MATH 721 PROBLEM SET 1

DUE ON TUESDAY, SEP. 22, IN CLASS

1. Prove Urysohn's lemma: let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $K \subseteq \Omega$ be a compact set. Prove that there is a function $\psi \in C_c^\infty(\Omega)$ such that $\psi(x) = 1$ for any $x \in K$.

2. Assume X is a measurable set and $f : X \to \mathbb{R}$ is a function. Prove that f is measurable if and only if the sets $E_r = \{x \in X : f(x) \geq r\}$ are measurable for any $r \in \mathbb{Q}$.

3. Prove that the set of points at which a sequence of measurable functions $f_n : X \to \mathbb{R}$ converges is a measurable set.

4. (a) Assume $f_n : X \to [0, \infty]$, $n = 1, 2, \ldots$, is a sequence of measurable functions and $f_1 \geq f_2 \geq f_3 \geq \ldots$. For any $x \in X$ let $f(x) = \lim_{n\to\infty} f_n(x)$. Prove that if $f_1 \in L^1(\mu)$ then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

(b) Show that the conclusion of part (a) may fail if the assumption $f_1 \in L^1(\mu)$ is omitted.

5. Assume X is a measure space and $f : X \to [0, \infty)$ is a measurable function such that $\int_X f \, d\mu < \infty$. Prove that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\int_X f \cdot \chi_E \, d\mu < \varepsilon$$

for any measurable set $E \subseteq X$ with $\mu(E) < \delta$.