,x)$ is of high frequency, the constant c (the low frequency part) induces a spatial shift of the high frequency part and the lack of uniform dependence on the initial data becomes evident. The construction of smooth, square-integrable analogs of (1.3) for (1.1) has been caried out in [19] in detail in the case $\alpha = 1$, see also [21]. We notice that the failure of uniform continuity is irrespective of the regularity assumption which is imposed on the initial data.

Tao has interpreted this phenomenon for the Benjamin-Ono equation as a gauge change, which opened the path to the satisfactory well-posedness result in [10] for the Benjamin-Ono equation, i.e. the case $\alpha = 1$. There the gauge change can be undone by a multiplication of projection to positive frequencies of the solution by a function $e^{i\phi}$. The linear part reduces to a Schrödinger equation for positive frequencies, and commuting $e^{i\phi}$ by the Schrödinger equation leads to a drift term which balances the worst low-high interaction

$$u_{\text{low}}\partial_x u_{\text{high}}$$

of the quadratic part.

The same ideas show that one encounters a pseudo-differential gauge transform for the dispersion generalized Benjamin-Ono equation (1.1). We do not pursue this pseudo-differential point of view, but it is advisable to keep it in mind. Instead we decompose the solution into small frequency bands of size $\sqrt{\lambda}$ at frequency λ. At this frequency scale the gauge change is essentially a multiplication by a purely imaginary phase function. We carry out bilinear estimates for these frequency bands and study the effect of the gauge transform. This is technical and painful. The main contribution of this paper is the demonstration that this circle of ideas can be carried through for a non trivial example. The phenomenon described above will most likely be encountered at other problems as well. Gaining a general understanding of it seems to be desirable and this is our aim.

The rest of the paper is organized as follows: in section 2 we reduce Theorem 1.1 to proving several apriori bounds on smooth solutions, and differences of smooth solutions, of the equation (1.1), on bounded time intervals. This reduction relies on energy-type estimates.

In section 3 we construct our main renormalization, which is the key step to further reducing the problem to perturbative analysis. After subtracting the low frequency component of the solution, which is essentially left unchanged by the evolution due to the null structure of the nonlinearity, we further decompose the solution into frequency blocks and multiply each frequency block by a suitable bounded factor. This renormalization leads to an infinite system of coupled equations satisfied by the frequency blocks. A similar construction was used by two of the authors in [10] for the Benjamin–Ono equation. However, in our situation, we need to renormalize each frequency block by a different factor, which leads to substantial technical difficulties in the perturbative analysis.

In section 4 we define our main normed spaces, and show that the main theorem can be reduced to proving the nonlinear estimates in Proposition 4.3.

The remaining sections are concerned with the proof of Proposition 4.3. We prove first frequency-localized bilinear estimates, see sections 5 and 6. Then we prove frequency-localized linear estimates on operators defined by multiplication by smooth bounded factors, see section 7, and frequency-localized commutator estimates, see section 9. Finally, we put all these estimates together in section 8 and 10 to complete the proof of Proposition 4.3.

2. Reduction to a priori estimates

We recall first a standard local well-posedness theorem:

Proposition 2.1. *Assume $\phi \in H_r^\infty$. Then there is $T = T(\|\phi\|_{H_r^2}) \in (0,1]$ and a unique solution $u = S_T^\infty(\phi) \in C((-T,T) : H_r^\infty)$ of the initial-value problem*

$$\begin{cases} \partial_t u + D^\alpha \partial_x u + \partial_x(u^2/2) = 0 \text{ on } \mathbb{R}_x \times (-T,T); \\ u(0) = \phi. \end{cases} \tag{2.1}$$

In addition, for any $\sigma \geq 2$,

$$\sup_{t\in(-T,T)} \|u(t)\|_{H_r^\sigma} \leq C(\sigma, \|\phi\|_{H_r^\sigma}, \sup_{t\in(-T,T)} \|u(t)\|_{H_r^2}).$$

Most of the paper is concerned with proving suitable *a priori* estimates on the solutions S_T^∞ constructed in Proposition 2.1. For Theorem 1.1 (a) we need the following estimate:

Proposition 2.2. *There is a constant $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ with the property that if $T \in (0,1]$, $\phi \in H_r^\infty$,*

$$\|\phi\|_{H_r^0} \leq \varepsilon_0, \tag{2.2}$$

and $u = S_T^\infty(\phi) \in C((-T,T) : H_r^\infty)$ is a solution of the initial-value problem (2.1), *then*

$$\sup_{t\in(-T,T)} \|u(t)\|_{H_r^2} \leq C\|\phi\|_{H_r^2}. \tag{2.3}$$

Theorem 1.1 (a) follows easily from Proposition 2.2: by scaling (i.e. replace ϕ by $\phi_\lambda(x) = \lambda^\alpha \phi(\lambda x)$ and u by $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1} t)$, $\lambda \ll 1$) it suffices to prove Theorem 1.1 (a) for $\phi \in H_r^\infty$ with $\|\phi\|_{H_r^0} \leq \varepsilon_0$. Using Proposition 2.1 and Proposition 2.2, we can construct the solution $S_1^\infty(\phi) \in C((-1,1) : H_r^\infty)$. Finally, we use the conservation law (1.2) to extend the solution to the entire real line.

To prove Theorem 1.1 (b) we need an additional estimate on differences of solutions. For any $\phi \in H_r^\infty$ and $N \in [1,\infty)$ let $\phi^N = \mathcal{F}^{-1}[\widehat{\phi}(\xi) \cdot \mathbf{1}_{[-N,N]}(\xi)] \in H_r^\infty$.

Proposition 2.3. *Assume $N \in [2,\infty)$, $\phi \in H_r^\infty$, and $\|\phi\|_{H_r^0} \leq \varepsilon_0$ (see* (2.2)*). Then*

$$\sup_{t\in(-1,1)} \|S^\infty(\phi)(t) - S^\infty(\phi^N)(t)\|_{H_r^0} \leq C\|\phi - \phi^N\|_{H_r^0}. \tag{2.4}$$

We show now that Proposition 2.2 and Proposition 2.3 imply Theorem 1.1 (b). Assume $\phi \in H_r^0$ is fixed,

$$\phi_n \in H_r^\infty \text{ and } \lim_{n\to\infty} \phi_n = \phi \text{ in } H_r^0.$$

For Theorem 1.1 (b) it suffices to prove that the sequence $S_T^\infty(\phi_n) \in C((-T,T) : H_r^\infty)$ is a Cauchy sequence in $C((-T,T) : H_r^0)$. By scaling, we may assume $\|\phi\|_{H_r^0} \leq \varepsilon_0/2$. Using the conservation law (1.2) it suffices to prove that for any $\delta > 0$ there is M_δ such that

$$\sup_{t\in(-1,1)} \|S^\infty(\phi_m)(t) - S^\infty(\phi_n)(t)\|_{H_r^0} \leq \delta \text{ for any } m,n \geq M_\delta. \tag{2.5}$$

We observe now that $\|\phi_n - \phi_n^N\|_{H_r^0} \leq \|\phi - \phi^N\|_{H_r^0} + \|\phi - \phi_n\|_{H_r^0}$. Thus we can fix $N = N(\delta,\phi) \geq 2$ and M_δ^1 such that $\|\phi_n - \phi_n^N\|_{H_r^0} \leq \delta/(4C)$ and $\|\phi_n\| \leq \varepsilon_0$ for any

$n \geq M^1_\delta$, where C is the constant in (2.4). Thus, using (2.4),

$$\sup_{t\in(-1,1)} \|S^\infty(\phi_n)(t) - S^\infty(\phi_n^N)(t)\|_{H^0_r} \leq \delta/4 \text{ for any } n \geq M^1_\delta. \tag{2.6}$$

It remains to estimate

$$\sup_{t\in(-1,1)} \|S^\infty(\phi_n^N)(t) - S^\infty(\phi_m^N)(t)\|_{H^0_r}.$$

Using standard energy estimates for the difference equation, we have

$$\begin{aligned}
&\sup_{t\in(-1,1)} \|S^\infty(\phi_n^N)(t) - S^\infty(\phi_m^N)(t)\|_{H^0_r}\\
&\leq \|\phi_n^N - \phi_m^N\|_{H^0_r} \cdot \exp\Big(C\int_{-1}^1 \|\partial_x(S^\infty(\phi_n^N))(t)\|_{L^\infty_x} + \|\partial_x(S^\infty(\phi_m^N))(t)\|_{L^\infty_x}\,dt\Big)\\
&\leq \|\phi_n - \phi_m\|_{H^0_r} \cdot \exp\Big(C \sup_{t\in(-1,1)} \big[\|S^\infty(\phi_n^N)(t)\|_{H^2_r} + \|S^\infty(\phi_m^N)(t)\|_{H^2_r}\big]\Big).
\end{aligned}$$

Using Proposition 2.2, it follows that

$$\begin{aligned}
\sup_{t\in(-1,1)} &\|S^\infty(\phi_n^N)(t) - S^\infty(\phi_m^N)(t)\|_{H^0_r}\\
&\leq \|\phi_n - \phi_m\|_{H^0_r} \cdot \exp\Big(C[\|\phi_n^N\|_{H^2_r} + \|\phi_m^N\|_{H^2_r}]\Big)\\
&\leq \|\phi_n - \phi_m\|_{H^0_r} \cdot \exp(CN^2).
\end{aligned} \tag{2.7}$$

The bound (2.5) follows from (2.6) and (2.7).

This completes the proof of Theorem 1.1. The rest of the paper is concerned with proving Proposition 2.2 and Proposition 2.3.

3. The main renormalization

The initial-value problem (1.1) cannot be analyzed perturbatively, due to the strong interactions between very low and high frequencies. In this section we construct a para-differential renormalization which allows us to recover information about the solution u of (1.1) indirectly, by analyzing perturbatively a system of equations satisfied by suitably renormalized frequency components of u, see (3.15) and (3.16).

Assume in this section that $T \in (0,1]$ and $u \in C((-T,T) : H^\infty_r)$ is a solution of the initial-value problem

$$\begin{cases} \partial_t u + D^\alpha\partial_x u + \partial_x(u^2/2) = 0 \text{ on } \mathbb{R}_x \times (-T,T);\\ u(0) = \phi. \end{cases} \tag{3.1}$$

Assume, in addition, that $\|\phi\|_{H^0_r} \leq \varepsilon_0$, for some sufficiently small constant ε_0 (compare with (2.2)). Let $\phi_{\text{low}} = \mathcal{F}^{-1}[\widehat{\phi}(\xi)\cdot \mathbf{1}_{[-1/2,1/2]}(\xi)] \in H^\infty_r$ and $\phi_{\text{high}} = \phi - \phi_{\text{low}}$. Let $v(x,t) = u(x,t) - \phi_{\text{low}}(x)$, so

$$\begin{cases} \partial_t v + D^\alpha\partial_x v + \phi_{\text{low}}\cdot\partial_x v = -\partial_x(v^2/2) - v\cdot\partial_x\phi_{\text{low}} - D^\alpha\partial_x\phi_{\text{low}} - \partial_x(\phi_{\text{low}}^2/2);\\ v(0) = \phi_{\text{high}}. \end{cases} \tag{3.2}$$

on $\mathbb{R}_x \times (-T,T)$.

We fix the increasing sequence $\{n_k\}_{k\in\mathbb{Z}}$

$$\begin{cases} n_0 = 0,\ n_1 = 4,\ n_{k+1} = n_k + n_k^{1/2} \text{ for } k = 1, 2, \dots; \\ n_{-k} = -n_k \text{ for } k = 1, 2, \dots. \end{cases} \tag{3.3}$$

Clearly, $|n_k| \approx |k|^2$ for $|k| \geq 1$. We fix also smooth functions $\chi_k : \mathbb{R} \to [0, 1]$, $k \in \mathbb{Z}$, χ_k supported in $[(2n_{k-1} + n_k)/3, (2n_{k+1} + n_k)/3]$ such that

$$\begin{cases} \sum_{k\in\mathbb{Z}} \chi_k \equiv 1; \\ |\partial_\xi^\sigma \chi_k| \leq C(1 + |n_k|)^{-\sigma/2} \text{ for any } k \in \mathbb{Z} \text{ and } \sigma = 0, 1, 2. \end{cases} \tag{3.4}$$

For $k \in \mathbb{Z}$ let

$$I_k = [(5n_{k-1} + n_k)/6, (5n_{k+1} + n_k)/6]. \tag{3.5}$$

Let P_k and $\widetilde{P}_k$, $k \in \mathbb{Z}$, denote the operators defined by the Fourier multipliers χ_k and $\mathbf{1}_{I_k}$ respectively.

We apply the operators P_k, $k \in \mathbb{Z} \setminus \{0\}$, to the identity (3.2); the result is

$$\begin{cases} \partial_t(P_k v) + D^\alpha \partial_x(P_k v) + \phi_{\text{low}} \cdot \partial_x(P_k v) = E_k \text{ on } \mathbb{R}_x \times (-T, T); \\ (P_k v)(0) = P_k(\phi_{\text{high}}), \end{cases} \tag{3.6}$$

where

$$E_k = [\phi_{\text{low}} \cdot \partial_x(P_k v) - P_k(\phi_{\text{low}} \cdot \partial_x v)] - P_k \partial_x(v^2/2) - P_k(v \cdot \partial_x \phi_{\text{low}}). \tag{3.7}$$

We apply also the operator P_0 to the identity (3.2); the result is

$$\begin{cases} \partial_t(P_0 v) + D^\alpha \partial_x(P_0 v) = R_0 \text{ on } \mathbb{R}_x \times (-T, T); \\ (P_0 v)(0) = P_0(\phi_{\text{high}}), \end{cases} \tag{3.8}$$

where

$$\begin{aligned} R_0 = &- P_0(\phi_{\text{low}} \cdot \partial_x v) - P_0 \partial_x(v^2/2) \\ &- P_0(v \cdot \partial_x \phi_{\text{low}}) - D^\alpha \partial_x P_0(\phi_{\text{low}}) - P_0 \partial_x(\phi_{\text{low}}^2/2). \end{aligned} \tag{3.9}$$

We define the smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ as the anti-derivative of ϕ_{low},

$$\Psi'(x) = \phi_{\text{low}}(x) \text{ and } \Psi(0) = 0. \tag{3.10}$$

For $k \in \mathbb{Z} \setminus \{0\}$ we define the functions

$$v_k(x, t) = P_k(v)(x, t) \cdot e^{-ia_k\Psi(x)} \text{ where } a_k = -n_k|n_k|^{-\alpha}/(\alpha + 1). \tag{3.11}$$

We substitute the identity $P_k(v) = e^{ia_k\Psi} v_k$ into (3.6); the result is

$$e^{ia_k\Psi}\partial_t v_k + e^{ia_k\Psi} D^\alpha \partial_x(v_k) + (\alpha + 1) D^\alpha v_k \cdot (ia_k\Psi') e^{ia_k\Psi} + \phi_{\text{low}} e^{ia_k\Psi} \partial_x(v_k) = E_k' \tag{3.12}$$

where

$$\begin{aligned} E_k' = &[e^{ia_k\Psi} D^\alpha \partial_x(v_k) + (\alpha + 1) D^\alpha v_k \cdot (ia_k\Psi') e^{ia_k\Psi} - D^\alpha \partial_x(e^{ia_k\Psi} v_k)] \\ &+ E_k - \phi_{\text{low}}(ia_k\Psi') e^{ia_k\Psi} \cdot v_k. \end{aligned}$$

We multiply (3.12) by $e^{-ia_k\Psi}$. Using the definition (3.11) of the coefficients a_k, it follows that

$$\begin{cases} \partial_t v_k + D^\alpha \partial_x(v_k) = R_k \text{ on } \mathbb{R}_x \times (-T, T); \\ v_k(0) = e^{-ia_k\cdot\Psi} \cdot P_k(\phi_{\text{high}}), \end{cases} \tag{3.13}$$

where

$$
\begin{aligned}
R_k = & - e^{-ia_k\Psi} P_k \partial_x (v^2/2) \\
& - \phi_{\text{low}}[\partial_x v_k - D^\alpha v_k \cdot (in_k|n_k|^{-\alpha})] \\
& - [e^{-ia_k\Psi} D^\alpha \partial_x (e^{ia_k\Psi} v_k) - D^\alpha \partial_x (v_k) - (\alpha+1) D^\alpha v_k \cdot (ia_k \Psi')] \\
& - e^{-ia_k\Psi}[P_k(\phi_{\text{low}} \cdot \partial_x v) - \phi_{\text{low}} \cdot \partial_x (P_k v)] \\
& - [ia_k \phi_{\text{low}}^2 \cdot v_k + e^{-ia_k\Psi} P_k (v \cdot \partial_x \phi_{\text{low}})].
\end{aligned}
\tag{3.14}
$$

To summarize, given a solution $u \in C((-T,T) : H_r^\infty)$ of the initial-value problem (3.1) we constructed functions $v_k \in C((-T,T) : H^\infty)$, $k \in \mathbb{Z}$, which solve the initial-value problems

$$
\begin{cases}
\partial_t v_k + D^\alpha \partial_x (v_k) = R_k \text{ on } \mathbb{R}_x \times (-T,T); \\
v_k(0) = e^{-ia_k \cdot \Psi} \cdot P_k(\phi_{\text{high}}),
\end{cases}
\tag{3.15}
$$

where, for simplicity of notation, $a_0 = 0$. The functions $R_k \in C((-T,T) : H^\infty)$ are defined in (3.9) for $k = 0$, and (3.14) for $k \neq 0$. In addition, $u = v + \phi_{\text{low}}$,

$$
v = \sum_{k \in \mathbb{Z}} e^{ia_k\Psi} v_k, \text{ and } v_k = e^{-ia_k\Psi} \widetilde{P}_k (e^{ia_k\Psi} v_k) \text{ for any } k \in \mathbb{Z}. \tag{3.16}
$$

4. Proof of the main theorem

The rest of the argument is based on perturbative analysis of the system of equations (3.15), for fixed ϕ_{low}. In this section we define our main normed spaces used for this perturbative analysis and show how to reduce Propositions 2.2 and 2.3 to the more technical Proposition 4.3 below. Proposition 4.3 will be proved in the remaining sections of the paper.

The normed spaces constructed in this section are very similar to those used by two of the authors in [10] for the analysis of the Benjamin–Ono equation. However, our spaces are adapted to the frequency intervals I_k constructed in section 3, instead of dyadic intervals, since they are used to measure the components v_k and their renormalizations.

Let $\eta_0 : \mathbb{R} \to [0,1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \{1, 2, \ldots\}$ let $\eta_k(\nu) = \eta_0(\nu/2^k) - \eta_0(\nu/2^{k-1})$. For $k \in \mathbb{Z}$ let $\widetilde{\eta}_k(\nu) = \eta_0(\nu/2^k) - \eta_0(\nu/2^{k-1})$. We define the sets $J_0 = [-2, 2]$, $J_k = \{\nu \in \mathbb{R} : |\nu| \in [2^{k-1}, 2^{k+1}]$, $k = 1, 2, \ldots$, and $\widetilde{J}_k = \{\nu \in \mathbb{R} : |\nu| \in [2^{k-1}, 2^{k+1}]$, $k \in \mathbb{Z}$. For $\xi \in \mathbb{R}$ let

$$
\omega(\xi) = -\xi|\xi|^\alpha. \tag{4.1}
$$

Recall the sequence n_k, the functions χ_k, and the intervals I_k, $k \in \mathbb{Z}$, defined in (3.3)–(3.5). We fix $\delta = (\alpha - 1)/100 \in (0, 1/100)$. We define the normed spaces $Z_k = Z_k(\mathbb{R} \times \mathbb{R})$, $k \in \mathbb{Z}$: for $|k| \geq 1$ (high frequencies) we define

$$
\begin{aligned}
Z_k = & \{f \in L^2 : f \text{ supported in } I_k \times \mathbb{R} \text{ and} \\
& \|f\|_{Z_k} = \sum_{j=0}^\infty 2^{j/2} \beta_{k,j} \|\eta_j(\tau - \omega(\xi)) f(\xi, \tau)\|_{L^2_{\xi,\tau}} < \infty\},
\end{aligned}
\tag{4.2}
$$

where

$$
\beta_{k,j} = 1 + (2^j / |n_k|^{\alpha+1})^{1/2-\delta}. \tag{4.3}
$$

Notice that $2^{j/2}\beta_{k,j} \approx 2^{j(1-\delta)}$ when k is small. For $k = 0$ (low frequencies) we define

$$X_0 = \{f \in L^2 : f \text{ supported in } I_0 \times \mathbb{R} \text{ and}$$
$$\|f\|_{X_0} = \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{2} 2^{j(1-\delta)} 2^{-k'} \|\eta_j(\tau)\widetilde{\eta}_{k'}(\xi) f(\xi,\tau)\|_{L^2_{\xi,\tau}} < \infty\} \tag{4.4}$$

and

$$Y_0 = \{f \in L^2 : f \text{ supported in } I_0 \times \mathbb{R} \text{ and}$$
$$\|f\|_{Y_0} = \sum_{j=0}^{\infty} 2^{j(1-\delta)} \|\mathcal{F}^{-1}[\eta_j(\tau) f(\xi,\tau)]\|_{L^1_x L^2_t} < \infty\}. \tag{4.5}$$

Finally, we define

$$Z_0 = X_0 + Y_0. \tag{4.6}$$

The normed spaces Z_k, $k \in \mathbb{Z}$, are our main spaces of functions defined in the Fourier space. They are similar to the spaces used in [10], but slightly simpler (we do not need the spaces Y_k for $|k| \geq 1$, compare to [10, Section 3], since the local smoothing phenomenon is not essential if $\alpha > 1$).

For $\sigma \in [0,\infty)$ we define the normed spaces $\widetilde{H}^\sigma = \widetilde{H}^\sigma(\mathbb{R})$, $\mathbf{F}^\sigma = \mathbf{F}^\sigma(\mathbb{R}\times\mathbb{R})$, and $\mathbf{N}^\sigma = \mathbf{N}^\sigma(\mathbb{R}\times\mathbb{R})$. We define first

$$\widetilde{H}^\sigma = \Big\{\phi \in H^\infty : \|\phi\|^2_{\widetilde{H}^\sigma} = \|\chi_0 \mathcal{F}(\phi)\|^2_{B_0} + \sum_{|k|\geq 1} (1+|n_k|)^{2\sigma} \|\chi_k \mathcal{F}(\phi)\|^2_{L^2_\xi} < \infty\Big\},$$
$$\|f\|_{B_0} = \inf_{f=g+h} \|\mathcal{F}^{-1}(g)\|_{L^1} + \sum_{k'=-\infty}^{2} 2^{-k'} \|\widetilde{\eta}_{k'} h\|_{L^2_\xi}. \tag{4.7}$$

Then we define

$$\mathbf{F}^\sigma = \Big\{u \in C(\mathbb{R} : \widetilde{H}^\sigma) : u \text{ supported in } \mathbb{R}_x \times [-4,4]$$
$$\text{and } \|u\|^2_{\mathbf{F}^\sigma} = \sum_{k\in\mathbb{Z}} (1+|n_k|)^{2\sigma} \|P_k u\|^2_{F_k} < \infty \tag{4.8}$$
$$\text{where } \|P_k u\|_{F_k} = \|\chi_k(\xi)\cdot\mathcal{F}(u)\|_{Z_k}\Big\},$$

and

$$\mathbf{N}^\sigma = \Big\{u \in C(\mathbb{R} : \widetilde{H}^\sigma) : u \text{ supported in } \mathbb{R}_x \times [-4,4]$$
$$\text{and } \|u\|^2_{\mathbf{N}^\sigma} = \sum_{k\in\mathbb{Z}} (1+|n_k|)^{2\sigma} \|P_k u\|^2_{N_k} < \infty \tag{4.9}$$
$$\text{where } \|P_k u\|_{N_k} = \|\chi_k(\xi)(\tau-\omega(\xi)+i)^{-1}\cdot\mathcal{F}(u)\|_{Z_k}\Big\}.$$

Finally, for any $T' \in (0,1]$, $\sigma \in [0,2]$ and $f \in C((-T',T') : \widetilde{H}^\sigma)$ we define

$$\|f\|_{\mathbf{F}^\sigma(T')} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times(-T',T')} \|\widetilde{f}\|_{\mathbf{F}^\sigma},$$

and

$$\|f\|_{\mathbf{N}^\sigma(T')} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times(-T',T')} \|\widetilde{f}\|_{\mathbf{N}^\sigma}.$$

It follows easily from the definitions that

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{\widetilde{H}^\sigma} \leq C\|u\|_{\mathbf{F}^\sigma} \tag{4.10}$$

if $\sigma \in [0,2]$ and $u \in \mathbf{F}^\sigma$, see [10, Lemma 4.2] for a similar proof. Thus $\mathbf{F}^\sigma \hookrightarrow C(\mathbb{R} : \widetilde{H}^\sigma)$.

For any $u \in C(\mathbb{R} : L^2)$ let $\widetilde{u}(.,t) \in C(\mathbb{R} : L^2)$ denote its partial Fourier transform with respect to the variable x. For $\phi \in H^\infty$ let $W(t)\phi \in C(\mathbb{R} : H^\infty)$ denote the solution of the free evolution given by

$$[W(t)\phi]^\sim(\xi,t) = e^{it\omega(\xi)}\widehat{\phi}(\xi), \tag{4.11}$$

where $\omega(\xi)$ is defined in (4.1). We record first an $\widetilde{H}^\sigma \to \mathbf{F}^\sigma$ homogeneous linear estimate.

Proposition 4.1. *If $\sigma \in [0,2]$ and $\phi \in \widetilde{H}^\sigma$ then*

$$\|\mathbf{1}_{(-1,1)}(t)\cdot(W(t)\phi)\|_{\mathbf{F}^\sigma(1)} \leq C\|\phi\|_{\widetilde{H}^\sigma}. \tag{4.12}$$

See, for example, [10, Lemma 5.1] for a similar proof. We need also an inhomogeneous $\mathbf{N}^\sigma \to \mathbf{F}^\sigma$ linear estimate, see, for example, [10, Lemma 5.2] for a similar proof.

Proposition 4.2. *If $\sigma \in [0,2]$, $T \in (0,1]$, and $u \in \mathbf{N}^\sigma(T)$ then*

$$\Big\|\int_0^t W(t-s)(u(s))\,ds\Big\|_{\mathbf{F}^\sigma(T)} \leq C\|u\|_{\mathbf{N}^\sigma(T)}.$$

In the rest of this section we use the notation in section 3. In particular, given a solution $u \in C((-T,T) : H_r^\infty)$, $T \in (0,1]$, of the initial-value problem (3.1) with $\|\phi\|_{H_r^0} \leq \varepsilon_0$, we constructed the functions $v_k \in C((-T,T) : H^\infty)$, $k \in \mathbb{Z}$, which solve the equations

$$\begin{cases}\partial_t v_k + D^\alpha\partial_x(v_k) = R_k \text{ on } \mathbb{R}_x \times (-T,T);\\ v_k(0) = e^{-ia_k\cdot\Psi}\cdot P_k(\phi_{\text{high}}).\end{cases} \tag{4.13}$$

Here $R_k \in C((-T,T) : H^\infty)$, $k \in \mathbb{Z}$, are as in (3.9) and (3.14), and a_k are defined in (3.11). Assume also that $u' \in C((-T,T) : H_r^\infty)$ is a solution of the initial-value problem

$$\begin{cases}\partial_t u' + D^\alpha\partial_x u' + \partial_x({u'}^2/2) = 0 \text{ on } \mathbb{R}_x \times (-T,T);\\ u'(0) = \phi'.\end{cases}$$

Assume, in addition, that

$$\|\phi'\|_{H_r^0} \leq \varepsilon_0 \quad \text{and} \quad \phi'_{\text{low}} = \phi_{\text{low}}, \tag{4.14}$$

where, as in section 3, $\phi'_{\text{low}} = \mathcal{F}^{-1}[\widehat{\phi'}(\xi)\cdot\mathbf{1}_{[-1/2,1/2]}(\xi)]$. Let v'_k, R'_k, n_k, a_k, Ψ be defined as in section 3, so

$$\begin{cases}\partial_t v'_k + D^\alpha\partial_x(v'_k) = R'_k \text{ on } \mathbb{R}_x \times (-T,T);\\ v'_k(0) = e^{-ia_k\cdot\Psi}\cdot P_k(\phi'_{\text{high}}).\end{cases} \tag{4.15}$$

Our main proposition concerning the nonlinearities R_k and R'_k is the following:

Proposition 4.3. *(a) For any $\sigma \in [0,2]$*

$$\sum_{k\in\mathbb{Z}} \|R_k\|^2_{\mathbf{N}^\sigma(T)} < \infty.$$

In addition, the mapping $\mathcal{N} : (0,T] \to [0,\infty)$,

$$\mathcal{N}(T') = \sum_{k\in\mathbb{Z}} \|R_k\|^2_{\mathbf{N}^0(T')}$$

is continuous and increasing on the interval $(0,T]$ and

$$\lim_{T'\to 0} \mathcal{N}(T') = 0.$$

(b) Assume $\sigma \in [0,2]$ and $T' \in [0,T]$. Then

$$\sum_{k\in\mathbb{Z}} \|R_k\|^2_{\mathbf{N}^\sigma(T')} \le C\big(\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^\sigma(T')}\big)\big(\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^0(T')} + \varepsilon_0^2\big) + C\|\phi\|^2_{H^\sigma}. \tag{4.16}$$

In addition,

$$\sum_{k\in\mathbb{Z}} \|R'_k - R_k\|^2_{\mathbf{N}^0(T')} \le C\big(\sum_{k\in\mathbb{Z}} \|v'_k - v_k\|^2_{\mathbf{F}^0(T')}\big)\big(\sum_{k\in\mathbb{Z}} \big(\|v_k\|^2_{\mathbf{F}^0(T')} + \|v'_k\|^2_{\mathbf{F}^0(T')}\big) + \varepsilon_0^2\big). \tag{4.17}$$

The proof of Proposition 4.3 will cover sections 5–10. In the rest of this section we show how to use this proposition to complete the proof of Theorem 1.1.

Proof of Proposition 2.2. It follows easily from the definitions that for $\sigma \in [0,2]$

$$\sum_{k\in\mathbb{Z}} \|e^{-ia_k\cdot\Psi} \cdot P_k(\phi_{\text{high}})\|^2_{\widetilde{H}^\sigma} \le C\|\phi_{\text{high}}\|^2_{H^\sigma}. \tag{4.18}$$

See also [10, Lemma10.1] for a similar proof. Using Propositions 4.1 and (4.2) and the equations (4.13) it follows that for any $\sigma \in [0,2]$ and $T' \in (0,T]$

$$\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^\sigma(T')} \le C\|\phi_{\text{high}}\|^2_{H^\sigma} + C\sum_{k\in\mathbb{Z}} \|R_k\|^2_{\mathbf{N}^\sigma(T')}. \tag{4.19}$$

We set $\sigma = 0$ and combine (4.19) and (4.16); it follows that

$$\mathcal{N}(T') \le C(\varepsilon_0^2 + \mathcal{N}(T'))^2 + C\varepsilon_0^2.$$

Using Proposition 4.3 (a) it follows that $\mathcal{N}(T') \le C\varepsilon_0^2$ for any $T' \in (0,T]$, provided that the constant ε_0 is taken sufficiently small. In particular

$$\sum_{k\in\mathbb{Z}} \|R_k\|^2_{\mathbf{N}^0(T)} \le C\varepsilon_0^2.$$

It follows from (4.19) with $\sigma = 0$ that

$$\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^0(T)} \le C\varepsilon_0^2. \tag{4.20}$$

Thus, using (4.16) with $\sigma = 2$

$$\sum_{k\in\mathbb{Z}} \|R_k\|^2_{\mathbf{N}^2(T)} \le C\varepsilon_0^2\big(\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^2(T)}\big) + C\|\phi\|^2_{H^2}.$$

Using (4.19) with $\sigma = 2$ it follows that

$$\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^2(T)} \le C\|\phi\|^2_{H^2} + C\varepsilon_0^2\big(\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^2(T)}\big),$$

therefore, assuming ε_0 sufficiently small,

$$\sum_{k\in\mathbb{Z}} \|v_k\|^2_{\mathbf{F}^2(T)} \le C\|\phi\|^2_{H^2}.$$

Using (4.10) it follows that for any $t \in (-T, T)$

$$\sum_{k\in\mathbb{Z}} \|v_k(t)\|^2_{\widetilde{H}^2} \le C\|\phi\|^2_{H^2}. \tag{4.21}$$

We use now (3.16)

$$u = \phi_{\mathrm{low}} + \sum_{k\in\mathbb{Z}} e^{ia_k\Psi} v_k, \text{ and } v_k = e^{-ia_k\Psi}\widetilde{P}_k(e^{ia_k\Psi}v_k) \text{ for any } k \in \mathbb{Z}.$$

to prove a bound on u. For any $t \in (-T, T)$ we have, using (4.21),

$$\begin{aligned}\|u(t)\|^2_{H^2} &\le C\|\phi_{\mathrm{low}}\|^2_{H^2} + C\sum_{k\in\mathbb{Z}} \|\widetilde{P}_k(e^{ia_k\Psi}v_k(t))\|^2_{H^2}\\ &\le C\|\phi_{\mathrm{low}}\|^2_{H^2} + \sum_{k\in\mathbb{Z}} \|v_k(t)\|^2_{H^2} \le C\|\phi\|^2_{H^2}.\end{aligned}$$

This completes the proof of Proposition 2.2. □

Proof of Proposition 2.3. Let $\phi' = \phi^N$, so (4.14) is verified, and subtract equations (4.15) and (4.13). The result is

$$\begin{cases}\partial_t(v'_k - v_k) + D^\alpha\partial_x(v'_k - v_k) = R'_k - R_k \text{ on } \mathbb{R}_x \times (-T, T);\\ (v'_k - v_k)(0) = e^{-ia_k\cdot\Psi}\cdot P_k((\phi' - \phi)_{\mathrm{high}}).\end{cases}$$

Using (4.18) we have

$$\sum_{k\in\mathbb{Z}} \|e^{-ia_k\cdot\Psi}\cdot P_k((\phi' - \phi)_{\mathrm{high}})\|^2_{\widetilde{H}^0} \le C\|(\phi' - \phi)_{\mathrm{high}}\|^2_{H^0}.$$

Using Proposition 4.1 and 4.2 it follows that

$$\sum_{k\in\mathbb{Z}} \|v'_k - v_k\|^2_{\mathbf{F}^0(T)} \le C\|(\phi' - \phi)_{\mathrm{high}}\|^2_{H^0} + \sum_{k\in\mathbb{Z}} \|R'_k - R_k\|^2_{\mathbf{N}^0(T)}.$$

Using (4.17) with $T' = T$ and (4.20),

$$\sum_{k\in\mathbb{Z}} \|R'_k - R_k\|^2_{\mathbf{N}^0(T)} \le C\varepsilon_0^2 \sum_{k\in\mathbb{Z}} \|v'_k - v_k\|^2_{\mathbf{F}^0(T)}.$$

It follows from the last two inequalities that

$$\sum_{k\in\mathbb{Z}} \|v'_k - v_k\|^2_{\mathbf{F}^0(T)} \le C\|\phi' - \phi\|^2_{L^2},$$

provided that ε_0 is sufficiently small. Using (4.10) it follows that

$$\sum_{k\in\mathbb{Z}} \|v'_k(t) - v_k(t)\|^2_{L^2} \le C\|\phi' - \phi\|^2_{L^2} \tag{4.22}$$

for any $t \in (-T, T)$. As in the proof of Proposition 2.2, it follows from (4.22) and (3.16) that $\|u'(t) - u(t)\|^2_{L^2} \le C\|\phi' - \phi\|^2_{L^2}$, which completes the proof of Proposition 2.3. □

In view of the results in section 2, the main theorem follows from Propositions 2.2 and 2.3. Thus it remains to prove Proposition 4.3, which is the goal of the rest of the paper.

5. Localized L^2 estimates

In this section we prove the localized L^2 estimates in Corollary 5.2. These estimates are similar to the estimates proved in [10, Section 6] in the study of the Benjamin-Ono equation. More general L^2 estimates of this type can be found in [25].

For $\xi_1, \xi_2 \in \mathbb{R}$ let

$$\Omega(\xi_1, \xi_2) = -\omega(\xi_1 + \xi_2) + \omega(\xi_1) + \omega(\xi_2), \tag{5.1}$$

where, as before, $\omega(\xi) = -\xi|\xi|^\alpha$. For compactly supported functions $f, g, h \in L^2(\mathbb{R} \times \mathbb{R})$ let

$$J(f,g,h) = \int_{\mathbb{R}^4} f(\xi_1,\mu_1)g(\xi_2,\mu_2)h(\xi_1+\xi_2,\mu_1+\mu_2+\Omega(\xi_1,\xi_2))\,d\xi_1 d\xi_2 d\mu_1 d\mu_2. \tag{5.2}$$

Given a triplet of real numbers $(\alpha_1, \alpha_2, \alpha_3)$ let $\min(\alpha_1, \alpha_2, \alpha_3)$, $\max(\alpha_1, \alpha_2, \alpha_3)$, and $\mathrm{med}(\alpha_1, \alpha_2, \alpha_3)$ denote the minimum, the maximum, and the median (more precisely, $\mathrm{med}(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3 - \max(\alpha_1, \alpha_2, \alpha_3) - \min(\alpha_1, \alpha_2, \alpha_3)$) of the numbers α_1, α_2, and α_3. We define the sets U_k, $k \in \mathbb{Z}$,

$$\begin{cases} U_k = \{\nu \in \mathbb{R} : |\nu| \in [(5n_{k-1} + n_k)/6, (5n_{k+1} + n_k)/6]\} \text{ if } k \in [1,\infty) \cap \mathbb{Z}; \\ U_k = \{\nu \in \mathbb{R} : |\nu| \in [2^{k+1}, 2^{k+3}]\} \text{ if } k \in (-\infty, 0] \cap \mathbb{Z}. \end{cases} \tag{5.3}$$

Clearly $U_k = I_k \cup I_{-k}$ if $k \geq 1$ and $U_k = \widetilde{J}_{k+2}$ if $k \leq 0$. For $k_1, k_2, k_3 \in \mathbb{Z}$ let

$$d_\alpha(k_1, k_2; k_3) = \inf\{\big||\xi_1|^\alpha - |\xi_2|^\alpha\big| : \xi_1 \in U_{k_1}, \xi_2 \in U_{k_2}, \xi_1 + \xi_2 \in U_{k_3}\}. \tag{5.4}$$

Lemma 5.1. *Assume $k_1, k_2, k_3 \in \mathbb{Z}$, $j_1, j_2, j_3 \in \mathbb{Z}_+$, and $f_{k_i}^{j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ are functions supported in $U_{k_i} \times J_{j_i}$, $i = 1, 2, 3$.*

(a) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$,

$$|J(f_{k_1}^{j_1}, f_{k_2}^{j_2}, f_{k_3}^{j_3})| \leq C \min(|U_{k_1}|, |U_{k_2}|, |U_{k_3}|)^{1/2} 2^{\min(j_1,j_2,j_3)/2} \prod_{i=1}^3 \|f_{k_i}^{j_i}\|_{L^2}. \tag{5.5}$$

(b) Assume that $\{i_1, i_2, i_3\}$ is a permutation of $\{1, 2, 3\}$. Then

$$|J(f_{k_1}^{j_1}, f_{k_2}^{j_2}, f_{k_3}^{j_3})| \leq C 2^{(j_1+j_2+j_3)/2} [2^{j_{i_3}} d_\alpha(k_{i_1}, k_{i_2}; k_{i_3})]^{-1/2} \prod_{i=1}^3 \|f_{k_i}^{j_i}\|_{L^2}. \tag{5.6}$$

(c) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$,

$$|J(f_{k_1}^{j_1}, f_{k_2}^{j_2}, f_{k_3}^{j_3})| \leq C 2^{\min(j_1,j_2,j_3)/2 + \mathrm{med}(j_1,j_2,j_3)/4} \prod_{i=1}^3 \|f_{k_i}^{j_i}\|_{L^2}. \tag{5.7}$$

Proof of Lemma 5.1. Let $A_{k_i}(\xi) = \Big[\int_{\mathbb{R}} |f_{k_i}^{j_i}(\xi,\mu)|^2\,d\mu\Big]^{1/2}$, $i = 1, 2, 3$. Using the Hölder inequality and the support properties of the functions $f_{k_i}^{j_i}$,

$$\begin{aligned} |J(f_{k_1}^{j_1}, f_{k_2}^{j_2}, f_{k_3}^{j_3})| &\leq C 2^{\min(j_1,j_2,j_3)/2} \int_{\mathbb{R}^2} A_{k_1}(\xi_1) A_{k_2}(\xi_2) A_{k_3}(\xi_1 + \xi_2)\,d\xi_1 d\xi_2 \\ &\leq C \min(|U_{k_1}|, |U_{k_2}|, |U_{k_3}|)^{1/2} 2^{\min(j_1,j_2,j_3)/2} \prod_{i=1}^3 \|f_{k_i}^{j_i}\|_{L^2}, \end{aligned} \tag{5.8}$$

which is part (a).

For part (b), using simple changes of variables and the fact that ω is odd,

$$|J(f,g,h)| = |J(g,f,h)| \text{ and } |J(f,g,h)| = |J(\widetilde{f},h,g)|, \tag{5.9}$$

where $\widetilde{f}(\xi,\mu) = f(-\xi,-\mu)$. Thus, by symmetry, in proving (5.6) we may assume that $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$. Let

$$B_{k_3}(\xi,\mu) = \Big[\frac{1}{2^{j_1}2^{j_2}}\int_{\mathbb{R}^2} |f_{k_3}^{j_3}(\xi,\mu+\alpha+\beta)|^2(1+\alpha/2^{j_1})^{-2}(1+\beta/2^{j_2})^{-2}\,d\alpha d\beta\Big]^{1/2}.$$

Clearly,

$$\|B_{k_3}\|_{L^2} = C\|f_{k_3}^{j_3}\|_{L^2} \text{ and } B_{k_3} \text{ is supported in } U_{k_3}\times\mathbb{R}. \tag{5.10}$$

Also, using Hölder's inequality,

$$\begin{aligned}
&|J(f_{k_1}^{j_1}, f_{k_2}^{j_2}, f_{k_3}^{j_3})|\\
&\leq C2^{(j_1+j_2)/2}\int_{\mathbb{R}^2} A_{k_1}(\xi_1)A_{k_2}(\xi_2)B_{k_3}(\xi_1+\xi_2,\Omega(\xi_1,\xi_2))\,d\xi_1 d\xi_2\\
&\leq C2^{(j_1+j_2)/2}\|A_{k_1}\|_{L^2}\|A_{k_2}\|_{L^2}\Big[\int_{U_{k_1}\times U_{k_2}} |B_{k_3}(\xi_1+\xi_2,\Omega(\xi_1,\xi_2))|^2\,d\xi_1 d\xi_2\Big]^{1/2}.
\end{aligned} \tag{5.11}$$

Thus, for (5.6), it suffices to prove that

$$\Big[\int_{U_{k_1}\times U_{k_2}} |B_{k_3}(\xi_1+\xi_2,\Omega(\xi_1,\xi_2))|^2\,d\xi_1 d\xi_2\Big]^{1/2} \leq C[d_\alpha(k_1,k_2;k_3)]^{-1/2}\|B_{k_3}\|_{L^2}. \tag{5.12}$$

Let $B'_{k_3}(\xi,\mu) = B_{k_3}(\xi,-\omega(\xi)+\mu)$, $\|B'_{k_3}\|_{L^2} = \|B_{k_3}\|_{L^2}$, B'_{k_3} supported in $U_{k_3}\times\mathbb{R}$. For (5.12) it suffices to prove that

$$\Big[\int_{U_{k_1}\times U_{k_2}} |B'_{k_3}(\xi_1+\xi_2,\omega(\xi_1)+\omega(\xi_2))|^2\,d\xi_1 d\xi_2\Big]^{1/2} \leq C[d_\alpha(k_1,k_2;k_3)]^{-1/2}\|B'_{k_3}\|_{L^2}. \tag{5.13}$$

We observe now that $|\omega'(\xi_1)-\omega'(\xi_2)| \geq C^{-1}d_\alpha(k_1,k_2;k_3)$ if $\xi_1\in U_{k_1}$, $\xi_2\in U_{k_2}$, and $\xi_1+\xi_2\in U_{k_3}$. The bound (5.13) follows.

For part (c), using part (a), we may assume

$$2^{\mathrm{med}\,(j_1,j_2,j_3)/2} \leq C^{-1}\min\,(|U_{k_1}|,|U_{k_2}|,|U_{k_3}|). \tag{5.14}$$

Using (5.9), we may also assume $j_1 = \min\,(j_1,j_2,j_3)$ and $j_2 = \mathrm{med}\,(j_1,j_2,j_3)$. Let

$$R_{j_2} = \{(\xi_1,\xi_2) : |\xi_1-\xi_2| \geq 2^{j_2/2}\}.$$

For the integral over $(\xi_1,\xi_2)\in {}^cR_{j_2} = \mathbb{R}^2\setminus R_{j_2}$ we use a bound similar to (5.8):

$$\begin{aligned}
&\Big|\int_{{}^cR_{j_2}\times\mathbb{R}^2} f_{k_1}^{j_1}(\xi_1,\mu_1)f_{k_2}^{j_2}(\xi_2,\mu_2)f_{k_3}^{j_3}(\xi_1+\xi_2,\mu_1+\mu_2+\Omega(\xi_1,\xi_2))\,d\xi_1 d\xi_2 d\mu_1 d\mu_2\Big|\\
&\qquad\leq C2^{j_1/2}\int_{{}^cR_{j_2}} A_{k_1}(\xi_1)A_{k_2}(\xi_2)A_{k_3}(\xi_1+\xi_2)\,d\xi_1 d\xi_2\\
&\qquad\leq C2^{j_1/2}\iint_{|\mu|\leq 2^{j_2/2}} A_{k_1}(\xi_2+\mu)A_{k_2}(\xi_2)A_{k_3}(2\xi_2+\mu)\,d\xi_2 d\mu\\
&\qquad\leq C2^{j_1/2}\int_{|\mu|\leq 2^{j_2/2}}\Big(\int_{\mathbb{R}} |A_{k_1}(\xi_2+\mu)|^2|A_{k_2}(\xi_2)|^2\,d\xi_2\Big)^{1/2}\|A_{k_3}\|_{L^2}d\mu\\
&\qquad\leq C2^{j_1/2}2^{j_2/4}\|A_{k_1}\|_{L^2}\|A_{k_2}\|_{L^2}\|A_{k_3}\|_{L^2},
\end{aligned}$$

which suffices for (5.7). For the integral over $(\xi_1,\xi_2)\in R_{j_2}$ we use a bound similar to (5.11)

$$\begin{split}&\Big|\int_{R_{j_2}\times\mathbb{R}^2} f_{k_1}^{j_1}(\xi_1,\mu_1)f_{k_2}^{j_2}(\xi_2,\mu_2)f_{k_3}^{j_3}(\xi_1+\xi_2,\mu_1+\mu_2+\Omega(\xi_1,\xi_2))\,d\xi_1 d\xi_2 d\mu_1 d\mu_2\Big|\\ &\leq C2^{(j_1+j_2)/2}\|A_{k_1}\|_{L^2}\|A_{k_2}\|_{L^2}\Big[\int_{R_{j_2}\cap(U_{k_1}\times U_{k_2})}|B_{k_3}(\xi_1+\xi_2,\Omega(\xi_1,\xi_2))|^2\,d\xi_1 d\xi_2\Big]^{1/2}.\end{split} \tag{5.15}$$

Using (5.14) and the identity $\omega'(\xi)=-(\alpha+1)|\xi|^\alpha$, we observe that

$$|\omega'(\xi_1)-\omega'(\xi_2)|\geq C^{-1}2^{j_2/2} \text{ if } (\xi_1,\xi_2)\in R_{j_2}\cap(U_{k_1}\times U_{k_2}) \text{ and } \xi_1+\xi_2\in U_{k_3}.$$

As before, it follows that

$$\Big[\int_{R_{j_2}\cap(U_{k_1}\times U_{k_2})}|B_{k_3}(\xi_1+\xi_2,\Omega(\xi_1,\xi_2))|^2\,d\xi_1 d\xi_2\Big]^{1/2}\leq C2^{-j_2/4}\|B_{k_3}\|_{L^2}.$$

The bound (5.7) follows by substituting this bound into (5.15). □

We restate now Lemma 5.1 in a form that is suitable for the bilinear estimates in the next section. For $k\in\mathbb{Z}$ and $j\in\mathbb{Z}_+$ let $V_k^j=\{(\xi,\tau)\in\mathbb{R}\times\mathbb{R}:\xi\in U_k \text{ and } \tau-\omega(\xi)\in J_j\}$.

Corollary 5.2. *Assume $k_1,k_2,k_3\in\mathbb{Z}$, $j_1,j_2,j_3\in\mathbb{Z}_+$, and $f_{k_i}^{j_i}\in L^2(\mathbb{R}^2)$ are functions supported in $V_{k_i}^{j_i}$, $i=1,2$.*

(a) For any $k_1,k_2,k_3\in\mathbb{Z}$ and $j_1,j_2,j_3\in\mathbb{Z}_+$,

$$\|\mathbf{1}_{V_{k_3}^{j_3}}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\leq C\min(|U_{k_1}|,|U_{k_2}|,|U_{k_3}|)^{1/2}2^{\min(j_1,j_2,j_3)/2}\prod_{i=1}^2\|f_{k_i}^{j_i}\|_{L^2}. \tag{5.16}$$

(b) If $\{i_1,i_2,i_3\}$ is a permutation of $\{1,2,3\}$ then

$$\|\mathbf{1}_{V_{k_3}^{j_3}}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\leq C2^{(j_1+j_2+j_3)/2}[2^{j_{i_3}}d_\alpha(k_{i_1},k_{i_2};k_{i_3})]^{-1/2}\prod_{i=1}^2\|f_{k_i}^{j_i}\|_{L^2}. \tag{5.17}$$

(c) For any $k_1,k_2,k_3\in\mathbb{Z}$ and $j_1,j_2,j_3\in\mathbb{Z}_+$,

$$\|\mathbf{1}_{V_{k_3}^{j_3}}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\leq C2^{\min(j_1,j_2,j_3)/2+\mathrm{med}(j_1,j_2,j_3)/4}\prod_{i=1}^2\|f_{k_i}^{j_i}\|_{L^2}. \tag{5.18}$$

Proof of Corollary 5.2. Clearly,

$$\|\mathbf{1}_{V_{k_3}^{j_3}}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}=\sup_{\|f\|_{L^2}=1}\Big|\int_{V_{k_3}^{j_3}}f\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\,d\xi d\tau\Big|.$$

Let $f_{k_3}^{j_3}=\mathbf{1}_{V_{k_3}^{j_3}}\cdot f$, and then $g_{k_i}^{j_i}(\xi,\mu)=f_{k_i}^{j_i}(\xi,\mu+\omega(\xi))$, $i=1,2,3$. The functions $g_{k_i}^{j_i}$ are supported in $U_{k_i}\times J_{j_i}$, $\|g_{k_i}^{j_i}\|_{L^2}=\|f_{k_i}^{j_i}\|_{L^2}$, and, using simple changes of variables,

$$\int_{V_{k_3}^{j_3}}f\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\,d\xi d\tau=J(g_{k_1}^{j_1},g_{k_2}^{j_2},g_{k_3}^{j_3}).$$

Corollary 5.2 follows from Lemma 5.1. □

6. Bilinear estimates

In this section we prove several L^2-based bilinear estimates. All of our estimates are based on Corollary 5.2. For $\rho \in [-1,1]$ we define the family of normed spaces X_0^ρ,

$$X_0^\rho = \{f \in L^2 : f \text{ supported in } I_0 \times \mathbb{R} \text{ and}$$
$$\|f\|_{X_0^\rho} = \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{2} 2^{j(1-\delta)} 2^{\rho k'} \|\eta_j(\tau - \omega(\xi))\widetilde{\eta}_{k'}(\xi) f(\xi,\tau)\|_{L^2_{\xi,\tau}} < \infty\}. \tag{6.1}$$

Clearly, $X_0^{-1} = X_0$ (compare with the definition (4.4)) and $X_0^\rho \hookrightarrow X_0^{\rho'}$ if $\rho \le \rho'$. In addition, it follows easily that

$$X_0^{-1} \hookrightarrow Z_0 \hookrightarrow X_0^{-1/2+\delta}. \tag{6.2}$$

For $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z} \setminus \{0\}$ we define the sets

$$D_k^j = \{(\xi,\tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k \text{ and } \tau - \omega(\xi) \in J_j\} \subseteq V_{|k|}^j. \tag{6.3}$$

For $j \in \mathbb{Z}_+$ and $k' \in (-\infty, 2] \cap \mathbb{Z}$ we define the sets

$$D_{0,k'}^j = \{(\xi,\tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_0 \cap \widetilde{J}_{k'} \text{ and } \tau - \omega(\xi) \in J_j\} \subseteq V_{k'-2}^j. \tag{6.4}$$

Using the definition, if $|k| \ge 1$ and $f_k \in Z_k$ then f_k can be written in the form

$$\begin{cases} f_k = \sum_{j=0}^{\infty} f_k^j; \\ \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|f_k^j\|_{L^2} = \|f_k\|_{Z_k}, \end{cases} \tag{6.5}$$

such that f_k^j is supported in D_k^j. If $f_0 \in X_0^\rho$ then f_0 can be written in the form

$$\begin{cases} f_0 = \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{2} f_{0,k'}^j; \\ \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{2} 2^{j(1-\delta)} 2^{\rho k'} \|f_{0,k'}^j\|_{L^2} = \|f_0\|_{X_0^\rho}, \end{cases} \tag{6.6}$$

such that $f_{0,k'}^j$ is supported in $D_{0,k'}^j$. The identities (6.5) and (6.6) are our main atomic decompositions of functions in Z_k, $k \ge 1$, and X_0^ρ.

We consider first Low × High → High interactions.

Lemma 6.1. *(a) Assume $k, k_1, k_2 \in \mathbb{Z}\setminus\{0\}$, $|n_k| \ge 2^{20}$, $|n_{k_1}| \le |n_k|/2^{10}$, $f_{k_1} \in Z_{k_1}$, and $f_{k_2} \in Z_{k_2}$. Then*

$$\begin{aligned}(1+|n_k|)\cdot\|\chi_k(\xi)(\tau-\omega(\xi)+i)^{-1}\cdot(f_{k_1} * f_{k_2})\|_{Z_k} \\ \le C(1+|n_{k_1}|)^{-1/2}(1+|n_k|)^{-\delta}\cdot\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}.\end{aligned} \tag{6.7}$$

(b) Assume $k, k_2 \in \mathbb{Z}\setminus\{0\}$, $|n_k| \ge 2^{20}$, $f_{k_2} \in Z_{k_2}$, and $f_0 \in X_0^\rho$, $\rho \in \{-1/2+\delta, \delta\}$. Then

$$\begin{aligned}(1+|n_k|)^{1/2-\rho+\delta}\cdot\|\chi_k(\xi)(\tau-\omega(\xi)+i)^{-1}\cdot(f_0 * f_{k_2})\|_{Z_k} \\ \le C(1+|n_k|)^{-\delta}\cdot\|f_0\|_{X_0^\rho}\|f_{k_2}\|_{Z_{k_2}}.\end{aligned} \tag{6.8}$$

Proof of Lemma 6.1. For part (a), we may assume $|n_{k_2} - n_k| \leq |n_k|/2^5$. Using (6.5), we may assume $f_{k_1} = f_{k_1}^{j_1}$ is supported in $D_{k_1}^{j_1}$ and $f_{k_2} = f_{k_2}^{j_2}$ is supported in $D_{k_2}^{j_2}$. For (6.7) it suffices to prove that

$$\begin{aligned} &|n_k| \cdot \sum_{j\in\mathbb{Z}_+} 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ &\leq C|n_{k_1}|^{-1/2}|n_k|^{-\delta} \cdot 2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2} \cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}. \end{aligned} \tag{6.9}$$

Using (5.17) and (5.4),

$$\|\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C2^{(j_1+j_2+j)/2}[(2^j+2^{j_2})|n_k|^\alpha + 2^{j_1}|n_{k_1}||n_k|^{\alpha-1}]^{-1/2}\Pi, \tag{6.10}$$

where $\Pi = \|f_{k_1}^{j_1}\|_{L^2}\cdot\|f_{k_2}^{j_2}\|_{L^2}$. Let $\Pi' = 2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2}\cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}$. Using (6.10), if $j = \max(j_1, j_2, j)$ then

$$|n_k| \cdot 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C\frac{2^{-j/2}\beta_{k,j}}{\beta_{k_1,j_1}\beta_{k_2,j_2}}|n_k|^{1-\alpha/2}\cdot\Pi'. \tag{6.11}$$

If $j_2 = \max(j_1, j_2, j)$ then

$$|n_k| \cdot 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C\frac{\beta_{k,j}}{2^{j_2/2}\beta_{k_1,j_1}\beta_{k_2,j_2}}|n_k|^{1-\alpha/2}\cdot\Pi'. \tag{6.12}$$

If $j_1 = \max(j_1, j_2, j)$ then

$$|n_k| \cdot 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C\frac{\beta_{k,j}}{2^{j_1/2}\beta_{k_1,j_1}\beta_{k_2,j_2}}|n_{k_1}|^{-1/2}|n_k|^{1-(\alpha-1)/2}\cdot\Pi'. \tag{6.13}$$

We observe now that for any $\xi_1, \xi_2 \in \mathbb{R}$

$$\frac{|\Omega(\xi_1,\xi_2)|}{\min(|\xi_1|,|\xi_2|,|\xi_1+\xi_2|)\cdot\max(|\xi_1|^\alpha,|\xi_2|^\alpha,|\xi_1+\xi_2|^\alpha)} \in [2^{-4}, 2^4]. \tag{6.14}$$

Thus, by examining the supports of the functions, $\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1} * f_{k_2}^{j_2}) \equiv 0$ unless

$$2^{\max(j_1,j_2,j)} \geq C^{-1}|n_{k_1}||n_k|^\alpha. \tag{6.15}$$

Thus, using (6.11) and (4.3),

$$\begin{aligned} &|n_k| \cdot \sum_{j\geq\max(j_1,j_2)} 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ &\leq C\sum_{2^j\geq C^{-1}|n_{k_1}||n_k|^\alpha} \frac{2^{-j/2}\beta_{k,j}}{\beta_{k_1,j_1}\beta_{k_2,j_2}}|n_k|^{1-\alpha/2}\cdot\Pi' \leq C|n_{k_1}|^{-1/2}|n_k|^{1-\alpha}\cdot\Pi'. \end{aligned}$$

If $j_2 \geq j_1$ then, using (6.12), (6.15), and (4.3)

$$\begin{aligned} &|n_k| \cdot \sum_{j\leq j_2} 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ &\leq C\sum_{j\leq j_2} \frac{\beta_{k,j}}{2^{j_2/2}\beta_{k_1,j_1}\beta_{k_2,j_2}}|n_k|^{1-\alpha/2}\cdot\Pi' \leq C|n_{k_1}|^{-1/2}|n_k|^{1-\alpha}\ln(2+|n_k|)\cdot\Pi'. \end{aligned}$$

Finally, if $j_1 \geq j_2$ then, using (6.13), (6.15), and (4.3)

$$\begin{aligned}|n_k| \cdot \sum_{j \leq j_1} & 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ &\leq C \sum_{j \leq j_1} \frac{\beta_{k,j}}{2^{j_1/2}\beta_{k_1,j_1}\beta_{k_2,j_2}} |n_{k_1}|^{-1/2}|n_k|^{1-(\alpha-1)/2} \cdot \Pi' \\ &\leq C|n_{k_1}|^{-1/2}|n_k|^{1-\alpha}\ln(2+|n_k|) \cdot \Pi'.\end{aligned}$$

The estimate (6.9) follows from the last three bounds.

For part (b), using (6.5) and (6.6) we may assume $f_0 = f_{0,k'}^{j_1}$ is supported in $D_{0,k'}^{j_1}$ and $f_{k_2} = f_{k_2}^{j_2}$ is supported in $D_{k_2}^{j_2}$. For (6.8) it suffices to prove that

$$\begin{aligned}(|n_k| + 2^{-k'/2}|n_k|^{1/2}|n_k|^{-19\delta}) \cdot \sum_{j \in \mathbb{Z}_+} 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ \leq C|n_k|^{-20\delta} \cdot 2^{j_1(1-\delta)}2^{-k'(1/2-\delta)}\|f_{0,k'}^{j_1}\|_{L^2} \cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}.\end{aligned} \tag{6.16}$$

Using (5.16),

$$\|\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C 2^{j_2/2}2^{k'/2} \cdot \|f_{0,k'}^{j_1}\|_{L^2} \cdot \|f_{k_2}^{j_2}\|_{L^2}, \tag{6.17}$$

and the bound (6.16) follows easily if $2^{k'}|n_k|^{1+30\delta} \leq 1$.

Assume that

$$2^{k'}|n_k|^{1+30\delta} \geq 1. \tag{6.18}$$

In this case $|n_k| \geq 2^{-k'/2}|n_k|^{1/2}|n_k|^{-19\delta}$. Using (5.17) and (5.4),

$$\|\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C 2^{(j_1+j_2+j)/2}[(2^j + 2^{j_2})|n_k|^\alpha]^{-1/2}\Pi, \tag{6.19}$$

where $\Pi = \|f_{0,k'}^{j_1}\|_{L^2} \cdot \|f_{k_2}^{j_2}\|_{L^2}$. Using (6.14), $\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2}) \equiv 0$ unless

$$2^{\max(j_1,j_2,j)} \geq C^{-1}2^{k'}|n_k|^\alpha. \tag{6.20}$$

Let $\Pi' = 2^{j_1(1-\delta)}2^{-k'(1/2-\delta)}\|f_{0,k'}^{j_1}\|_{L^2} \cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}$. If $j = \max(j_1, j_2, j)$ then, using (6.19),

$$|n_k| \cdot 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C \frac{2^{-j/2}\beta_{k,j}}{2^{j_1(1/2-\delta)}\beta_{k_2,j_2}} 2^{k'(1/2-\delta)}|n_k|^{1-\alpha/2} \cdot \Pi'. \tag{6.21}$$

If $j_2 = \max(j_1, j_2, j)$ then, using (6.19),

$$|n_k| \cdot 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C \frac{\beta_{k,j}}{2^{j_1(1/2-\delta)}2^{j_2/2}\beta_{k_2,j_2}} 2^{k'(1/2-\delta)}|n_k|^{1-\alpha/2} \cdot \Pi'. \tag{6.22}$$

If $j_1 = \max(j_1, j_2, j)$ then, using (6.17)

$$|n_k| \cdot 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{0,k'}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq C \frac{2^{-j/2}\beta_{k,j}}{2^{j_1(1-\delta)}\beta_{k_2,j_2}} 2^{k'(1-\delta)}|n_k| \cdot \Pi'. \tag{6.23}$$

Thus, using (6.21), (6.20), and (4.3),

$$|n_k|\cdot\sum_{j\geq\max(j_1,j_2)}2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j}\cdot(f_{0,k'}^{j_1}*f_{k_2}^{j_2})\|_{L^2}$$
$$\leq C\sum_{2^j\geq C^{-1}2^{k'}|n_k|^\alpha}\frac{2^{-j/2}\beta_{k,j}}{2^{j_1(1/2-\delta)}\beta_{k_2,j_2}}2^{k'(1/2-\delta)}|n_k|^{1-\alpha/2}\cdot\Pi'\leq C2^{-\delta k'}|n_k|^{1-\alpha}\cdot\Pi'.$$

If $j_2\geq j_1$ then, using (6.22), (6.20), and (4.3)

$$|n_k|\cdot\sum_{j\leq j_2}2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j}\cdot(f_{0,k'}^{j_1}*f_{k_2}^{j_2})\|_{L^2}$$
$$\leq C\sum_{j\leq j_2}\frac{\beta_{k,j}}{2^{j_1(1/2-\delta)}2^{j_2/2}\beta_{k_2,j_2}}2^{k'(1/2-\delta)}|n_k|^{1-\alpha/2}\cdot\Pi'$$
$$\leq C2^{-\delta k'}|n_k|^{1-\alpha}\ln(2+|n_k|)\cdot\Pi'.$$

Finally, if $j_1\geq j_2$ then, using (6.23), (6.20), and (4.3)

$$|n_k|\cdot\sum_{j\leq j_1}2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j}\cdot(f_{0,k'}^{j_1}*f_{k_2}^{j_2})\|_{L^2}$$
$$\leq C\sum_{j\leq j_1}\frac{2^{-j/2}\beta_{k,j}}{2^{j_1(1-\delta)}\beta_{k_2,j_2}}2^{k'(1-\delta)}|n_k|\cdot\Pi'\leq C|n_k|^{1-\alpha+\alpha\delta}\cdot\Pi'.$$

The estimate (6.16) follows from the last three bounds and (6.18). □

We consider now High × High → Low interactions.

Lemma 6.2. *Assume* $k_1,k_2,k\in\mathbb{Z}\backslash\{0\}$, $\min(|n_{k_1}|,|n_{k_2}|)\geq 2^{10}(1+|n_k|)$, $f_{k_1}\in Z_{k_1}$, *and* $f_{k_2}\in Z_{k_2}$. *Then*

$$(1+|n_k|)\cdot\|\chi_k(\xi)(\tau-\omega(\xi)+i)^{-1}\cdot(f_{k_1}*f_{k_2})\|_{Z_k}$$
$$\leq C(1+|n_k|)^{-1/2}(1+|n_{k_1}|+|n_{k_2}|)^{-\delta}\cdot\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}. \tag{6.24}$$

In addition, if $\min(|n_{k_1}|,|n_{k_2}|)\geq 2^{10}$, $f_{k_1}\in Z_{k_1}$, *and* $f_{k_2}\in Z_{k_2}$ *then*

$$\|\chi_0(\xi)(\tau-\omega(\xi)+i)^{-1}\cdot(f_{k_1}*f_{k_2})\|_{X_0^{-1/2+\delta}}$$
$$\leq C(1+|n_{k_1}|+|n_{k_2}|)^{-\delta}\cdot\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}. \tag{6.25}$$

Proof of Lemma 6.2. Clearly, we may assume $|n_{k_1}/n_{k_2}|\in[1/2,2]$ and $n_{k_1}\cdot n_{k_2}<0$. Using (6.5), we may assume $f_{k_1}=f_{k_1}^{j_1}$ is supported in $D_{k_1}^{j_1}$ and $f_{k_2}=f_{k_2}^{j_2}$ is supported in $D_{k_2}^{j_2}$.

For (6.24) it suffices to prove that

$$|n_k|\cdot\sum_{j\in\mathbb{Z}_+}2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}$$
$$\leq C|n_k|^{-1/2}|n_{k_1}|^{-\delta}\cdot 2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2}\cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}. \tag{6.26}$$

In view of (6.14), $\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\equiv 0$ unless

$$2^{\max(j_1,j_2,j)}\geq C^{-1}|n_k||n_{k_1}|^\alpha. \tag{6.27}$$

Using (5.17),

$$\|\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\leq C2^{(j_1+j_2+j)/2}[2^{\max(j_1,j_2)}|n_{k_1}|^\alpha+2^j|n_k||n_{k_1}|^{\alpha-1}]^{-1/2}\Pi,\tag{6.28}$$

where $\Pi=\|f_{k_1}^{j_1}\|_{L^2}\cdot\|f_{k_2}^{j_2}\|_{L^2}$. Using (6.27) and (6.28),

$$\begin{aligned}|n_k|\cdot&\sum_{j\geq\max(j_1,j_2)}2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\\&\leq C\sum_{2^j\geq C^{-1}|n_k||n_{k_1}|^\alpha}2^{-j/2}\beta_{k,j}\cdot|n_k|^{1/2}|n_{k_1}|^{-(\alpha-1)/2}\cdot2^{(j_1+j_2)/2}\Pi\\&\leq C|n_k|^{-1/2}|n_{k_1}|^{-(\alpha-1)/2}\cdot2^{(j_1+j_2)/2}\Pi.\end{aligned}$$

Using again (6.27) and (6.28),

$$\begin{aligned}|n_k|\cdot&\sum_{j\leq\max(j_1,j_2)}2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\\&\leq C\sum_{j\leq\max(j_1,j_2)}\beta_{k,j}\cdot2^{-\max(j_1,j_2)/2}|n_k\|n_{k_1}|^{-\alpha/2}\cdot2^{(j_1+j_2)/2}\Pi\\&\leq C\log(2+|n_{k_1}|)|n_{k_1}|^{-\alpha/2}\cdot2^{(j_1+j_2)/2}\Pi,\end{aligned}$$

since in this last estimate we may assume $2^{\max(j_1,j_2)}\geq C^{-1}|n_k||n_{k_1}|^\alpha$. The bound (6.26) follows from the last two estimates.

For (6.25) it suffices to prove that

$$\begin{aligned}\sum_{k'=-\infty}^{2}\sum_{j\in\mathbb{Z}_+}&2^{-j\delta}2^{-k'(1/2-\delta)}\|\mathbf{1}_{D_{0,k'}^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\\&\leq C|n_{k_1}|^{-\delta}\cdot2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2}\cdot2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}.\end{aligned}\tag{6.29}$$

In view of (6.14), $\mathbf{1}_{D_{0,k'}^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\equiv0$ unless

$$2^{\max(j_1,j_2,j)}\geq C^{-1}2^{k'}|n_{k_1}|^\alpha.\tag{6.30}$$

Using (5.16),

$$\|\mathbf{1}_{D_{0,k'}^j}\cdot(f_{k_1}^{j_1}*f_{k_2}^{j_2})\|_{L^2}\leq C2^{k'/2}2^{-\max(j_1,j_2)/2}\cdot2^{(j_1+j_2)/2}\Pi,\tag{6.31}$$

where $\Pi=\|f_{k_1}^{j_1}\|_{L^2}\cdot\|f_{k_2}^{j_2}\|_{L^2}$. Clearly,

$$\sum_{2^{\max(j_1,j_2,j)}\geq C^{-1}2^{k'}|n_{k_1}|^\alpha}2^{-j\delta}\cdot2^{-\max(j_1,j_2)/2}\leq C(1+2^{k'}|n_{k_1}|^\alpha)^{-\delta},$$

and the bound (6.29) follows from (6.31). □

Finally, we consider interactions of comparable frequencies.

Lemma 6.3. *Assume* $k_1,k_2\in\mathbb{Z}$, $k\in\mathbb{Z}\setminus\{0\}$, $(1+|n_{k_i}|)/(1+|n_k|)\in[2^{-20},2^{20}]$, $i=1,2$, $f_{k_1}\in Z_{k_1}$, *and* $f_{k_2}\in Z_{k_2}$. *Then*

$$\begin{aligned}(1+|n_k|)\cdot&\|\chi_k(\xi)(\tau-\omega(\xi)+i)^{-1}\cdot(f_{k_1}*f_{k_2})\|_{Z_k}\\&\leq C\Lambda(k_1,k_2,k)(1+|n_k|)^{-\delta}\cdot\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}},\end{aligned}\tag{6.32}$$

where, with $A = \min\big(\big||n_{k_1}| - |n_{k_2}|\big|, \big||n_k| - |n_{k_1}|\big|, \big||n_k| - |n_{k_2}|\big|\big)$,

$$\Lambda(k_1, k_2, k) = \begin{cases} 1 & \text{if } A \leq 2^{50}(1+|n_k|)^{1/2}; \\ A^{-1/2} & \text{if } A > 2^{50}(1+|n_k|)^{1/2}. \end{cases}$$

In addition, if $k_1, k_2 \in \mathbb{Z}$, $1 + |n_{k_i}| \in [2^{-20}, 2^{20}]$, $i = 1, 2$, $f_{k_1} \in Z_{k_1}$, *and* $f_{k_2} \in Z_{k_2}$, *then*

$$\|\chi_0(\xi)(\tau - \omega(\xi) + i)^{-1} \cdot (f_{k_1} * f_{k_2})\|_{X_0^{-1/2+\delta}} \leq C\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}. \tag{6.33}$$

Proof of Lemma 6.3. We analyze two cases: $|n_k| \geq 2^{100}$ and $|n_k| \leq 2^{100}$.

Case 1: $|n_k| \geq 2^{100}$. In view of the hypothesis, $|n_{k_i}| \geq 2^{70}$, $i = 1, 2$. Using (6.5), we may assume $f_{k_1} = f_{k_1}^{j_1}$ is supported in $D_{k_1}^{j_1}$ and $f_{k_2} = f_{k_2}^{j_2}$ is supported in $D_{k_2}^{j_2}$. For (6.32) it suffices to prove that

$$\begin{aligned} &|n_k| \cdot \sum_{j \in \mathbb{Z}_+} 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ &\leq C\Lambda(k_1, k_2, k)|n_k|^{-\delta} \cdot 2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2} \cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}. \end{aligned} \tag{6.34}$$

In view of (6.14), $\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2}) \equiv 0$ unless

$$2^{\max(j_1,j_2,j)} \geq C^{-1}|n_k|^{\alpha+1} \text{ and } \beta_{k,j} \leq C\max(\beta_{k_1,j_1}, \beta_{k_2,j_2}). \tag{6.35}$$

Using (5.18),

$$\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq 2^{(j_1+j_2+j)/2}2^{-\max(j_1,j_2,j)/2} \cdot \Pi,$$

where $\Pi = \|f_{k_1}^{j_1}\|_{L^2} \cdot \|f_{k_2}^{j_2}\|_{L^2}$. Thus, using (6.35),

$$\begin{aligned} &|n_k| \cdot \sum_{j \in \mathbb{Z}_+} 2^{-j/2}\beta_{k,j}\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \\ &\leq C\ln(2 + |n_k|)|n_k|^{-(\alpha-1)/2} \cdot 2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2} \cdot 2^{j_2/2}\beta_{k_2,j_2}\|f_{k_2}^{j_2}\|_{L^2}. \end{aligned} \tag{6.36}$$

It remains to prove the bound (6.34) in the case $A > 2^{50}(1 + |n_k|)^{1/2}$. In this case, using (5.17),

$$\|\mathbf{1}_{D_k^j} \cdot (f_{k_1}^{j_1} * f_{k_2}^{j_2})\|_{L^2} \leq 2^{(j_1+j_2+j)/2}[2^{\max(j_1,j_2,j)}A|n_k|^{\alpha-1}]^{-1/2} \cdot \Pi.$$

The bound (6.34) follows from (6.35).

Case 2: $|n_k| \leq 2^{100}$. Since $|n_{k_i}| \leq C$, $i = 1, 2$, for (6.32) we have to prove that

$$\|\chi_k(\xi)(\tau - \omega(\xi) + i)^{-1} \cdot (f_{k_1} * f_{k_2})\|_{Z_k} \leq C\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}. \tag{6.37}$$

It follows from the definitions and (6.2) that

$$\sum_{j \in \mathbb{Z}} 2^{j(1-\delta)}\|\eta_j(\tau - \omega(\xi))f_{k_l}\|_{L^2} \leq C\|f_{k_l}\|_{Z_{k_l}}, \tag{6.38}$$

for $l = 1, 2$, since $|n_{k_l}| \leq C$. The bound (6.37) follows easily using (5.16). The bound (6.33) also follows from (6.38) and (5.16). This completes the proof of the lemma. ∎

7. Multiplication by smooth bounded functions

In this section we consider operators on Z_k given by convolutions with Fourier transforms of certain smooth bounded functions. As in [10], for integers $N \geq 100$ we define the space of *admissible factors*

$$\begin{aligned} S_N^\infty = \{&m : \mathbb{R}^2 \to \mathbb{C} : m \text{ supported in } \mathbb{R}\times[-10,10] \text{ and} \\ &\|m\|_{S_N^\infty} = \sum_{\sigma_1=0}^{N} \|\partial_t^{\sigma_1} m\|_{L^\infty} + \sum_{\sigma_1=0}^{N}\sum_{\sigma_2=1}^{N} \|\partial_t^{\sigma_1}\partial_x^{\sigma_2} m\|_{L^2} < \infty\}. \end{aligned} \tag{7.1}$$

Notice that bounded functions such as $\eta_0(t)e^{iq\Psi}$, $q \in \mathbb{R}$, Ψ as in (3.10), are in S_N^∞. We also define the space of *restricted admissible factors*

$$\begin{aligned} S_N^2 = \{&m : \mathbb{R}^2 \to \mathbb{C} : m \text{ supported in } \mathbb{R}\times[-4,4] \text{ and} \\ &\|m\|_{S_N^2} = \sum_{\sigma_1=0}^{N}\sum_{\sigma_2=0}^{N} \|\partial_t^{\sigma_1}\partial_x^{\sigma_2} m\|_{L^2} < \infty\}. \end{aligned} \tag{7.2}$$

It is easy to see that bounded functions such as $\eta_0(t)\phi_{\mathrm{low}}e^{iq\Psi}$, $q \in \mathbb{R}$ (with the notation in section 3) are in S_N^2. Using the Sobolev embedding theorem, it is easy to verify the following properties:

$$\begin{cases} S_N^2 \subseteq S_{N-10}^\infty; \\ S_N^\infty \cdot S_N^\infty \subseteq S_{N-10}^\infty; \\ S_N^2 \cdot S_N^\infty \subseteq S_{N-10}^2; \\ \partial_x S_N^\infty \subseteq S_{N-10}^2. \end{cases} \tag{7.3}$$

For $k \in \mathbb{Z}$ we define

$$M_k^{\mathrm{high}} = \bigcup_{2^{j+20} \geq |n_k|^\alpha} J_j \text{ and } M_k^{\mathrm{low}} = (M_k^{\mathrm{high}})^c, \tag{7.4}$$

and

$$Z_k^{\mathrm{high}} = \big\{f_k \in Z_k : f_k \text{ is supported in } \{\tau - \omega(\xi) \in M_k^{\mathrm{high}}\}\big\}. \tag{7.5}$$

Clearly, $Z_k^{\mathrm{high}} = Z_k$ if $|n_k|^\alpha \leq 2^{20}$. The main result in this section is the following proposition.

Proposition 7.1. *(a) Assume $k_1, k_2 \in \mathbb{Z}$, $f_{k_1}^{\mathrm{high}} \in Z_{k_1}^{\mathrm{high}}$, $\epsilon \in \{-1, 0\}$, and $m \in S_{100}^\infty$. Then*

$$\begin{aligned} &\Big\|\chi_{k_2}(\xi_2)(\tau_2 - \omega(\xi_2) + i)^\epsilon \cdot (f_{k_1}^{\mathrm{high}} * \mathcal{F}(m))\Big\|_{Z_{k_2}} \\ &\qquad \leq C(1+|k_1-k_2|)^{-60}\ln(2+|n_{k_1}|)\|m\|_{S_{100}^\infty} \cdot \|(\tau_1 - \omega(\xi_1) + i)^\epsilon \cdot f_{k_1}^{\mathrm{high}}\|_{Z_{k_1}}. \end{aligned} \tag{7.6}$$

(b) Assume $k_1, k_2 \in \mathbb{Z}$, $k_1 \neq 0$, $f_{k_1} \in Z_{k_1}$, $\epsilon \in \{-1, 0\}$, and $m' \in S_{100}^2$. Then

$$\begin{aligned} &\Big\|\chi_{k_2}(\xi_2)(\tau_2 - \omega(\xi_2) + i)^\epsilon \cdot (f_{k_1} * \mathcal{F}(m'))\Big\|_{Z_{k_2}} \\ &\qquad \leq C(1+|k_1-k_2|)^{-60}\ln(2+|n_{k_1}|)\|m'\|_{S_{100}^2} \cdot \|(\tau_1 - \omega(\xi_1) + i)^\epsilon \cdot f_{k_1}\|_{Z_{k_1}}. \end{aligned} \tag{7.7}$$

In addition, if $f_0 \in X_0^0$ *(see definition* (6.1)*) then*

$$\begin{split}\Big\|\chi_{k_2}(\xi_2)(\tau_2-\omega(\xi_2)+i)^\epsilon\cdot(f_0\ast\mathcal{F}(m'))\Big\|_{Z_{k_2}}\\ \leq C(1+|k_2|)^{-60}\|m'\|_{S^2_{100}}\cdot\|(\tau_1-\omega(\xi_1)+i)^\epsilon\cdot f_0\|_{X_0^0}.\end{split} \tag{7.8}$$

The rest of this section is concerned with the proof of Proposition 7.1. The proofs in this section are similar to the proofs in [10, Section 9]. We may assume $\|m\|_{S^\infty_{100}}=\|m'\|_{S^2_{100}}=1$. For any $j''\in\mathbb{Z}_+$ and $k''\in\mathbb{Z}$ let

$$\begin{split}m_{k'',j''}&=\mathcal{F}^{-1}\big[\eta_{j''}(\tau)\widetilde{\eta}_{k''}(\xi)\mathcal{F}(m)\big],\\ m'_{k'',j''}&=\mathcal{F}^{-1}\big[\eta_{j''}(\tau)\widetilde{\eta}_{k''}(\xi)\mathcal{F}(m')\big].\end{split} \tag{7.9}$$

Let $m_{\leq k'',j''}=\sum_{k'''\leq k''}m_{k''',j''}$ and $m'_{\leq k'',j''}=\sum_{k'''\leq k''}m'_{k''',j''}$. Using (7.1) and (7.2), for any $j''\in\mathbb{Z}_+$ and $k''\in\mathbb{Z}$,

$$\begin{cases}\|m_{\leq k'',j''}\|_{L^\infty_{x,t}}\leq C2^{-80j''};\\ 2^{k''}\|m_{k'',j''}\|_{L^2_{x,t}}+\|m_{k'',j''}\|_{L^\infty_{x,t}}\leq C(1+2^{k''})^{-80}2^{-80j''},\end{cases} \tag{7.10}$$

and

$$\|m'_{\leq k'',j''}\|_{L^2_{x,t}}+(1+2^{k''})^{-80}\|m'_{k'',j''}\|_{L^2_{x,t}}\leq C2^{-80j''}. \tag{7.11}$$

We prove the proposition in several steps.

Step 1: proof of (7.6) **in the case** $k_1=k_2=0$. The estimate (7.6) in this case is the main reason for defining the space Z_0 as in (4.6), instead of, for example, $Z_0=X_0^{-1/2+\delta}$. Using the definition (4.6) it is easy to see that

$$\|(\tau-\omega(\xi)+i)^\epsilon h\|_{Z_0}\approx\|(\tau+i)^\epsilon h\|_{Z_0}. \tag{7.12}$$

Therefore, we have to prove that

$$\Big\|\chi_0(\xi_2)(\tau_2+i)^\epsilon\cdot(f_0\ast\mathcal{F}(m))\Big\|_{Z_0}\leq C\|(\tau_1+i)^\epsilon\cdot f_0\|_{Z_0} \tag{7.13}$$

for any $f_0\in Z_0$, $\epsilon\in\{0,-1\}$.

Assume first that $(\tau_1+i)^\epsilon f_0\in X_0=X_0^{-1}$. Using the representation (6.6), we may assume that $f_0=f^{j_1}_{0,k'}$ is an L^2 function supported in $D^{j_1}_{0,k'}$, $k'\leq 2$, $j_1\geq 0$,

$$\|(\tau_1+i)^\epsilon f_0\|_{X_0}\approx 2^{\epsilon j_1}2^{j_1(1-\delta)}2^{-k'}\|f^{j_1}_{0,k'}\|_{L^2}.$$

We decompose

$$m=\sum_{j''=0}^\infty m_{\leq k'-10,j''}+\sum_{k''=k'-9}^\infty\sum_{j''=0}^\infty m_{k'',j''}.$$

For (7.13) it suffices to prove that

$$\begin{split}&\sum_{j''=0}^\infty\Big\|\chi_0(\xi_2)(\tau_2+i)^\epsilon\cdot(f^{j_1}_{0,k'}\ast\mathcal{F}(m_{\leq k'-10,j''}))\Big\|_{X_0}\\ &+\sum_{k''=k'-9}^{10}\sum_{j''=0}^\infty\Big\|\chi_0(\xi_2)(\tau_2+i)^\epsilon\cdot(f^{j_1}_{0,k'}\ast\mathcal{F}(m_{k'',j''}))\Big\|_{Y_0}\\ &\leq C2^{\epsilon j_1}2^{j_1(1-\delta)}2^{-k'}\|f^{j_1}_{0,k'}\|_{L^2}\end{split} \tag{7.14}$$

Using the definition (4.4), the first sum in the left-hand is dominated by

$$C\sum_{j''=0}^{\infty}\sum_{j_2=0}^{\infty}2^{\epsilon j_2}2^{j_2(1-\delta)}2^{-k'}\|\eta_{j_2}(\tau_2)\cdot(f_{0,k'}^{j_1}*\mathcal{F}(m_{\leq k'-10,j''}))\|_{L^2}$$

We observe that $\eta_{j_2}(\tau_2)\cdot(f_{0,k'}^{j_1}*\mathcal{F}(m_{\leq k'-10,j''}))\equiv 0$ unless

$$(j_2,j'')\in L_{j_1}^C=\{(j_2,j'')\in\mathbb{Z}_+\times\mathbb{Z}_+:\,|j_1-j_2|\leq C\text{ or }j_1,j_2\leq j''+C\}\tag{7.15}$$

for some constant C. Thus, using (7.10) and Plancherel theorem, the expression above is dominated by

$$\sum_{(j_2,j'')\in L_{j_1}^C}2^{\epsilon j_2}2^{j_2(1-\delta)}2^{-k'}\|f_{0,k'}^{j_1}\|_{L^2}\|m_{\leq k'-10,j''}\|_{L^\infty}\leq C2^{\epsilon j_1}2^{j_1(1-\delta)}2^{-k'}\|f_{0,k'}^{j_1}\|_{L^2},$$

as desired. Similarly, the second sum in the left-hand side of (7.14) is dominated by

$$\begin{aligned}&C\sum_{k''=k'-9}^{10}\sum_{(j_2,j'')\in L_{j_1}^C}2^{\epsilon j_2}2^{j_2(1-\delta)}\|\mathcal{F}^{-1}(f_{0,k'}^{j_1})\cdot m_{k'',j''}))\|_{L_x^1L_t^2}\\ \leq&C\|f_{0,k'}^{j_1}\|_{L^2}\sum_{k''=k'-9}^{10}\sum_{(j_2,j'')\in L_{j_1}^C}2^{\epsilon j_2}2^{j_2(1-\delta)}\|m_{k'',j''}\|_{L_x^2L_t^\infty}\\ \leq&C2^{\epsilon j_1}2^{j_1(1-\delta)}2^{-k'}\|f_{0,k'}^{j_1}\|_{L^2},\end{aligned}$$

where in the last inequality we use $\|m_{k'',j''}\|_{L_x^2L_t^\infty}\leq C2^{-k''}2^{-70j''}$, compare with (7.10).

It remains to prove (7.13) in the case $(\tau_1+i)^\epsilon f_0\in Y_0$. Using the definition (4.5), we may assume $f_0=g_0^{j_1}$ is supported in $I_0\times J_{j_1}$ and

$$\|(\tau_1+i)^\epsilon f_0\|_{Y_0}\approx 2^{\epsilon j_1}2^{j_1(1-\delta)}\|\mathcal{F}^{-1}(g_0^{j_1})\|_{L_x^1L_t^2}.$$

For (7.13) it suffices to prove that

$$\sum_{j''\geq 0}\|\chi_0(\xi_2)(\tau_2+i)^\epsilon\cdot(g_0^{j_1}*\mathcal{F}(m_{\leq 10,j''})\|_{Y_0}\leq C2^{\epsilon j_1}2^{j_1(1-\delta)}\|\mathcal{F}^{-1}(g_0^{j_1})\|_{L_x^1L_t^2}.$$

Using Plancherel theorem and (7.10), the left-hand side is dominated by

$$C\sum_{(j_2,j'')\in L_{j_1}^C}2^{j_2(1-\delta)}2^{\epsilon j_2}\|\mathcal{F}^{-1}(g_0^{j_1})\cdot m_{\leq 10,j''}\|_{L_x^1L_t^2}\leq C2^{\epsilon j_1}2^{j_1(1-\delta)}\|\mathcal{F}^{-1}(g_0^{j_1})\|_{L_x^1L_t^2},$$

as desired.

Step 2: proof of (7.8) **in the case** $k_2=0$. Using the representation (6.6), we may assume that $f_0=f_{0,k'}^{j_1}$ is an L^2 function supported in $D_{0,k'}^{j_1}$, $k'\leq 2$, $j_1\geq 0$,

$$\|(\tau_1-\omega(\xi_1)+i)^\epsilon f_0\|_{X_0^0}\approx 2^{\epsilon j_1}2^{j_1(1-\delta)}\|f_{0,k'}^{j_1}\|_{L^2}.$$

In view of (7.12) it suffices to prove that

$$\sum_{j''=0}^{\infty}\|\chi_0(\xi_2)(\tau_2+i)^\epsilon\cdot(f_{0,k'}^{j_1}*\mathcal{F}(m'_{\leq 10,j''}))\|_{Y_0}\leq C2^{\epsilon j_1}2^{j_1(1-\delta)}\|f_{0,k'}^{j_1}\|_{L^2}.\tag{7.16}$$

Using Plancherel theorem and (4.5), the left-hand side of (7.16) is dominated by

$$C\|f^{j_1}_{0,k'}\|_{L^2}\sum_{(j_2,j'')\in L^C_{j_1}}2^{j_2(1-\delta)}2^{\epsilon j_2}\|m'_{\leq 10,j''}\|_{L^2_xL^\infty_t},$$

and the bound (7.16) follows since $\|m'_{\leq 10,j''}\|_{L^2_xL^\infty_t}\leq C2^{-70j''}$, compare with (7.11).

Step 3: proof of (7.6) **in the case** $k_1,k_2\in\mathbb{Z}\setminus\{0\}$**,** $|k_1-k_2|\leq 10$**.** In view of the definition of $Z^{\text{high}}_{k_1}$ and (6.5), we may assume that $f^{\text{high}}_{k_1}=f^{j_1}_{k_1}$ is an L^2 function supported in $D^{j_1}_{k_1}$, $2^{j_1+20}\geq|n_{k_1}|^\alpha$, $\|(\tau_1-\omega(\xi_1)+i)^\epsilon f^{\text{high}}_{k_1}\|_{Z_{k_1}}\approx 2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}$. We write

$$m=\sum_{j''=0}^\infty m_{\leq -100,j''}+\sum_{k''=-99}^\infty\sum_{j''=0}^\infty m_{k'',j''}. \tag{7.17}$$

For (7.6) it suffices to prove that for $\epsilon\in\{-1,0\}$

$$\begin{aligned}&\sum_{j''\geq 0}\big\|\chi_{k_2}(\xi_2)(\tau_2-\omega(\xi_2)+i)^\epsilon\cdot[f^{j_1}_{k_1}*\mathcal{F}(m_{\leq -100,j''})](\xi_2,\tau_2)\big\|_{Z_{k_2}}\\&+\sum_{k''\geq -99}\sum_{j''\geq 0}\big\|\chi_{k_2}(\xi_2)(\tau_2-\omega(\xi_2)+i)^\epsilon\cdot[f^{j_1}_{k_1}*\mathcal{F}(m_{k'',j''})](\xi_2,\tau_2)\big\|_{Z_{k_2}}\\&\leq C\ln(2+|n_{k_1}|)2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}.\end{aligned} \tag{7.18}$$

To bound the first sum in (7.18), we use (6.14) and $2^{j_1+20}\geq|n_{k_1}|^\alpha$ to conclude that $\mathbf{1}_{D^{j_2}_{k_2}}\cdot[f^{j_1}_{k_1}*\mathcal{F}(m_{\leq -100,j''})](\xi_2,\tau_2)\equiv 0$ unless $(j_2,j'')\in L^C_{j_1}$, see definition (7.15). Using Plancherel theorem and (7.10),

$$\big|\big|f^{j_1}_{k_1}*\mathcal{F}(m_{\leq -100,j''})\big|\big|_{L^2_{\xi_2,\tau_2}}\leq C2^{-80j''}\|f^{j_1}_{k_1}\|_{L^2}.$$

Thus the first sum in (7.18) is dominated by

$$C\sum_{(j_2,j'')\in L^C_{j_1}}2^{\epsilon j_2}2^{j_2/2}\beta_{k_2,j_2}2^{-80j''}\|f^{j_1}_{k_1}\|_{L^2},$$

which suffices (recall that $|k_1-k_2|\leq 10$).

To bound the second sum in (7.18) assume first that $\epsilon=0$. As before, we use (6.14) to conclude that $\mathbf{1}_{D^{j_2}_{k_2}}\cdot[f^{j_1}_{k_1}*\mathcal{F}(m_{k'',j''})](\xi_2,\tau_2)\equiv 0$ unless

$$|j_1-j_2|\leq 4\ \text{ or }\ j_1,j_2\leq\log_2(|n_{k_1}|^\alpha)+k''+j''+C. \tag{7.19}$$

Using Plancherel theorem and (7.10),

$$\big|\big|f^{j_1}_{k_1}*\mathcal{F}(m_{k'',j''})\big|\big|_{L^2_{\xi_2,\tau_2}}\leq C2^{-80k''}2^{-80j''}\|f^{j_1}_{k_1}\|_{L^2}. \tag{7.20}$$

Thus, using $j_1+C\geq\log_2(|n_{k_1}|^\alpha)$, the second sum in (7.18) is dominated by

$$\begin{aligned}&C\sum_{k''\geq -99}\sum_{j''\geq 0}2^{-80k''}2^{-80j''}\|f^{j_1}_{k_1}\|_{L^2}\sum_{j_2\leq j_1+k''+j''+C}2^{j_2/2}\beta_{k_2,j_2}\\&\leq C2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}.\end{aligned}$$

We bound now the second sum in (7.18) when $\epsilon=-1$. The main difficulty is the presence of the indices $j_2\ll j_1$. In fact, for indices $j_2\geq j_1-10$, the argument above applies since the left-hand side is multiplied by 2^{-j_2} and the right-hand side is multiplied by 2^{-j_1}. In view of (7.19), it suffices to prove that

$$\begin{aligned}&\sum_{k''+j''\geq j_1-\log_2(|n_{k_1}|^{\alpha})-C}\ \sum_{j_2\leq j_1-10}2^{-j_2/2}\beta_{k_2,j_2}\big|\big|\mathbf{1}_{D^{j_2}_{k_2}}\cdot[f^{j_1}_{k_1}*\mathcal{F}(m_{k'',j''})]\big|\big|_{L^2}\\&\leq C\ln(2+|n_{k_1}|)2^{-j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}.\end{aligned}\tag{7.21}$$

If $j_1\geq\log_2(|n_{k_1}|^{\alpha+1})-C$ the bound (7.21) follows easily from (7.20). Assuming $j_1\leq\log_2(|n_{k_1}|^{\alpha+1})-C$, the sum over $k''\geq\log_2(|n_{k_1}|)-C$ in (7.21) is bounded easily using again (7.20). If $k''\leq\log_2(|n_{k_1}|)-C$ then, using Corollary 5.2 (b)

$$\big|\big|\mathbf{1}_{D^{j_2}_{k_2}}\cdot[f^{j_1}_{k_1}*\mathcal{F}(m_{k'',j''})]\big|\big|_{L^2}\leq C2^{10(j''+k'')}2^{j_2/2}|n_{k_1}|^{-\alpha/2}\|m_{k'',j''}\|_{L^2}\|f^{j_1}_{k_1}\|_{L^2}.$$

The bound (7.21) follows using (7.10).

Step 4: proof of (7.7) **in the case** $k_2\neq 0$**,** $|k_1-k_2|\leq 10$**.** In view of (6.5), we may assume that $f_{k_1}=f^{j_1}_{k_1}$ is an L^2 function supported in $D^{j_1}_{k_1}$, $\|(\tau_1-\omega(\xi_1)+i)^{\epsilon}f_{k_1}\|_{Z_{k_1}}\approx 2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}$. In view of the case analyzed earlier, we may assume that

$$2^{j_1}\leq|n_{k_1}|^{\alpha}.$$

For (7.7) it suffices to prove that for $\epsilon\in\{-1,0\}$

$$\begin{aligned}&\sum_{k''\in\mathbb{Z}}\sum_{j''\geq 0}\big\|\chi_{k_2}(\xi_2)(\tau_2-\omega(\xi_2)+i)^{\epsilon}\cdot[f^{j_1}_{k_1}*\mathcal{F}(m'_{k'',j''})]\big\|_{Z_{k_2}}\\&\leq C\ln(2+|n_{k_1}|)2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}.\end{aligned}$$

Using the definition of the Z_k spaces, this is equivalent to proving that

$$\begin{aligned}&\sum_{k''\in\mathbb{Z}}\sum_{j''\geq 0}\sum_{j_2\geq 0}2^{\epsilon j_2}2^{j_2/2}\beta_{k_2,j_2}\|\mathbf{1}_{D^{j_2}_{k_2}}\cdot[f^{j_1}_{k_1}*|\mathcal{F}(m'_{k'',j''})|]\|_{L^2}\\&\leq C\ln(2+|n_{k_1}|)2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}.\end{aligned}\tag{7.22}$$

This follows using the bound (5.16) for $2^{k''}|n_{k_1}|^{\alpha}\leq 1$ and $2^{k''}\geq|n_{k_1}|/100$, and the bound (5.17) for $2^{k''}\in[|n_{k_1}|^{-\alpha},|n_{k_1}|/100]$.

Step 5: proof of (7.6) **and** (7.7) **in the case** $k_1,k_2\in\mathbb{Z}\setminus\{0\}$**,** $|k_1-k_2|\geq 10$**.** Clearly, it suffices to prove the stronger bound (7.7). In view of (6.5), we may assume that $f_{k_1}=f^{j_1}_{k_1}$ is an L^2 function supported in $D^{j_1}_{k_1}$, $\|(\tau_1-\omega(\xi_1)+i)^{\epsilon}f_{k_1}\|_{Z_{k_1}}\approx 2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}$. It suffices to prove that

$$\begin{aligned}&\sum_{2^{k''}\geq(|n_{k_1}|+|n_{k_2}|)^{1/2}}\ \sum_{j''\geq 0}\big\|\chi_{k_2}(\xi_2)(\tau_2-\omega(\xi_2)+i)^{\epsilon}\cdot[f^{j_1}_{k_1}*\mathcal{F}(m'_{k'',j''})]\big\|_{Z_{k_2}}\\&\leq C|k_1-k_2|^{-60}2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2}.\end{aligned}$$

Using (5.16) and (6.14), it suffices to prove that

$$\begin{aligned}&\sum_{2^{k''}\geq(|n_{k_1}|+|n_{k_2}|)^{1/2}}\ \sum_{j_2,j''}2^{\epsilon j_2}2^{j_2/2}\beta_{k_2,j_2}\|f^{j_1}_{k_1}\|_{L^2}2^{10k''+10j''}\|m'_{k'',j''}\|_{L^2}\\&\qquad\leq C|k_1-k_2|^{-60}2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f^{j_1}_{k_1}\|_{L^2},\end{aligned}\tag{7.23}$$

where the sum over j_2 and j'' is taken over the set

$$\{(j_2,j'')\in\mathbb{Z}_+\times\mathbb{Z}_+:|j_2-j_1|\leq C\text{ or }j_1,j_2\leq 10k''+10j''+C\}.$$

The bound (7.23) follows easily using (7.11).

Step 6: proof of (7.6) **and** (7.7) **in the case** $k_2 = 0$**,** $k_1 \neq 0$**.** In view of (7.12) and the discussion in Steps 3, 4, and 5, it suffices to prove that

$$\|\chi_0(\xi_2/2^{10})(\tau_2+i)^\epsilon(f_{k_1} * \mathcal{F}(m'))\|_{Z_0} \leq C|k_1|^{-60}\|(\tau_1-\omega(\xi_1)+i)^\epsilon f_{k_1}\|_{Z_{k_1}}$$

for any $f_{k_1} \in Z_{k_1}$, $\epsilon \in \{-1,0\}$. In view of (6.5), we may assume that $f_{k_1} = f_{k_1}^{j_1}$ is an L^2 function supported in $D_{k_1}^{j_1}$, $\|(\tau_1-\omega(\xi_1)+i)^\epsilon f_{k_1}\|_{Z_{k_1}} \approx 2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2}$. It suffices to prove that

$$\begin{aligned}\sum_{2^{k''+10}\geq|n_{k_1}|}\sum_{j''\geq 0}\sum_{j_2\geq 0} 2^{j_2(1-\delta)}2^{\epsilon j_2}\|\mathcal{F}^{-1}[\chi_0(\xi_2/2^{10})\eta_{j_2}(\tau_2)(f_{k_1}^{j_1} * \mathcal{F}(m'_{k'',j''}))]\|_{L^1_xL^2_t}\\ \leq C|k_1|^{-60}2^{\epsilon j_1}2^{j_1/2}\beta_{k_1,j_1}\|f_{k_1}^{j_1}\|_{L^2},\end{aligned} \tag{7.24}$$

where the restriction $2^{k''+10}\geq|n_{k_1}|$ may be assumed due to the support property of $f_{k_1}^{j_1}$. Using (7.11) and the support properties,

$$\begin{aligned}\|\mathcal{F}^{-1}[\chi_0(\xi_2/2^{10})\eta_{j_2}(\tau_2)(f_{k_1}^{j_1} * \mathcal{F}(m'_{k'',j''}))]\|_{L^1_xL^2_t} &\leq C\|f_{k_1}^{j_1}\|_{L^2}\|m'_{k'',j''}\|_{L^2_xL^\infty_t}\\ &\leq C2^{-70(k''+j'')}\|f_{k_1}^{j_1}\|_{L^2},\end{aligned}$$

and $\|\mathcal{F}^{-1}[\chi_0(\xi_2/2^{10})\eta_{j_2}(\tau_2)(f_{k_1}^{j_1} * \mathcal{F}(m'_{k'',j''}))]\|_{L^1_xL^2_t} = 0$ unless

$$|j_1-j_2|\leq 4 \quad \text{or} \quad j_1,j_2\leq j''+\log_2(|n_{k_1}|^\alpha)+C.$$

The bound (7.24) follows easily, using also $2^{j_1/2}\beta_{k_1,j_1}\geq 2^{j_1(1-\delta)}|n_{k_1}|^{-1}$.

Step 7: proof of (7.6) **and** (7.8) **in the case** $k_2\neq 0$**,** $k_1 = 0$**.** In view of (6.2) and the discussion in Steps 3, 4, and 5, it suffices to prove that

$$\|\chi_{k_2}(\xi_2)(\tau_2-\omega(\xi_2)+i)^\epsilon(f_0 * \mathcal{F}(m'))\|_{Z_{k_2}} \leq C|k_2|^{-60}\|(\tau_1-\omega(\xi_1)+i)^\epsilon f_0\|_{X_0^0}$$

for any $f_0\in X_0^0$ supported in $\{(\xi_1,\tau_1) : |\xi_1|\leq 2^{-20}\}$, $\epsilon\in\{-1,0\}$. In view of (6.6), we may assume that $f_0 = f_{0,k'}^{j_1}$ is an L^2 function supported in $D_{0,k'}^{j_1}$, $k'\leq -10$, $\|(\tau_1-\omega(\xi_1)+i)^\epsilon f_0\|_{X_0^0}\approx 2^{\epsilon j_1}2^{j_1(1-\delta)}\|f_{0,k'}^{j_1}\|_{L^2}$. It suffices to prove that

$$\begin{aligned}\sum_{2^{k''+10}\geq|n_{k_2}|}\sum_{j''\geq 0}\sum_{j_2\geq 0} 2^{j_2/2}\beta_{k_2,j_2}2^{\epsilon j_2}\|\mathbf{1}_{D_{k_2}^{j_2}}\cdot(f_{0,k'}^{j_1} * \mathcal{F}(m'_{k'',j''}))\|_{L^2}\\ \leq C|k_2|^{-60}2^{\epsilon j_1}2^{j_1(1-\delta)}\|f_{0,k'}^{j_1}\|_{L^2},\end{aligned} \tag{7.25}$$

where the restriction $2^{k''+10}\geq|n_{k_2}|$ may be assumed due to support properties. Using (7.11) and Plancherel theorem we have

$$\|\mathbf{1}_{D_{k_2}^{j_2}}\cdot(f_{0,k'}^{j_1} * \mathcal{F}(m'_{k'',j''}))\|_{L^2}\leq C2^{-70(k''+j'')}\|f_{0,k'}^{j_1}\|_{L^2}.$$

Using support properties we have $\|\mathbf{1}_{D_{k_2}^{j_2}}\cdot(f_{0,k'}^{j_1} * \mathcal{F}(m'_{k'',j''}))\|_{L^2} = 0$ unless

$$|j_1-j_2|\leq 4 \quad \text{or} \quad j_1,j_2\leq j''+\log_2(|n_{k_2}|^\alpha)+C.$$

The bound (7.25) follows easily, using also $2^{j_2/2}\beta_{k_2,j_2}\leq C2^{j_2(1-\delta)}$.

8. The main technical lemma

In this section we combine the estimates in sections 6 and 7 to prove our main global estimate. We define

$$\begin{aligned}&\|\cdot\|_{\widetilde{F}_k}=\|\cdot\|_{F_k}\text{ for }k\neq 0\text{ and }\|\cdot\|_{\widetilde{F}_0}=\|\mathcal{F}(\cdot)\|_{X_0^0},\\&\|\cdot\|_{\widetilde{N}_k}=\|\cdot\|_{N_k}\text{ for }k\neq 0\text{ and }\|\cdot\|_{\widetilde{N}_0}=\|(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(\cdot)\|_{X_0^0},\end{aligned}\tag{8.1}$$

see (6.1). These norms are clearly controlled by $\|\cdot\|_{F_k}$ and $\|\cdot\|_{N_k}$ respectively. Moreover,

$$\|\partial_x(\cdot)\|_{F_k}\leq C(1+|n_k|)\|\cdot\|_{\widetilde{F}_k},\qquad \|\partial_x(\cdot)\|_{N_k}\leq C(1+|n_k|)\|\cdot\|_{\widetilde{N}_k}.$$

Lemma 8.1. *Assume $\sigma\in[0,2]$ and $\Psi:\mathbb{R}\to\mathbb{R}$ is defined as in (3.10). For any $(k_1,k_2,k)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ assume that $a_{k_1,k_2,k}\in[-4,4]$, $w_{k_1,k_2,k},w'_{k_2,k_1,k}\in C(\mathbb{R}:\widetilde{H}^\sigma)$ are supported in $\mathbb{R}_x\times[-4,4]$, $\mathcal{F}(w_{k_1,k_2,k})\in Z_{k_1}$, $\mathcal{F}(w'_{k_2,k_1,k})\in Z_{k_2}$,*

$$\sup_{k_2,k\in\mathbb{Z}}\|\mathcal{F}(w_{k_1,k_2,k})\|_{Z_{k_1}}=\Gamma_{k_1}\ \text{and}\ \sup_{k_1,k\in\mathbb{Z}}\|\mathcal{F}(w'_{k_2,k_1,k})\|_{Z_{k_2}}=\Gamma'_{k_2}.$$

Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma+\delta/4}\Big(&\sum_{k_1,k_2\in\mathbb{Z}}\|\partial_xP_k(e^{ia_{k_1,k_2,k}\Psi}w_{k_1,k_2,k}w'_{k_2,k_1,k})\|_{N_k}\\&+\|P_k(\partial_x(e^{ia_{k_1,k_2,k}\Psi})w_{k_1,k_2,k}w'_{k_2,k_1,k})\|_{N_k}\Big)^2\\&\leq C\Big(\sum_{k_1\in\mathbb{Z}}(1+|n_{k_1}|)^{-\delta/4}{\Gamma_{k_1}}^2\Big)\Big(\sum_{k_2\in\mathbb{Z}}(1+|n_{k_2}|)^{2\sigma-\delta/4}{\Gamma'_{k_2}}^2\Big)\\&+C\Big(\sum_{k_1\in\mathbb{Z}}(1+|n_{k_1}|)^{2\sigma-\delta/4}{\Gamma_{k_1}}^2\Big)\Big(\sum_{k_2\in\mathbb{Z}}(1+|n_{k_2}|)^{-\delta/4}{\Gamma'_{k_2}}^2\Big).\end{aligned}\tag{8.2}$$

Proof of Lemma 8.1. Assume first that

$$a_{k_1,k_2,k}=0\text{ for any }k_1,k_2,k\in\mathbb{Z}.\tag{8.3}$$

In this case we use only the dyadic estimates in section 6. For any $k\in\mathbb{Z}$ let

$$Q_k=\{(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}:(I_{k_1}+I_{k_2})\cap I_k\neq\emptyset\text{ and }|n_{k_1}|\leq|n_{k_2}|\}.$$

With J_l as in section 4, it suffices to prove the (slightly stronger) estimate

$$\begin{aligned}&\sum_{l=0}^\infty 2^{(2\sigma+2+\delta/2)l}\sum_{n_k\in J_l}\big(\sum_{(k_1,k_2)\in Q_k}\|P_k(w_{k_1,k_2,k}\cdot w'_{k_2,k_1,k})\|_{\widetilde{N}_k}\big)^2\\&\quad\leq C\big(\sum_{l_1=0}^\infty 2^{-(\delta/2)l_1}\sum_{n_{k_1}\in J_{l_1}}{\Gamma_{k_1}}^2\big)\big(\sum_{l_2=0}^\infty 2^{(2\sigma-\delta/2)l_2}\sum_{n_{k_2}\in J_{l_2}}{\Gamma'_{k_2}}^2\big).\end{aligned}\tag{8.4}$$

We fix now $l\in\mathbb{Z}_+$ and estimate

$$2^{2l}\sum_{n_k\in J_l}\big(\sum_{(k_1,k_2)\in Q_k}\|P_k(w_{k_1,k_2,k}\cdot w'_{k_2,k_1,k})\|_{\widetilde{N}_k}\big)^2.$$

We split the set $Q_k = Q'_k \cup Q''_k \cup Q'''_k$, where we define the three subsets according to the conditions of Lemma 6.1, Lemma 6.2, and Lemma 6.3:

$$Q'_k = \begin{cases} \{(k_1,k_2)\in Q_k : |n_{k_1}| \leq |n_k|/2^{10}\} & \text{if } |n_k|\geq 2^{20}\\ \emptyset & \text{if } |n_k|<2^{20}\end{cases}$$
$$Q''_k = \{(k_1,k_2)\in Q_k : |n_{k_1}|\geq 2^{10}(1+|n_k|)\}$$
$$Q'''_k = \{(k_1,k_2)\in Q_k : (1+|n_{k_i}|)/(1+|n_k|)\in[2^{-20},2^{20}] \text{ for } i=1,2\}$$

Using Lemma 6.1 we estimate

$$\begin{aligned}&2^{2l}\sum_{n_k\in J_l}\Big(\sum_{(k_1,k_2)\in Q'_k}\|P_k(w_{k_1,k_2,k}\cdot w'_{k_2,k_1,k})\|_{\widetilde{N}_k}\Big)^2\\ &\leq C2^{-2\delta l}\sum_{n_k\in J_l}\Big(\sum_{l_1\leq l-10}\sum_{l_2\in[l-5,l+5]}2^{-l_1/2}\Sigma'(l_1,l_2,n_k)\Big)^2\\ &\leq C2^{-3l\delta/2}\sum_{n_k\in J_l}\sum_{l_1\leq l-10}\sum_{l_2\in[l-5,l+5]}2^{-l_1}\Sigma'(l_1,l_2,n_k)^2,\end{aligned}\tag{8.5}$$

where

$$\Sigma'(l_1,l_2,n_k)=\sum_{n_{k_1}\in J_{l_1},n_{k_2}\in J_{l_2},|n_{k_1}+n_{k_2}-n_k|\leq 2^{l/2+10}}\Gamma_{k_1}\Gamma'_{k_2}.$$

We observe that for any $n_k\in J_l$

$$\big|\{(n_{k_1},n_{k_2})\in J_{l_1}\times J_{l_2} : |n_{k_1}+n_{k_2}-n_k|\leq 2^{l/2+10}\}\big|\leq C2^{l_1/2}.$$

Indeed, for any $n_{k_1}\in J_{l_1}$ there are at most C numbers $n_{k_2}\in J_{l_2}$ for which $|n_{k_1}+n_{k_2}-n_k|\leq 2^{l/2+10}$. Moreover, we observe that for fixed k_1,k_2

$$\big|\{n_k\in J_l : |n_{k_1}+n_{k_2}-n_k|\leq 2^{l/2+10}\}\big|\leq C.$$

Thus

$$\sum_{n_k\in J_l}\Sigma'(l_1,l_2,n_k)^2\leq C2^{l_1/2}\Big(\sum_{n_{k_1}\in J_{l_1}}{\Gamma_{k_1}}^2\Big)\Big(\sum_{n_{k_2}\in J_{l_2}}{\Gamma'_{k_2}}^2\Big),$$

which shows that the left-hand side of (8.5) is dominated by

$$C2^{-3l\delta/2}\Big(\sum_{l_1=0}^{\infty}2^{-l_1/2}\sum_{n_{k_1}\in J_{l_1}}{\Gamma_{k_1}}^2\Big)\Big(\sum_{l_2\in[l-5,l+5]}\sum_{n_{k_2}\in J_{l_2}}{\Gamma'_{k_2}}^2\Big).\tag{8.6}$$

Using now Lemma 6.2 we estimate

$$\begin{aligned}&2^{2l}\sum_{n_k\in J_l}\Big(\sum_{(k_1,k_2)\in Q''_k}\|P_k(w_{k_1,k_2,k}\cdot w'_{k_2,k_1,k})\|_{\widetilde{N}_k}\Big)^2\\ &\leq C2^{-l}\sum_{n_k\in J_l}\Big(\sum_{l_1,l_2\geq l+5,|l_1-l_2|\leq 2}2^{-\delta l_1}\Sigma''(l_1,l_2,n_k)\Big)^2\\ &\leq C2^{-(1+\delta)l}\sum_{n_k\in J_l}\sum_{l_1,l_2\geq l+5,|l_1-l_2|\leq 2}2^{-\delta l_1}\Sigma''(l_1,l_2,n_k)^2,\end{aligned}\tag{8.7}$$

where

$$\Sigma''(l_1,l_2,n_k)=\sum_{n_{k_1}\in J_{l_1},n_{k_2}\in J_{l_2},|n_{k_1}+n_{k_2}-n_k|\leq 2^{l_1/2+10}}\Gamma_{k_1}\Gamma'_{k_2}.$$

The Cauchy-Schwarz inequality implies

$$\Sigma''(l_1,l_2,n_k)^2 \leq \Big(\sum_{n_{k_1}\in J_{l_1}} {\Gamma_{k_1}}^2\Big)\Big(\sum_{n_{k_2}\in J_{l_2}} {\Gamma'_{k_2}}^2\Big).$$

Since $|\{n_k : n_k \in J_l\}| \leq C2^{l/2}$ the left-hand side of (8.7) is dominated by

$$C2^{-(1/2+\delta)l}\Big(\sum_{l_1=l+5}^{\infty} 2^{-(\delta/2)l_1}\sum_{n_{k_1}\in J_{l_1}} {\Gamma_{k_1}}^2\Big)\Big(\sum_{l_2=l+5}^{\infty} 2^{-(\delta/2)l_2}\sum_{n_{k_2}\in J_{l_2}} {\Gamma'_{k_2}}^2\Big). \tag{8.8}$$

Finally, using Lemma 6.3 we estimate

$$\begin{aligned}
&2^{2l}\sum_{n_k\in J_l}\Big(\sum_{(k_1,k_2)\in Q_k'''}\|P_k(w_{k_1,k_2,k}\cdot w'_{k_2,k_1,k})\|_{\widetilde{N}_k}\Big)^2\\
&\leq C2^{-2\delta l}\sum_{n_k\in J_l}\Big[\sum_{\substack{1+|n_{k_1}|,1+|n_{k_2}|\in[2^{l-40},2^{l+40}]\\ |n_{k_1}+n_{k_2}-n_k|\leq C2^{l/2+10}}}\Lambda(k_1,k_2,k)\Gamma_{k_1}\Gamma'_{k_2}\Big]^2\\
&\leq C2^{-2\delta l}\Big[\sum_{1+|n_{k_1}|\in[2^{l-40},2^{l+40}]}{\Gamma_{k_1}}^2\Big]\Big[\sum_{1+|n_{k_2}|\in[2^{l-40},2^{l+40}]}{\Gamma'_{k_2}}^2\Big].
\end{aligned} \tag{8.9}$$

The bound (8.4) follows from (8.6), (8.8), and (8.9).

We remove now the hypothesis (8.3). Let $\Gamma(\sigma)$ denote the right-hand side of (8.2). Since $\partial_x(e^{ia_{k_1,k_2,k}\Psi})\eta_0(t/4)\in S^2_{100}$, it follows from Proposition 7.1 (b) (with $\epsilon=-1$) that

$$\begin{aligned}
&\|P_k(\partial_x(e^{ia_{k_1,k_2,k}\Psi})w_{k_1,k_2,k}w'_{k_2,k_1,k})\|_{N_k}\\
&\leq C\sum_{\nu\in\mathbb{Z}}(1+|\nu|)^{-60}\ln(2+|n_{k+\nu}|)\|P_{k+\nu}(w_{k_1,k_2,k}w'_{k_2,k_1,k})\|_{\widetilde{N}_{k+\nu}}
\end{aligned}$$

for any $k,k_1,k_2\in\mathbb{Z}$. Thus, using (8.4),

$$\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma+2+\delta/4}\Big(\sum_{k_1,k_2\in\mathbb{Z}}\|P_k(\partial_x(e^{ia_{k_1,k_2,k}\Psi})w_{k_1,k_2,k}w'_{k_2,k_1,k})\|_{N_k}\Big)^2\leq\Gamma(\sigma). \tag{8.10}$$

To control the first term in the right-hand of (8.2) we decompose the functions $w_{k_1,k_2,k}$ and $w'_{k_2,k_1,k}$ into high and low modulation components according to (7.4)

$$\begin{aligned}
w_{k_1,k_2,k} &= u_{k_1,k_2,k}+v_{k_1,k_2,k}\\
&=\mathcal{F}^{-1}(\mathbf{1}_{M^{\text{high}}_{k_1}}(\tau-\omega(\xi))\mathcal{F}(w_{k_1,k_2,k}))+\mathcal{F}^{-1}(\mathbf{1}_{M^{\text{low}}_{k_1}}(\tau-\omega(\xi))\mathcal{F}(w_{k_1,k_2,k})),
\end{aligned}$$

and

$$\begin{aligned}
w'_{k_2,k_1,k} &= u'_{k_2,k_1,k}+v'_{k_2,k_1,k}\\
&=\mathcal{F}^{-1}(\mathbf{1}_{M^{\text{high}}_{k_2}}(\tau-\omega(\xi))\mathcal{F}(w'_{k_2,k_1,k}))+\mathcal{F}^{-1}(\mathbf{1}_{M^{\text{low}}_{k_2}}(\tau-\omega(\xi))\mathcal{F}(w'_{k_2,k_1,k})).
\end{aligned}$$

It follows from Proposition 7.1 (a) (with $\epsilon=0$) that, for any $\nu\in\mathbb{Z}$

$$\sup_{k_2,k\in\mathbb{Z}}\|\mathcal{F}(P_{k_1+\nu}(e^{ia_{k_1,k_2,k}\Psi}\eta_0(t/4)u_{k_1,k_2,k}))\|_{Z_{k_1+\nu}}\leq C(1+|\nu|)^{-50}\ln(2+|n_{k_1}|)\Gamma_{k_1}.$$

Thus, using (8.4) with $\widetilde{u}_{k_1+\nu,k_1,k_2,k} = P_{k_1+\nu}(e^{ia_{k_1,k_2,k}\Psi}\eta_0(t/4)u_{k_1,k_2,k})$

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma+\delta/4}\Big(\sum_{k_1,k_2\in\mathbb{Z}}\|\partial_x P_k(e^{ia_{k_1,k_2,k}\Psi}u_{k_1,k_2,k}w'_{k_2,k_1,k})\|_{N_k}\Big)^2\\&\leq\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma+\delta/4}\Big(\sum_{\nu,k_1,k_2\in\mathbb{Z}}\|\partial_x P_k(\widetilde{u}_{k_1+\nu,k_1,k_2,k}w'_{k_2,k_1,k})\|_{N_k}\Big)^2\\&\leq\Gamma(\sigma),\end{aligned}\tag{8.11}$$

as desired. Similarly,

$$\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma+\delta/4}\Big(\sum_{k_1,k_2\in\mathbb{Z}}\|\partial_x P_k(e^{ia_{k_1,k_2,k}\Psi}v_{k_1,k_2,k}u'_{k_2,k_1,k}\eta_0(t/4))\|_{N_k}\Big)^2\leq\Gamma(\sigma).\tag{8.12}$$

Finally, to control the contribution of $v_{k_1,k_2,k}v'_{k_2,k_1,k}$ we make the observation that the product of two functions of low modulation has high modulation:

$$\mathcal{F}(P_{k'}(v_{k_1,k_2,k}v'_{k_2,k_1,k}))\in Z_{k'}^{\mathrm{high}}\text{ for any }k'\in\mathbb{Z}.$$

This follows from (6.14) (recall that $Z_k^{\mathrm{high}} = Z_k$ if $|n_k|^\alpha \leq 2^{20}$). It follows from Proposition 7.1 (a) (with $\epsilon = -1$) that

$$\begin{aligned}&\|\partial_x P_k(e^{ia_{k_1,k_2,k}\Psi}\eta_0(t/4)v_{k_1,k_2,k}v'_{k_2,k_1,k})\|_{N_k}\\&\leq\|P_k(\partial_x(e^{ia_{k_1,k_2,k}\Psi})\eta_0(t/4)v_{k_1,k_2,k}v'_{k_2,k_1,k})\|_{N_k}\\&+C\sum_{\nu\in\mathbb{Z}}(1+|\nu|)^{-50}\ln(2+|n_{k+\nu}|)\|\partial_x P_{k+\nu}(v_{k_1,k_2,k}v'_{k_2,k_1,k})\|_{N_{k+\nu}}.\end{aligned}$$

Thus, using (8.4) and (8.10)

$$\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma+\delta/4}\Big(\sum_{k_1,k_2\in\mathbb{Z}}\|\partial_x P_k(e^{ia_{k_1,k_2,k}\Psi}\eta_0(t/4)v_{k_1,k_2,k}v'_{k_2,k_1,k})\|_{N_k}\Big)^2\leq\Gamma(\sigma).\tag{8.13}$$

The lemma follows from (8.10), (8.11), (8.12), and (8.13). ∎

9. Commutator estimates

We prove now several commutator estimates. Recall the definitions (8.1).

Lemma 9.1. *Assume that $R(D) = \partial_x^{\sigma_1}D^{\sigma_2}$ for $\sigma_1 \in \{0,1\}$ and $1 < \sigma_2 < 2$ or $\sigma_2 = 0$. Assume further that $m, m' \in S^\infty_{150}$, $\|m\|_{S^\infty_{150}} + \|m'\|_{S^\infty_{150}} \leq 1$. Then, for any $\sigma \in [0,2]$ and $k,\mu\in\mathbb{Z}$*

$$\begin{aligned}&(1+|\mu|)^{40}(1+|n_{k+\mu}|)^{2\sigma}\|P_{k+\mu}[mP_kR(D)(m'w)-P_kR(D)(mm'w)]\|^2_{F_{k+\mu}}\\&\leq C\sum_{\nu\in\mathbb{Z}}(1+|\nu|)^{-40}(1+|n_{k+\nu}|)^{2\sigma+2\sigma_1+2\sigma_2-1}\ln^2(2+|n_{k+\nu}|)\|P_{k+\nu}w\|^2_{\widetilde{F}_{k+\nu}},\end{aligned}\tag{9.1}$$

and

$$\begin{aligned}&(1+|\mu|)^{40}(1+|n_{k+\mu}|)^{2\sigma}\|P_{k+\mu}[mP_kR(D)(m'w)-P_kR(D)(mm'w)]\|^2_{N_{k+\mu}}\\&\leq C\sum_{\nu\in\mathbb{Z}}(1+|\nu|)^{-40}(1+|n_{k+\nu}|)^{2\sigma+2\sigma_1+2\sigma_2-1}\ln^2(2+|n_{k+\nu}|)\|P_{k+\nu}w\|^2_{\widetilde{N}_{k+\nu}}.\end{aligned}\tag{9.2}$$

Proof of Lemma 9.1. We decompose $w = \sum_{\nu\in\mathbb{Z}} P_{k+\nu}w$ and define the function

$$q(\xi) = (i\xi)^{\sigma_1}|\xi|^{\sigma_2}\chi_k(\xi).$$

We calculate

$$\begin{aligned}
&\mathcal{F}[P_{k+\mu}[mP_kR(D)(m'P_{k+\nu}w) - P_kR(D)(mm'P_{k+\nu}w)]](\xi,\tau)\\
=&C\chi_{k+\mu}(\xi)\int_{\mathbb{R}\times\mathbb{R}} \mathcal{F}(m)(\xi_1,\tau_1)\mathcal{F}(m'P_{k+\nu}w)(\xi-\xi_1,\tau-\tau_1)[q(\xi)-q(\xi-\xi_1)]\,d\xi_1d\tau_1\\
=&C\int_{\mathbb{R}\times\mathbb{R}} \mathcal{F}(P_{k+\nu}w)(\xi-\xi_2,\tau-\tau_2)\cdot K(\xi_2,\tau_2,\xi)\,d\xi_2d\tau_2\\
=&C\int_{I_{k+\mu}} \mathrm{H}(\xi-\gamma)\Big[\int_{\mathbb{R}^2} \mathcal{F}(P_{k+\nu}w)(\xi-\xi_2,\tau-\tau_2)\cdot K'(\xi_2,\tau_2,\gamma)\,d\xi_2d\tau_2\Big]d\gamma,
\end{aligned}$$

where H denotes the Heaviside-function and

$$\begin{aligned}
K(\xi_2,\tau_2,\xi) = \int_{\mathbb{R}\times\mathbb{R}} &\mathcal{F}(m)(\xi_1,\tau_1)\mathcal{F}(m')(\xi_2-\xi_1,\tau_2-\tau_1)\\
&\cdot[q(\xi)-q(\xi-\xi_1)]\chi_{k+\mu}(\xi)\,d\xi_1d\tau_1;
\end{aligned}\tag{9.3}$$

and

$$\begin{aligned}
K'(\xi_2,\tau_2,\gamma) = \int_{\mathbb{R}\times\mathbb{R}} &\mathcal{F}(m)(\xi_1,\tau_1)\mathcal{F}(m')(\xi_2-\xi_1,\tau_2-\tau_1)\\
&\cdot\partial_\gamma[(q(\gamma)-q(\gamma-\xi_1))\chi_{k+\mu}(\gamma)]\,d\xi_1d\tau_1.
\end{aligned}\tag{9.4}$$

Case 1: $k+\mu\neq 0$**.** By definition of the norms it follows for $\epsilon\in\{0,-1\}$

$$\begin{aligned}
&\|(\tau-\omega(\xi)+i)^\epsilon\mathcal{F}[P_{k+\mu}[mP_kR(D)(m'P_{k+\nu}w) - P_kR(D)(mm'P_{k+\nu}w)]]\|_{Z_{k+\mu}}\\
\leq& C\int_{I_{k+\mu}} n(\gamma)d\gamma,
\end{aligned}$$

where

$$n(\gamma) := \Big\|(\tau-\omega(\xi)+i)^\epsilon\int_{\mathbb{R}^2}\mathcal{F}(P_{k+\nu}w)(\xi-\xi_2,\tau-\tau_2)\cdot K'(\xi_2,\tau_2,\gamma)\,d\xi_2d\tau_2\Big\|_{Z_{k+\mu}}.$$

For $\gamma\in I_{k+\mu}$ fixed it is easy to see that

$$\mathcal{F}^{-1}(K'(.,.,\gamma)) = Cm'\cdot\mathcal{F}^{-1}\big[\mathcal{F}(m)(\xi_1,\tau_1)\cdot\partial_\gamma[(q(\gamma)-q(\gamma-\xi_1))\chi_{k+\mu}(\gamma)]\big]$$

is a restricted admissible factor and

$$\|\mathcal{F}^{-1}(K'(.,.,\gamma))\|_{S^2_{100}} \leq C(1+|\mu|)^{-40}(1+|n_{k+\mu}|)^{\sigma_1+\sigma_2-1}.$$

The bounds (9.1) and (9.2) follow from estimate (7.7) in the case $|k+\nu|\geq 1$ and from (7.8) in the case $k+\nu=0$, combined with the Cauchy-Schwarz inequality. Recall that the integration in γ is over an interval of length $\approx(1+|n_{k+\mu}|)^{1/2}$.

Case 2: $k+\mu=0$ **and** $|k|\geq 2$**.** We use the following decomposition: For any $k'\in\mathbb{Z}$ define

$$m_{k'} = \mathcal{F}^{-1}\big[\widetilde{\eta}_{k'}\mathcal{F}(m)\big]$$

and set $m_{\leq k'} = \sum_{k''\leq k'} m_{k''}$, $m_{>k'} = \sum_{k''>k'} m_{k''}$. If m satisfies (7.1) we obtain

$$\|m_{>k'}\|_{S^2_{100}} \leq C2^{-k'}(1+2^{k'})^{-80}.$$

We have

$$\begin{aligned}
&P_{k+\mu}[mP_kR(D)(m'P_{k+\nu}w) - P_kR(D)(mm'P_{k+\nu}w)]\\
=&P_0[m_{>k'}P_kR(D)(m'P_{k+\nu}w)]
\end{aligned}\tag{9.5}$$

for $k' = \log_2(|n_k|) - 10$. We apply (7.7)

$$\begin{aligned} &\|(\tau-\omega(\xi)+i)^\epsilon \mathcal{F} P_0[m_{>k'} P_k R(D)(m' P_{k+\nu} w)]](\xi,\tau)\|_{Z_0} \\ &\leq C(1+|k|)^{-50}(1+|n_k|)^{-80}\|\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}(m' P_{k+\nu} w)(\tau,\xi)\|_{Z_k}. \end{aligned} \tag{9.6}$$

If $|\nu| \geq 2$ we repeat the same argument with m':

$$\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}(m' P_{k+\nu} w)(\tau,\xi) = \chi_k(\xi)(\tau+\omega(\xi)+i)^\epsilon \mathcal{F}(m'_{>k'} P_{k+\nu} w)(\tau,\xi).$$

with $k' = \log_2(1+|n_\nu|) - 10$, and we apply (7.7) if $k+\nu \neq 0$ and (7.8) otherwise.

If $|\nu| \leq 1$ we can afford to use the crude bound

$$\begin{aligned} &\|\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}(m' P_{k+\nu} w)(\tau,\xi)\|_{Z_k} \\ &\leq C(1+|n_k|^{\alpha+1})\|\chi_{k+\nu}(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F} w(\tau,\xi)\|_{Z_{k+\nu}}, \end{aligned} \tag{9.7}$$

which is straightforward, compare (7.6) and its proof for the high modulation case. In conjunction with (9.6) this finishes the discussion of Case 2.

Case 3: $k+\mu = 0$ and $|k| \leq 1$.

Subcase 3a: $|\nu| \geq 3$. Define $\nu' = \log_2(1+|n_\nu|) - 10$. It suffices to consider

$$\|\chi_0(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[mR(D)P_k[m'_{>\nu'} P_{k+\nu} w] - R(D)P_k[[mm']_{>\nu'} P_{k+\nu} w]\|_{Z_0}.$$

We apply the triangle inequality and obtain the estimate

$$\begin{aligned} &\|\chi_0(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[mR(D)P_k[m'_{>\nu'} P_{k+\nu} w]]\|_{Z_0} \\ \leq &C\|\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[m'_{>\nu'} P_{k+\nu} w]\|_{Z_k} \end{aligned}$$

for the first contribution by applying (7.6). Due to

$$\|m_{>\nu'}\|_{S^2_{100}} \leq C(1+|n_\nu|)^{-80}$$

we can now apply (7.7) to conclude further

$$\begin{aligned} &\|\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[m'_{>\nu'} P_{k+\nu} w]\|_{Z_k} \\ &\leq C(1+|\nu|)^{-60}(1+|n_\nu|)^{-60}\|\chi_{k+\nu}(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F} w\|_{Z_{k+\nu}}, \end{aligned}$$

which is sufficient. For the second contribution we directly use the estimate (7.7) and obtain

$$\begin{aligned} &\|\chi_0(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[R(D)P_k[[mm']_{\nu'} P_{k+\nu} w]\|_{Z_0} \\ &\leq C(1+|\nu|)^{-60}(1+|n_\nu|)^{-60}\|\chi_{k+\nu}(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F} w\|_{Z_{k+\nu}}, \end{aligned}$$

because

$$\|[mm']_{>\nu'}\|_{S^2_{100}} \leq C(1+|n_\nu|)^{-80}.$$

Subcase 3b: $|\nu| \leq 2$. The only issue here is the structure low frequency component Z_0 of the norms. We decompose $m = m_{\leq -10} + m_{>-10}$ and $m' P_{k+\nu} w = [m' P_{k+\nu} w]_{\leq -20} + [m' P_\nu w]_{>-20}$.

Contribution i): $m_{\leq -10}$ and $[m' P_{k+\nu} w]_{\leq -20}$. In the case where $\sigma_1 = \sigma_2 = 0$ we have

$$m_{\leq -10} R(D) P_k [m' P_{k+\nu} w]_{\leq -20} - R(D) P_k m_{\leq -10} [m' P_{k+\nu} w]_{\leq -20} = 0,$$

and if $\sigma_1 = 1$, $\sigma_2 = 0$ we obtain

$$\begin{aligned} &m_{\leq -10} R(D) P_k [m' P_{k+\nu} w]_{\leq -20} - R(D) P_k m_{\leq -10} [m' P_{k+\nu} w]_{\leq -20} \\ = &-(\partial_x m_{\leq -10}) P_k [m' P_{k+\nu} w]_{\leq -20}, \end{aligned}$$

hence we can assume $k = 0$ and the presence of P_k is redundant. In this case, we decompose

$$m' = m'_{\leq -30} + m'_{>-30}, \quad P_{k+\nu} w = P_{k+\nu} w_{\leq -30} + P_{k+\nu} w_{>-30}.$$

For the first contribution ($m'_{\leq -30}$ and $P_{k+\nu} w_{\leq -30}$) we obtain the bound

$$\|(\partial_x m_{\leq -10}) m'_{\leq -30} [P_{k+\nu} w]_{\leq -30}\|_{Z_0} \leq C \|P_{k+\nu} w\|_{\widetilde{Z}_{k+\nu}}$$

by using $\|\partial_x m_{\leq -10} \cdot m'_{\leq -30}\|_{S^2_{100}} \leq 1$ and (7.8). For the second contribution ($m'_{>-30}$ and $P_{k+\nu} w$) we obtain the bound

$$\|(\partial_x m_{\leq -10}) P_k [m'_{>-30} P_{k+\nu} w]_{\leq -20}\|_{Z_0} \leq C \|P_{k+\nu} w\|_{\widetilde{Z}_{k+\nu}}$$

where we successively use (7.7) or (7.8) as well as

$$\|\partial_x m_{\leq -10}\|_{S^2_{100}} \leq 1, \quad \|m'_{>-30}\|_{S^2_{100}} \leq 1.$$

Concerning the third contribution ($m'_{\leq -30}$ and $[P_{k+\nu} w]_{>-30}$) we successively apply (7.8) and (7.6) and we observe that $\|[P_{k+\nu} w]_{>-30}\|_{Z_0} \leq C \|P_{k+\nu} w\|_{\widetilde{Z}_0}$.

If $\sigma_1 + \sigma_2 > 1$ we apply the triangle inequality and use (7.6) for the first term

$$\begin{aligned} &\|\chi_0(\xi)(\tau - \omega(\xi) + i)^\epsilon \mathcal{F}[m_{\leq -10} R(D) P_k [m' P_{k+\nu} w]_{\leq -20}]\|_{Z_0} \\ \leq{}& C \|\chi_0(\xi)(\tau - \omega(\xi) + i)^\epsilon \mathcal{F}[R(D)[m' P_{k+\nu} w]_{\leq -20}\|_{Z_0} \\ \leq{}& C \sum_{j=0}^{\infty} 2^{j\epsilon} 2^{(1-\delta)j} \|\eta_j(\tau) \mathcal{F}[m' P_{k+\nu} w]\|_{L^2_{\tau,\xi}}, \end{aligned}$$

We decompose in modulation and use Plancherel (similarly to Step 1 in the proof of Proposition 7.1) to obtain the estimate

$$\begin{aligned} &\sum_{j=0}^{\infty} 2^{j\epsilon} 2^{(1-\delta)j} \|\eta_j(\tau) \mathcal{F}[m' P_{k+\nu} w]\|_{L^2_{\tau,\xi}} \\ \leq{}& C \sum_{j_1=0}^{\infty} 2^{j_1\epsilon} 2^{(1-\delta)j_1} \|\chi_{k+\nu}(\xi) \eta_{j_1}(\tau) \mathcal{F} w(\tau, \xi)\|_{L^2_{\tau,\xi}}, \end{aligned} \tag{9.8}$$

where we exploit that $m' \in S^\infty_{100}$, which is sufficient. Concerning the second term

$$\begin{aligned} &\|\chi_0(\xi)(\tau - \omega(\xi) + i)^\epsilon \mathcal{F}[R(D) P_k m_{\leq -10} [m' P_{k+\nu} w]_{\leq -20}]\|_{Z_0} \\ \leq{}& C \sum_{j=0}^{\infty} 2^{\epsilon j} 2^{(1-\delta)j} \|\eta_j(\tau) \chi_0(\xi) \mathcal{F} m_{\leq -10} [m' P_{k+\nu} w]_{\leq -20}\|_{L^2} \\ \leq{}& C \sum_{j_1=0}^{\infty} 2^{j_1\epsilon} 2^{(1-\delta)j_1} \|\chi_{k+\nu}(\xi) \eta_{j_1}(\tau) \mathcal{F} w(\tau, \xi)\|_{L^2_{\tau,\xi}}, \end{aligned}$$

as in (9.8), using $m_{\leq -10}, m' \in S^\infty_{100}$.

Contribution ii): $m_{>-10}$ and $[m' P_{k+\nu} w]_{\leq -20}$. We apply the triangle inequality. Note that $\|m_{>-10}\|_{S^2_{100}} \leq 1$ and there is only a contribution from the first term if

$k = 0$. Note that for $|\xi| \leq 2^{-20}$ the term vanishes. We obtain the bound

$$\begin{aligned}
&\|\chi_0(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[m_{>-10}R(D)P_k[m'P_{k+\nu}w]_{\leq -20}]\|_{Z_0}\\
\leq& C\sum_{j=0}^{\infty} 2^{j\epsilon}2^{(1-\delta)j}\|\chi_0(\xi)\eta_j(\tau)\mathcal{F}[m_{>-10}R(D)[m'P_{k+\nu}w]_{\leq -20}\|_{L^2_{\tau,\xi}}\\
\leq& C\sum_{j_1=0}^{\infty} 2^{\epsilon j_1}2^{(1-\delta)j_1}\|\chi_{k+\nu}(\xi)\eta_{j_1}(\tau)\mathcal{F}w(\tau,\xi)\|_{L^2_{\tau,\xi}},
\end{aligned}$$

by applying (9.8) twice. The second term can be treated similarly.

Contribution iii): m and $[m'P_{k+\nu}w]_{>-20}$. Again, we apply the triangle inequality. For the first term we apply (7.6) and use the definition of the spaces

$$\begin{aligned}
&\|\chi_0(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[mR(D)P_k[m'P_{k+\nu}w]_{>-20}]\|_{Z_0}\\
\leq& C\|\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[R(D)m'P_{k+\nu}w]_{>-20}\|_{Z_k}\\
\leq& C\sum_{j=0}^{\infty} 2^{j\epsilon}2^{(1-\delta)j}\|\chi_k(\xi)\eta_j(\tau)\mathcal{F}[m'P_{k+\nu}w]\|_{L^2_{\tau,\xi}},
\end{aligned}$$

and apply (9.8). Concerning the second term we apply (7.6) to obtain

$$\begin{aligned}
&\|\chi_0(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[P_kR(D)[m[m'P_{k+\nu}w]_{>-20}]]\|_{Z_0}\\
\leq& C\sum_{k'\in\mathbb{Z}}(1+|k'|)^{-20}\|\chi_{k'}(\xi)(\tau-\omega(\xi)+i)^\epsilon \mathcal{F}[m'P_{k+\nu}w]_{>-20}]\|_{Z_{k'}}\\
\leq& C\sum_{j=0}^{\infty} 2^{j\epsilon}2^{(1-\delta)j}\|\eta_j(\tau)\mathcal{F}[m'P_{k+\nu}w]\|_{L^2_{\tau,\xi}}
\end{aligned}$$

The claim follows from (9.8). □

Additionally, we will need a higher order commutator estimates. Let us define

$$\begin{aligned}
&\left[D^\alpha\partial_x;m'\right]_{(3)}w\\
:=&D^\alpha\partial_x(m'w)-m'D^\alpha\partial_x w-(\alpha+1)\partial_x(m')D^\alpha w+\tfrac{\alpha(\alpha+1)}{2}\partial_x^2(m')D^{\alpha-2}\partial_x w.
\end{aligned}$$

Lemma 9.2. *Let $\sigma \in [0,2]$. Assume that $R(D) = \partial_x D^\alpha$ for $1 < \alpha < 2$. Assume further that $m, m' \in S^\infty_{201}$, $\|m\|_{S^\infty_{201}} + \|m'\|_{S^\infty_{201}} \leq 1$. Then, for any $k, \mu \in \mathbb{Z}$, $k \neq 0$,*

$$\begin{aligned}
&(1+|\mu|)^{40}(1+|n_{k+\mu}|)^{2\sigma}\|P_{k+\mu}[mP_k\left[D^\alpha\partial_x;m'\right]_{(3)}w]\|^2_{N_{k+\mu}}\\
\leq&C(\|\partial_x m\|^2_{S^2_{200}}+\|\partial_x m'\|^2_{S^2_{200}})\\
&\cdot\sum_{\nu\in\mathbb{Z}}(1+|\nu|)^{-40}(1+|n_{k+\nu}|)^{2\sigma+2\alpha-3}\ln^2(2+|n_{k+\nu}|)\|P_{k+\nu}w\|^2_{\widetilde{N}_{k+\nu}}.
\end{aligned}\tag{9.9}$$

Proof of Lemma 9.2. We decompose $w = \sum_{\nu\in\mathbb{Z}} w_{k,\nu}$ where $w_{k,\nu} = P_{k+\nu}w$.

Case 1: $1 \leq |k| \leq 10$**.** We apply (7.6) in order to obtain

$$(1+|\mu|)^{40}\|P_{k+\mu}[mP_k\left[D^\alpha\partial_x;m'\right]_{(3)}w_{k,\nu}]\|_{N_{k+\mu}} \leq C\|P_k\left[D^\alpha\partial_x;m'\right]_{(3)}w_{k,\nu}\|_{N_k},$$

Further, since $k \neq 0$ and

$$||\xi|^\alpha\xi-|\xi-\xi_1|^\alpha(\xi-\xi_1)| \leq C|\xi_1|(|\xi|^\alpha+|\xi-\xi_1|^\alpha)$$

we have

$$\begin{aligned}&\|P_k[D^\alpha\partial_x(m'w_{k,\nu})-m'D^\alpha\partial_x w_{k,\nu}]\|_{N_k}\\ \le&C(1+|n_{k+\nu}|)^\alpha\|\chi_k(\xi)(\tau-\omega(\xi)+i)^\epsilon|\mathcal{F}\partial_x m'|*|\mathcal{F}w_{k,\nu}|\|_{Z_k}\\ \le&C(1+|\nu|)^{-40}\|\partial_x m'\|_{S^2_{200}}\|w_{k,\nu}\|_{\widetilde{N}_{k+\nu}}\end{aligned}$$

where in the last step we have used (7.8) in case $k+\nu=0$ and (7.7) otherwise. For the other two terms we have

$$\|\partial_x(m')D^\alpha w_{k,\nu}\|_{N_k}+\|\partial_x^2(m')D^{\alpha-2}\partial_x w_{k,\nu}\|_{N_k}\le C(1+\nu)^{-40}\|\partial_x m'\|_{S^2_{200}}\|w_{k,\nu}\|_{\widetilde{N}_{k+\nu}},$$

by (7.8) in case $k+\nu=0$ and (7.7) otherwise.

Case 2: $k+\mu=0$ and $|k|>10$. In this case we may replace m by $m_{\ge 0}$ and use (7.7) to obtain the upper bound

$$\begin{aligned}&\|P_{k+\mu}[mP_k\left[D^\alpha\partial_x;m'\right]_{(3)}w_{k,\nu}\|^2_{N_{k+\mu}}\\ \le&C(1+|k|)^{-90}\|m_{\ge 0}\|^2_{S^2_{150}}\|P_k\left[D^\alpha\partial_x;m'\right]_{(3)}w_{k,\nu}\|^2_{N_k},\end{aligned}$$

and observe that $\|m_{\ge 0}\|_{S^2_{150}}\le C\|\partial_x m\|_{S^2_{200}}$. We apply the triangle inequality and bound each term individually, using Proposition 7.1.

Case 3: $k+\mu\neq 0$, $k+\nu=0$ and $|k|>10$. In this case we may replace m' by $m'_{\ge 0}$. We use the crude bound (similar to (9.7))

$$\begin{aligned}&(1+|\mu|)^{40}\|P_{k+\mu}[mP_k\left[D^\alpha\partial_x;m'\right]_{(3)}w_{k,\nu}\|^2_{N_{k+\mu}}\\ \le&C(1+|n_k|)^{\alpha+1}\|P_k\left[D^\alpha\partial_x;m'_{\ge 0}\right]_{(3)}w_{k,\nu}\|^2_{N_k}.\end{aligned}$$

We apply the triangle inequality and use $\|m'_{\ge 0}\|_{S^2_{150}}\le C\|\partial_x m'\|_{S^2_{200}}$ and (7.8) to bound each of the four terms individually. We obtain

$$\|P_k\left[D^\alpha\partial_x;m'_{\ge 0}\right]_{(3)}w_{k,\nu}\|^2_{N_k}\le C\|\partial_x m'\|^2_{S^2_{200}}(1+|\nu|)^{-80}\|w_{k,\nu}\|^2_{\widetilde{N}_{k+\nu}}.$$

Case 4: $k+\mu\neq 0$ and $k+\nu\neq 0$ and $|k|>10$. For the smoothed out (at $\xi=0$) symbol $q(\xi)=i\xi|\xi|^\alpha(1-\eta_0)(2^{10}\xi)$ we calculate

$$\begin{aligned}&q(\xi-\xi_1)-q(\xi-\xi_2)-q'(\xi-\xi_2)(\xi_2-\xi_1)-\tfrac{1}{2}q''(\xi-\xi_2)(\xi_2-\xi_1)^2\\ =&(\xi_2-\xi_1)^3 I(\xi-\xi_2,\xi_2-\xi_1),\end{aligned}$$

where

$$I(\xi-\xi_2,\xi_2-\xi_1):=\int_0^1 q'''(\xi-\xi_2+s(\xi_2-\xi_1))\frac{(1-s)^2}{2}ds.$$

We obtain

$$\begin{aligned}&\mathcal{F}[P_{k+\mu}[mP_k\left[D^\alpha\partial_x;m'\right]_{(3)}w_{k,\nu}](\xi,\tau)=C\chi_{k+\mu}(\xi)\\ &\int_{I_{k+\nu}}\Big[\int_{\mathbb{R}^2}\mathbf{1}_{I_{k+\nu}}\mathcal{F}(w)(\xi-\xi_2,\tau-\tau_2)\mathrm{H}(\xi-\xi_2-\gamma)\cdot K(\xi_2,\tau_2,\gamma)\,d\xi_2 d\tau_2\Big]d\gamma,\end{aligned}$$

where H denotes the Heaviside-function and

$$\begin{aligned}K(\xi_2,\tau_2,\gamma)=\int_{|\xi_1|\ge|\mu|-2^{20}}&\mathcal{F}(m)(\xi_1,\tau_1)\mathcal{F}(\partial_x^3 m')(\xi_2-\xi_1,\tau_2-\tau_1)\\ &\cdot\partial_\gamma[\chi_k(\gamma+\xi_2-\xi_1)\chi_{k+\nu}(\gamma)I(\gamma,\xi_2-\xi_1)]\,d\xi_1 d\tau_1.\end{aligned}$$

For fixed $\gamma \in I_{k+\nu}$ the function $\mathcal{F}^{-1}K(\cdot,\cdot,\gamma)$ is a restricted admissible factor satisfying

$$\|\mathcal{F}^{-1}(K'(\cdot,\cdot,\gamma))\|_{S^2_{100}} \leq C\|\partial_x m'\|_{S^2_{200}}(1+|n_k|)^{\alpha-2}(1+|\nu|)^{-40}(1+|\mu|)^{-40}.$$

We have

$$\begin{aligned}&\|P_{k+\mu}[\left[D^\alpha \partial_x; m'\right]_{(3)}]\|_{N_{k+\mu}}\\ \leq& C\int_{I_{k+\nu}} \left\| \frac{\chi_{k+\mu}(\xi)}{\tau-\omega(\xi)+i}\int_{\mathbb{R}^2} \mathbf{1}_{I_{k+\nu}}\mathcal{F}(w)(\xi-\xi_2,\tau-\tau_2)\cdot K(\xi_2,\tau_2,\gamma)\,d\xi_2 d\tau_2\right\|_{Z_{k+\mu}} d\gamma.\end{aligned}$$

Finally, we apply (7.7) to the integrand for fixed γ and use the fact $|I_{k+\nu}| \leq C|n_{k+\nu}|^{\frac{1}{2}}$. □

Moreover, we will need a more specific commutator type estimate which makes use of the bilinear estimates from Section 6. Let us define an extension of the low frequency part of the initial data $\widetilde{\phi_{\rm low}}(x,t) := \eta_0(t/4)\phi_{\rm low}(x)$. Recall that $\|\widetilde{\phi_{\rm low}}\|_{L^2} \leq C\varepsilon_0$.

Lemma 9.3. *Assume that $m, m' \in S^\infty_{201}$, $\|m\|_{S^\infty_{201}} + \|m'\|_{S^\infty_{201}} \leq 1$. Then, for any $\sigma \in [0,2]$ and $k \in \mathbb{Z}\setminus\{0\}$, $\mu \in \mathbb{Z}$,*

$$\begin{aligned}&(1+|\mu|)^{40}(1+|n_{k+\mu}|)^{2\sigma}\|P_{k+\mu}[m[P_k(\widetilde{\phi_{\rm low}}\partial_x(m'w) - \widetilde{\phi_{\rm low}}P_k\partial_x(m'w)]]\|^2_{N_{k+\mu}}\\ &\leq C\varepsilon_0^2\cdot\sum_{\nu\in\mathbb{Z}}(1+|n_{k+\nu}|)^{2\sigma}\|P_{k+\nu}w\|^2_{F_{k+\nu}}.\end{aligned} \tag{9.10}$$

Proof of Lemma 9.3. We decompose $m'w = \sum_{\nu'\in\mathbb{Z}} P_{k+\nu'}(m'w)$ and there are nontrivial contributions only if $|\nu'| \leq 5$. It suffices to consider the case where $k+\mu \neq 0$ and $|k| > 10$ because otherwise the estimate follows from (7.7) and (7.8). We replace m with $m_{\geq k'}$ for $k' = \log_2(1+|n_\mu|) - 10$ in case $|\mu| \geq 10$. We compute the Fourier transform

$$\begin{aligned}&\mathcal{F}[P_{k+\mu}[m[P_k(\widetilde{\phi_{\rm low}}\partial_x P_{k+\nu'}(m'w)) - \widetilde{\phi_{\rm low}}P_k\partial_x P_{k+\nu'}(m'w)]]](\xi,\tau)\\ =&C\chi_{k+\mu}(\xi)\int_{I_{k+\nu'}} I(\xi,\tau,\gamma)d\gamma,\end{aligned}$$

where H denotes the Heaviside-function, $I(\xi,\tau,\gamma)$ is defined as

$$I(\xi,\tau,\gamma) := \int_{\mathbb{R}^2}\mathcal{F}\partial_x(m'w)(\xi-\xi_2,\tau-\tau_2)\mathrm{H}(\xi-\xi_2-\gamma)\mathcal{F}M(\xi_2,\tau_2,\gamma)\,d\xi_2 d\tau_2$$

and

$$\begin{aligned}\mathcal{F}M(\xi_2,\tau_2,\gamma) = \int &\mathcal{F}(m)(\xi_1,\tau_1)\mathcal{F}(\widetilde{\phi_{\rm low}})(\xi_2-\xi_1,\tau_2-\tau_1)\\ &\cdot\partial_\gamma[(\chi_k(\gamma+\xi_2-\xi_1)-\chi_k(\gamma))\chi_{k+\nu'}(\gamma)]\,d\xi_1 d\tau_1.\end{aligned}$$

It follows

$$\begin{aligned}
&\sum_{|\nu'|\leq 5} \|\mathcal{F}[P_{k+\mu}[m[P_k(\widetilde{\phi_{\rm low}}\partial_x P_{k+\nu'}(m'w)) - \widetilde{\phi_{\rm low}} P_k \partial_x P_{k+\nu'}(m'w)]](\xi,\tau)\|_{Z_{k+\mu}} \\
\leq& C \sum_{|\nu'|\leq 5} \int_{I_{k+\nu'}} \Big\| \chi_{k+\mu}(\xi)\mathcal{F}[\partial_x(m'w)M(\cdot,\cdot,\gamma)](\tau,\xi)\Big\|_{Z_{k+\mu}} d\gamma \\
\leq& C \sup_{|\nu'|\leq 5} \Big[\int_{I_{k+\nu'}} \Big\| \chi_{k+\mu}(\xi)\mathcal{F}[\partial_x(w)m'M(\cdot,\cdot,\gamma)](\tau,\xi)\Big\|_{Z_{k+\mu}} d\gamma \\
&+ \int_{I_{k+\nu'}} \Big\| \chi_{k+\mu}(\xi)\mathcal{F}[w\partial_x(m')M(\cdot,\cdot,\gamma)](\tau,\xi)\Big\|_{Z_{k+\mu}} d\gamma \Big]
\end{aligned}$$

Let M'_γ denote either $m'M(\cdot,\cdot,\gamma)$ or $\partial_x(m')M(\cdot,\cdot,\gamma)$. For $\gamma \in I_{k+\nu'}$ and $|\nu'| \leq 5$ one can show that

$$\|\chi_{k_1}\mathcal{F}M'_\gamma\|_{Z_{k_1}} \leq C\varepsilon_0(1+|n_k|)^{-1}(1+|k_1|)^{-60}(1+|\mu|)^{-60}$$

if $k_1 \neq 0$, and

$$\|\chi_0\mathcal{F}M'_\gamma\|_{X_0^\delta} \leq C\varepsilon_0(1+|n_k|)^{-1}(1+|\mu|)^{-60}.$$

We decompose $w = \sum_{\nu\in\mathbb{Z}} P_{k+\nu}w$ and apply Lemmas 6.1-6.3 and obtain for $\mu \in \mathbb{Z}$, $\mu + k \neq 0$ and $\sigma' \in \{0,1\}$:

$$\begin{aligned}
&\Big\| \chi_{k+\mu}(\xi)\mathcal{F}[P_{k+\nu}(\partial_x^{\sigma'}w)M'_\gamma](\tau,\xi)\Big\|_{Z_{k+\mu}} \\
\leq& C\varepsilon_0(1+|\mu-\nu|)^{-60}(1+|\mu|)^{-60}(1+|n_{k+\nu}|)^{1/2-\delta}(1+|n_k|)^{-1}\|P_{k+\nu}w\|_{F_{k+\nu}}
\end{aligned}$$

Note that $|I_{k+\nu'}| \leq C|n_{k+\nu'}|^{\frac12} \approx |n_k|^{\frac12}$ for $|\nu'| \leq 5$. The claim follows by summing up with respect to ν and Cauchy-Schwarz. □

10. Proof of Proposition 4.3

The properties in Part (a) are standard, cf. [12, Lemma 4.2] and its proof. We will only show the a priori estimate (4.16) because the estimate (4.17) for differences is very similar (recall that $\phi_{\rm low} = \phi'_{\rm low}$).

We need to estimate the following expressions, see (3.9) and (3.14),

$$R_0 = -P_0\partial_x(\phi_{\rm low}\cdot v) - P_0\partial_x(v^2/2) - D^\alpha\partial_x P_0(\phi_{\rm low}) - P_0\partial_x(\phi_{\rm low}^2/2), \tag{10.1}$$

and for $k \in \mathbb{Z}\setminus\{0\}$

$$R_k = R_k^{(1)} + R_k^{(2)} + R_k^{(3)} + R_k^{(4)} + R_k^{(5)},$$

where

$$R_k^{(1)} := -\, e^{-ia_k\Psi}P_k\partial_x(v^2/2) \tag{10.2}$$
$$R_k^{(2)} := -\, \phi_{\rm low}[\partial_x v_k - D^\alpha v_k \cdot (in_k|n_k|^{-\alpha})]] \tag{10.3}$$
$$R_k^{(3)} := -\, [e^{-ia_k\Psi}D^\alpha\partial_x(e^{ia_k\Psi}v_k) - D^\alpha\partial_x(v_k) - (\alpha+1)D^\alpha v_k\cdot(ia_k\Psi')] \tag{10.4}$$
$$R_k^{(4)} := -\, e^{-ia_k\Psi}[P_k(\phi_{\rm low}\cdot\partial_x v) - \phi_{\rm low}\cdot\partial_x(P_k v)] \tag{10.5}$$
$$R_k^{(5)} := -\, [ia_k\phi_{\rm low}^2\cdot v_k + e^{-ia_k\Psi}P_k(v\cdot\partial_x\phi_{\rm low})]. \tag{10.6}$$

We fix extensions $\widetilde{\widetilde{v}}_k$ of the functions v_k such that $\|\widetilde{\widetilde{v}}_k\|_{\mathbf{F}^{\sigma'}} \leq C\|v_k\|_{\mathbf{F}^{\sigma'}(T')}$, $\sigma' \in \{0, \sigma\}$, and $\operatorname{supp} \widetilde{\widetilde{v}}_k \subset \mathbb{R}_x \times [-4, 4]$. For any interval $[a, b] \subseteq \mathbb{R}$ let

$$P_{[a,b]} = \sum_{k \in \mathbb{Z} \cap [a,b]} P_k.$$

By (3.16) and the commutator estimate (9.1) the function

$$\widetilde{v}_k = e^{-ia_k\Psi} P_{[k-1,k+1]}(e^{ia_k\Psi}\widetilde{\widetilde{v}}_k)$$

is another extension of v_k with the properties, supported in $\mathbb{R}_x \times [-4, 4]$ and verifying

$$\begin{aligned} &\|\widetilde{v}_k\|_{\mathbf{F}^{\sigma'}} \leq C\|v_k\|_{\mathbf{F}^{\sigma'}(T')}, \qquad \sigma = \{0, \sigma'\}, \\ &\widetilde{v}_k = e^{-ia_k\Psi} P_{[k-2,k+2]}(e^{ia_k\Psi}\widetilde{v}_k). \end{aligned} \tag{10.7}$$

We define

$$\widetilde{v} = \sum_{k \in \mathbb{Z}} e^{ia_k\Psi}\widetilde{v}_k, \tag{10.8}$$

We look at each of the contributions (10.1)-(10.6) separately.

Contribution of (10.1): Recall the definition $\widetilde{\phi_{\text{low}}}(x, t) = \phi_{\text{low}}(x)\eta_0(t/4)$. We define the extension

$$\widetilde{R}_0 = -P_0\partial_x(\widetilde{\phi_{\text{low}}} \cdot \widetilde{v}) - P_0\partial_x(\widetilde{v}^2/2) - D^\alpha \partial_x P_0(\widetilde{\phi_{\text{low}}}) - P_0\partial_x(\widetilde{\phi_{\text{low}}}^2/2),$$

Obviously, it holds

$$\begin{aligned} &\|P_0[D^\alpha \partial_x \widetilde{\phi_{\text{low}}} + \partial_x \widetilde{\phi_{\text{low}}}^2/2]\|_{\mathbf{N}^\sigma} \\ &\leq C \sum_{k'=-\infty}^{2} 2^{-k'} \big[\|\widetilde{\eta}_{k'} \mathcal{F}[D^\alpha \partial_x \widetilde{\phi_{\text{low}}}]\|_{L^2_{\xi,\tau}} + \|\widetilde{\eta}_{k'} \mathcal{F}[P_0\partial_x \widetilde{\phi_{\text{low}}}^2]\|_{L^2_{\xi,\tau}}\big] \\ &\leq C(\|\phi_{\text{low}}\|_{L^2} + \|\phi_{\text{low}}\|_{L^2}^2). \end{aligned}$$

To estimate the contribution from the first two terms, we estimate

$$\begin{aligned} &\|P_0\partial_x(\widetilde{\phi_{\text{low}}} \cdot \widetilde{v}) + P_0\partial_x(\widetilde{v}^2/2)\|_{\mathbf{N}^\sigma} \\ &\leq C \sum_{k'=-\infty}^{2} \big[\|\widetilde{\eta}_{k'} \mathcal{F}[\widetilde{\phi_{\text{low}}} \cdot \widetilde{v}]\|_{L^2_{\xi,\tau}} + \|\widetilde{\eta}_{k'} \mathcal{F}[\widetilde{v}^2]\|_{L^2_{\xi,\tau}}\big] \\ &\leq C\varepsilon_0\|\widetilde{v}\|_{L^\infty_t L^2_x} + C\|\widetilde{v}\|^2_{L^\infty_t L^2_x} \\ &\leq C\varepsilon_0\big[\sum_{k \in \mathbb{Z}} \|\widetilde{v}_k\|^2_{L^\infty_t L^2_x}\big]^{1/2} + C\sum_{k \in \mathbb{Z}} \|\widetilde{v}_k\|^2_{L^\infty_t L^2_x}. \end{aligned}$$

Using (4.10), the two estimates above imply

$$\|\widetilde{R}_0\|^2_{\mathbf{N}^\sigma} \leq C\|\phi\|^2_{H^0} + C\sum_{k \in \mathbb{Z}} \|\widetilde{v}_k\|^2_{\mathbf{F}^0}\big(\varepsilon_0^2 + \sum_{k \in \mathbb{Z}} \|\widetilde{v}_k\|^2_{\mathbf{F}^0}\big). \tag{10.9}$$

Contribution of (10.2): We define

$$\widetilde{R}_k^{(1)} = -e^{-ia_k\Psi} P_k\partial_x(\widetilde{v}^2/2).$$

This is an extension of $R_k^{(1)}$. An application of the commutator estimate (9.2) yields

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{0\}}\|\widetilde{R}_k^{(1)}\|_{\mathbf{N}^\sigma}^2 \leq & C\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma}\|\partial_x P_k(e^{-ia_k\Psi}(\widetilde{v}^2)\|_{N_k}^2 \\ & +C\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma}\|\partial_x P_k[\widetilde{v}^2]\|_{N_k}^2\end{aligned} \tag{10.10}$$

For $k',\nu'\in\mathbb{Z}$ we define $w_{k',\nu'}=P_{k'}\widetilde{v}_{k'+\nu'}$, so

$$\widetilde{v}=\sum_{k',\nu'\in\mathbb{Z}}e^{ia_{k'+\nu'}\Psi}w_{k',\nu'}.$$

Using Lemma 8.1 (we ignore the $\delta/4$ gains) and this identity, the right-hand side of (10.10) is dominated by

$$C\sum_{\nu_1,\nu_2\in\mathbb{Z}}(1+|\nu_1|)^2(1+|\nu_2|)^2\Big(\sum_{k_1\in\mathbb{Z}}\|w_{k_1,\nu_1}\|_{F_{k_1}}^2\Big)\Big(\sum_{k_2\in\mathbb{Z}}(1+|n_{k_2}|)^{2\sigma}\|w_{k_2,\nu_2}\|_{F_{k_2}}^2\Big)$$

For $|\nu|\leq 10$ fixed and $\sigma'\in\{0,\sigma\}$ we estimate simply

$$\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma'}\|w_{k,\nu}\|_{F_k}^2=\sum_{k\in\mathbb{Z}}(1+|n_{k-\nu}|)^{2\sigma'}\|P_{k-\nu}\widetilde{v}_k\|_{F_{k-\nu}}^2\leq\sum_{k\in\mathbb{Z}}\|\widetilde{v}_k\|_{\mathbf{F}^{\sigma'}}^2.$$

For $|\nu|\geq 11$ and $\sigma'\in\{0,\sigma\}$ we estimate, using (10.7) and Lemma 9.1

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}(1+|n_k|)^{2\sigma'}\|w_{k,\nu}\|_{F_k}^2\\ =&\sum_{k\in\mathbb{Z}}(1+|n_{k-\nu}|)^{2\sigma'}\|P_{k-\nu}[e^{-ia_k\Psi}P_{[k-2,k+2]}(e^{ia_k\Psi}\widetilde{v}_k)]\|_{F_{k-\nu}}^2\\ \leq& C(1+|\nu|)^{-40}\sum_{k\in\mathbb{Z}}\|\widetilde{v}_k\|_{\mathbf{F}^{\sigma'}}^2\end{aligned}$$

Therefore

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\|\widetilde{R}_k^{(1)}\|_{\mathbf{N}^\sigma}^2\leq C\big(\sum_{k\in\mathbb{Z}}\|\widetilde{v}_k\|_{\mathbf{F}^0}^2\big)\big(\sum_{k\in\mathbb{Z}}\|\widetilde{v}_k\|_{\mathbf{F}^\sigma}^2\big). \tag{10.11}$$

Contribution of (10.3): We define the extension

$$\widetilde{R}_k^{(2)}:=-\widetilde{\phi_{\mathrm{low}}}[\partial_x\widetilde{v}_k-D^\alpha\widetilde{v}_k\cdot(in_k|n_k|^{-\alpha})]$$

where we set $\widetilde{\phi_{\mathrm{low}}}(x,t):=\eta_0(t/4)\phi_{\mathrm{low}}(x)$. We note that for small $\delta>0$

$$\|\widetilde{\phi_{\mathrm{low}}}\|_{S^2_{150}}+\|\mathcal{F}\widetilde{\phi_{\mathrm{low}}}\|_{X_0^\delta}\leq C\varepsilon_0.$$

We define

$$\widetilde{u}_k:=D_k\widetilde{v}_k \text{ where } D_k:=\partial_x-(in_k|n_k|^{-\alpha})\cdot D^\alpha$$

and we decompose

$$\widetilde{u}_k=\sum_{\nu\in\mathbb{Z}}\widetilde{u}_{k,\nu} \text{ where } \widetilde{u}_{k,\nu}=P_\nu\widetilde{u}_k.$$

Now, by definition

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\|\widetilde{R}_k^{(2)}\|_{\mathbf{N}^\sigma}^2\leq\sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{k_1\in\mathbb{Z}}(1+|n_{k_1}|)^{2\sigma}\left(\sum_{|\nu-k_1|\leq 5}\|P_{k_1}[\widetilde{\phi_{\mathrm{low}}}\widetilde{u}_{k,\nu}]\|_{N_{k_1}}\right)^2.$$

If $|k_1| \leq 10$ the estimates (7.7) and (7.8) imply that

$$\sum_{|\nu-k_1|\leq 5} \|P_{k_1}[\widetilde{\phi_{\rm low}}\widetilde{u}_{k,\nu}]\|_{N_{k_1}} \leq C\varepsilon_0 \sum_{|\nu|\leq 15} \|\widetilde{u}_{k,\nu}\|_{F_\nu} \leq C\varepsilon_0 \|\widetilde{v}_k\|_{\mathbf{F}^\sigma}$$

If $|k_1| > 10$ we use estimates (6.7) and (6.8) to obtain

$$(1+|n_{k_1}|)^\sigma \sum_{|\nu-k_1|\leq 5} \|P_{k_1}[\widetilde{\phi_{\rm low}}\widetilde{u}_{k,\nu}]\|_{N_{k_1}} \leq C(1+|n_{k_1}|)^{\sigma-1/2-\delta}\varepsilon_0 \sum_{|\nu-k_1|\leq 5} \|\widetilde{u}_{k,\nu}\|_{F_\nu}.$$

The symbol of $P_\nu D_k$

$$m(\xi) := \chi_\nu(\xi)\left(i\xi - \frac{|\xi|^\alpha}{|n_k|^\alpha} i n_k\right)$$

satisfies $|m(\xi)| \leq C\chi_\nu(\xi)(1+|k-\nu|)^{2\alpha}(1+|n_\nu|)^{\frac{1}{2}}$ by definition of the sequence n_k, see (3.3). Therefore we conclude that

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \|\widetilde{R}_k^{(2)}\|_{\mathbf{N}^\sigma}^2 \leq C\varepsilon_0^2 \sum_{k\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} (1+|n_{k_1}|)^{2\sigma}(1+|k-k_1|)^{10}\|P_{k_1}\widetilde{v}_k\|_{F_{k_1}}^2.$$

and we use (10.7) and the commutator estimate (9.1) to obtain

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \|\widetilde{R}_k^{(2)}\|_{\mathbf{N}^\sigma}^2 \leq C\varepsilon_0^2 \sum_{k\in\mathbb{Z}} \|\widetilde{v}_k\|_{\mathbf{F}^\sigma}^2. \tag{10.12}$$

Contribution of (10.4): As above, we define an extension

$$\widetilde{R}_k^{(3)} := -[e^{-ia_k\Psi}D^\alpha\partial_x(e^{ia_k\Psi}\widetilde{v}_k) - D^\alpha\partial_x(\widetilde{v}_k) - (\alpha+1)D^\alpha\widetilde{v}_k\cdot(ia_k\Psi')]$$

for $k\neq 0$. We use the property (10.7) and apply the commutator estimate (9.9) to obtain

$$\begin{aligned}\|\widetilde{R}_k^{(3)}\|_{\mathbf{N}^\sigma}^2 \leq & C\varepsilon_0^2|a_k|^2\sum_{\nu\in\mathbb{Z}}(1+|n_{k+\nu}|)^{2\sigma+2\alpha-5/2}(1+|\nu|)^{-40}\|P_{k+\nu}\widetilde{v}_k\|_{\widetilde{N}_{k+\nu}}^2 \\ &+ C\|e^{-ia_k\Psi}\partial_x^2(e^{ia_k\Psi})D^{\alpha-2}\partial_x\widetilde{v}_k\|_{\mathbf{N}^\sigma}^2.\end{aligned}$$

Since $|a_k| = |n_k|^{1-\alpha}$, the first term is bounded by $\varepsilon_0^2\|\widetilde{v}_k\|_{\mathbf{N}^\sigma}$. Concerning the second term, we note that the restriction of $e^{-ia_k\Psi}\partial_x^2(e^{ia_k\Psi})$ to the time interval $[-4,4]$ is a restricted admissible factor with norm less than $C\varepsilon_0|a_k|$ and estimate (7.7) yields

$$\|P_{[k-2,k+2]}[e^{-ia_k\Psi}\partial_x^2(e^{ia_k\Psi})D^{\alpha-2}\partial_x\widetilde{v}_k]\|_{\mathbf{N}^\sigma} \leq C\varepsilon_0\|\widetilde{v}_k\|_{\mathbf{N}^\sigma}.$$

if $|k|\leq 5$, and for $|k|>5$ Lemma 8.1 implies that

$$\sum_{|\mu|\leq 2}(1+|n_{k+\mu}|)^{2\sigma}\|P_{k+\mu}[e^{-ia_k\Psi}\partial_x^2(e^{ia_k\Psi})D^{\alpha-2}\partial_x\widetilde{v}_k]\|_{N_{k+\mu}}^2 \leq C\varepsilon_0\|\widetilde{v}_k\|_{\mathbf{F}^\sigma}.$$

In conclusion, we obtain

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \|\widetilde{R}_k^{(3)}\|_{\mathbf{N}^\sigma}^2 \leq C\varepsilon_0^2 \sum_{k\in\mathbb{Z}} \|\widetilde{v}_k\|_{\mathbf{F}^\sigma}^2. \tag{10.13}$$

Contribution of (10.5): As above, we define the extension

$$\widetilde{R}_k^{(4)} := -e^{-ia_k\Psi}[P_k(\widetilde{\phi_{\rm low}}\cdot\partial_x\widetilde{v}) - \phi_{\rm low}\cdot\partial_x(P_k\widetilde{v})],$$

see (10.8). Using (10.7) and Lemma 9.3 we obtain

$$\begin{aligned}\|\widetilde{R}_k^{(4)}\|_{\mathbf{N}^\sigma} \leq& \sum_{\substack{k_1\in\mathbb{Z}\\|k_1-k|\leq 5}} \|e^{-ia_k\Psi}[P_k(\widetilde{\phi_{\mathrm{low}}}\partial_x(e^{ia_{k_1}\Psi}\widetilde{v}_{k_1})) - \phi_{\mathrm{low}}\partial_x(P_k(e^{ia_{k_1}\Psi}\widetilde{v}_{k_1}))]\|_{\mathbf{N}^\sigma}\\ \leq& C\varepsilon_0 \sum_{\substack{k_1\in\mathbb{Z}\\|k_1-k|\leq 5}} \|\widetilde{v}_{k_1}\|_{\mathbf{F}^\sigma}.\end{aligned}$$

Summing up with respect to k yields

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \|\widetilde{R}_k^{(4)}\|_{\mathbf{N}^\sigma}^2 \leq C\varepsilon_0^2 \sum_{k\in\mathbb{Z}} \|\widetilde{v}_k\|_{\mathbf{F}^\sigma}^2. \tag{10.14}$$

Contribution of (10.6): We define the extension

$$\widetilde{R}_k^{(5)} := -[ia_k\widetilde{\phi_{\mathrm{low}}}^2\cdot\widetilde{v}_k + e^{-ia_k\Psi}P_k(\widetilde{v}\cdot\partial_x\widetilde{\phi_{\mathrm{low}}})].$$

Estimates (7.7) and (7.8) imply that

$$\|ia_k\widetilde{\phi_{\mathrm{low}}}^2\cdot\widetilde{v}_k\|_{\mathbf{N}^\sigma} \leq C\varepsilon_0^2\|\widetilde{v}_k\|_{\mathbf{N}^\sigma}.$$

Concerning the second term we use (10.7) and (9.2) and obtain

$$\begin{aligned}&\|e^{-ia_k\Psi}P_k(\widetilde{v}\cdot\partial_x\widetilde{\phi_{\mathrm{low}}})\|_{\mathbf{N}^\sigma}\\ \leq& \sum_{\substack{k_1\in\mathbb{Z}\\|k_1-k|\leq 5}} \|e^{-ia_k\Psi}P_k(\widetilde{v}_{k_1}\cdot e^{ia_{k_1}\Psi}\partial_x\widetilde{\phi_{\mathrm{low}}})\|_{\mathbf{N}^\sigma}\\ \leq& C\sum_{\substack{k_1\in\mathbb{Z}\\|k_1-k|\leq 5}} \sup_{a_{k,k_1}\in[-4,4]} \|\widetilde{v}_{k_1}\cdot e^{ia_{k,k_1}\Psi}\partial_x\widetilde{\phi_{\mathrm{low}}}\|_{\mathbf{N}^\sigma}\end{aligned}$$

It follows from (7.7) and (7.8) that

$$\sup_{a_{k,k_1}\in[-4,4]} \|P_0[\widetilde{v}_{k_1}\cdot e^{ia_{k,k_1}\Psi}\partial_x\widetilde{\phi_{\mathrm{low}}}]\|_{\mathbf{N}^\sigma} \leq C\varepsilon_0\|\widetilde{v}_{k_1}\|_{\mathbf{N}^\sigma}.$$

Moreover, we have the trivial bound

$$\begin{aligned}&\sup_{a_{k,k_1}\in[-4,4]} \|(I-P_0)[\widetilde{v}_{k_1}\cdot e^{ia_{k,k_1}\Psi}\partial_x\widetilde{\phi_{\mathrm{low}}}]\|_{\mathbf{N}^\sigma}\\ \leq& C\sup_{a_{k,k_1}\in[-4,4]} \|\widetilde{v}_{k_1}\cdot e^{ia_{k,k_1}\Psi}\partial_x\widetilde{\phi_{\mathrm{low}}}\|_{L_t^\infty H_x^\sigma} \leq C\varepsilon_0\|\widetilde{v}_{k_1}\|_{\mathbf{F}^\sigma}.\end{aligned}$$

By summing up with respect to k we conclude

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \|\widetilde{R}_k^{(5)}\|_{\mathbf{N}^\sigma}^2 \leq C\varepsilon_0^2 \sum_{k\in\mathbb{Z}} \|\widetilde{v}_k\|_{\mathbf{F}^\sigma}^2. \tag{10.15}$$

In summary, the estimate (4.16) now follows from (10.11), (10.12), (10.13), (10.14) and (10.15) and (10.9).

References

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations II. The KdV-equation. Geom. Funct. Anal. **3** (1993), 209–262.

[2] N. Burq and F. Planchon, On well-posedness for the Benjamin-Ono equation, Math. Ann. **340** (2008), 497–542.

[3] M. Christ, J. Colliander, and T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. **125** (2003), 1235–1293.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc. **16** (2003), 705–749.
[5] J. Colliander, C. E. Kenig, and G. Staffilani, Local well-posedness for dispersion-generalized Benjamin-Ono equations, Differential Integral Equations **16** (2003), 1441–1472.
[6] J. Ginibre and G. Velo, Smoothing properties and existence of solutions for the generalized Benjamin-Ono equation, J. Differ. Equations **93** (1991), no. 1, 150–212.
[7] Z. Guo, Global well-posedness of Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$, arXiv:0810.3445.
[8] Z. Guo, Local well-posedness for dispersion generalized Benjamin-Ono equations in Sobolev spaces, arXiv:0812.1825.
[9] S. Herr, Well-posedness for equations of Benjamin-Ono type, Illinois J. Math. **51** (2007), 951–976.
[10] A. D. Ionescu and C. E. Kenig, Global well-posedness of the Benjamin–Ono equation in low-regularity spaces, J. Amer. Math. Soc. **20** (2007), no. 3, 753–798.
[11] A. D. Ionescu and C. E. Kenig, Complex-valued solutions of the Benjamin-Ono equation, Contemporary Mathematics **428** (2007), 61–74.
[12] A. D. Ionescu and C. E. Kenig and D. Tataru, Global well-posedness of the KP-I initial-value problem in the energy space, Invent. math. **173** (2008), 265–304.
[13] R. J. Iorio, On the Cauchy problem for the Benjamin–Ono equation, Comm. Partial Differential Equations **11** (1986), 1031–1081.
[14] C. E. Kenig and K. D. Koenig, On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, Math. Res. Lett. **10** (2003), 879–895.
[15] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation, J. Amer. Math. Soc. **4** (1991), 323–347.
[16] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. **46** (1993), 527–620.
[17] C. E. Kenig, G. Ponce, and L. Vega, A bilinear estimate with applications to the KdV equation. J. Amer. Math. Soc. **9** (1996), 573–603.
[18] H. Koch and N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$, Int. Math. Res. Not. **2003** (2003), 1449–1464.
[19] H. Koch and N. Tzvetkov, Nonlinear wave interactions for the Benjamin–Ono equation, Int. Math. Res. Not. **2005** (2005), 1833–1847.
[20] L. Molinet and F. Ribaud, *On global well-posedness for a class of nonlocal dispersive wave equations*, Discrete Cont. Dyn. Syst. **15** (2006), no. 2, 657–668.
[21] L. Molinet, J.-C. Saut, and N. Tzvetkov, Ill-posedness issues for the Benjamin–Ono and related equations, SIAM J. Math. Anal. **33** (2001), 928–988.
[22] G. Ponce, On the global well-posedness of the Benjamin–Ono equation, Differential Integral Equations **4** (1991), 527–542.
[23] J.-C. Saut, Sur quelques généralisations de l'équation de Korteweg-de Vries, J. Math. Pures Appl. **58** (1979), 21–61.
[24] H.F. Smith and D. Tataru, Sharp local well-posedness results for the nonlinear wave equation, Ann. of Math. (2) **162** (2005), no. 1, 291–366.
[25] T. Tao, Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equations, Amer. J. Math. **123** (2001), 839–908.
[26] T. Tao, Global well-posedness of the Benjamin–Ono equation in $H^1(\mathbb{R})$, J. Hyperbolic Differ. Equ. **1** (2004), 27–49.

Universität Bonn
E-mail address: herr@math.uni-bonn.de

University of Wisconsin–Madison
E-mail address: ionescu@math.wisc.edu

University of Chicago
E-mail address: cek@math.uchicago.edu

Universität Bonn
E-mail address: koch@math.uni-bonn.de