MATH 721 PROBLEM SET 3

DUE ON TUESDAY, OCT. 20, IN CLASS

1. A positive measure μ on a measure space $(X, \mathcal{M})$ is called σ-finite if X is a countable union of sets X_i with $\mu(X_i) < \infty$. Prove that μ is σ-finite if and only if there exists $f \in L^1(d\mu)$ such that $f(x) > 0$ for any $x \in X$.

2. Let μ be a regular Borel measure on a compact metric space X; assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

3. Construct an uncountable compact set $E \subseteq \mathbb{R}$ of Lebesgue measure 0. Construct a totally disconnected compact set $K \subseteq \mathbb{R}$ with $m(K) > 0$ (K has no connected subset of more than one point).

4. Construct a sequence of continuous functions f_n on $[0, 1]$ such that $0 \leq f_n \leq 1$,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0,$$

but the sequence $f_n(x)$ does not converge for any $x \in [0, 1]$.

5. Let F be a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta_F(x)$ denote the distance of x to F. i.e. $\delta_F(x) = \inf\{|x - y| : y \in F\}$.
(i) Prove that δ_F is Lipschitz continuous, in fact

$$|\delta_F(x) - \delta_F(y)| \leq |x - y|.$$

(ii) Let

$$M(x) = \int \frac{\delta_F(y)}{|x - y|^2} dy.$$

Show that $M(x) < \infty$ for almost every $x \in F$.

6. (Carathéodory's criterion) Assume X is a space and $\mu : P(X) \to [0, \infty]$ is an *outer measure*, i.e. a positive function that satisfies the properties

$$\mu(\emptyset) = 0,$$
$$\mu(A) \leq \mu(B) \qquad \text{if } A \subseteq B \subseteq X,$$
$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i) \qquad \text{for any sets } A_1, A_2, \ldots.$$

Let

$$\mathcal{M} = \{A \subseteq X : \mu(E) = \mu(E \cap A) + \mu(E \cap {}^cA) \text{ for any } E \subseteq X\}.$$

Show that $\mathcal{M}$ is a σ-algebra and $(X, \mathcal{M}, \mu)$ is a measure space.