MATH 721 PROBLEM SET 2

DUE ON TUESDAY, OCT. 6, IN CLASS

1. (Egoroff's theorem) Assume $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$ (such a measure space is called a *probability space*). Assume $\{f_n\}_{n=1,2,\ldots} : X \to \mathbb{C}$ is a sequence of complex-valued measurable functions and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$. Prove that for every $\varepsilon > 0$ there is a set $A_\varepsilon \in \mathcal{M}$ with $\mu(A_\varepsilon) > 1 - \varepsilon$ such that the sequence f_n converges uniformly to f on the set A_ε.

2. Let X denote an uncountable set and let

$$\mathcal{M} = \{E \subseteq X : E \text{ is at most countable or } X \setminus E \text{ is at most countable}\}.$$

Let $\mu : \mathcal{M} \to [0, \infty]$ be defined by $\mu(E) = 0$ if E is at most countable and $\mu(E) = 1$ if E is not countable. Prove that $(X, \mathcal{M}, \mu)$ is a measure space and describe the set of measurable functions and their integrals.

3. (Layer cake representation) Assume $(X, \mathcal{M}, \mu)$ is a measure space. For any measurable function $f : X \to [0, \infty]$ define the function $F : [0, \infty) \to [0, \infty]$ by

$$F(t) = \mu(\{x \in X : f(x) > t\}).$$

Prove that F is measurable and

$$\int_X f \, d\mu = \int_{[0,\infty)} F(t) \, dt.$$

4. For any $\varepsilon > 0$ show that there is an open set $E \subseteq \mathbb{R}$ such that E is dense in $\mathbb{R}$ and $\mu(E) < \varepsilon$ (μ denotes the standard Lebesgue measure on $\mathbb{R}$).

5. Define a set $E \subseteq [0, 1]$ with the property that for any $x \in \mathbb{R}$ there is a unique $y \in E$ with the property that $x - y \in \mathbb{Q}$. Show that the set E is not a Borel set.

6. (a) Give an example of a sequence of continuous functions $f_n \in L^1([0, 1])$, $n = 1, 2, \ldots$, with the following properties:

1. $\lim_{n\to\infty} f_n(x) = 1$ for any $x \in [0, 1]$;
2. $\int_0^1 |f_n(x)| \, dx = 2$ for any $n = 1, 2, \ldots$.

(b) Show that if f_n are as in part (a) then

$$\lim_{n\to\infty}\int_0^1 |f_n(x)-1|\,dx = 1.$$