MATH 721 PROBLEM SET 4

DUE ON THURSDAY, OCT. 29, IN CLASS

1. Assume that $\phi : (a, b) \to \mathbb{R}$ is a continuous function such that

$$\phi((x + y)/2) \leq (1/2)(\phi(x) + \phi(y))$$

for any $x, y \in (a, b)$. Show that ϕ is convex.

2. For some measures spaces, the relation $p \leq q$ implies $L^p \subseteq L^q$; for others the inclusion is reversed. There are also spaces for which L^p does not contain L^q if $p \neq q$. Give examples of these situations, and find conditions on the measure spaces under which these situations will occur.

3. On the interval $[-1, 1]$ consider the standard Banach spaces $L^1 = L^1(m)$ and $L^2 = L^2(m)$.
(a) Let f_j be a sequence of functions in L^2. Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and

$$|\,||f_j||_{L^2} - \sqrt{2}| \leq 2^{-j}.$$

Show that $\lim_{j\to\infty} f_j(x) = 1$ a. e. $x \in [-1, 1]$.
(b) Discuss what happens if we drop the hypothesis that $f_j \geq 0$.

4. Suppose that $f \in L^p(0, \infty)$, $1 < p < \infty$. Show that

$$\lim_{x\to\infty} x^{1/p} \int_x^\infty \frac{|f(t)|}{t}\, dt = 0.$$

5. (Hanner's inequality) Assume $p \in [1, 2]$ and $f, g \in L^p(\mu)$. Show that

$$\|f + g\|_{L^p}^p + \|f - g\|_{L^p}^p \geq (\|f\|_{L^p} + \|g\|_{L^p})^p + (\|f\|_{L^p} - \|g\|_{L^p})^p.$$

Show that the inequality becomes an identity when $p = 2$ (called the *parallelogram identity*).

6. Show that if $p \leq q \leq r \in [1, \infty]$ then

$$L^p(\mu) \cap L^r(\mu) \subset L^q(\mu).$$

Define a canonical norm on the space $L^p(\mu) \cap L^r(\mu)$ and prove that the inclusion map above is continuous.