

MATH 721 PROBLEM SET 5

DUE ON THURSDAY, NOV. 12, IN CLASS

1. (Projection on convex sets in L^p) Assume (X, \mathcal{M}, μ) is a measure space, $p \in (1, \infty)$, E is a closed, convex set in $L^p(\mu)$, and $f \in L^p(\mu)$. Then there is a unique $g_0 \in E$ such that

$$\|f - g_0\|_{L^p} = \inf_{g \in E} \|f - g\|_{L^p}.$$

In addition, for any $g \in E$,

$$\operatorname{Re} \int_X (g - g_0) \cdot \overline{F} d\mu \leq 0 \quad \text{where} \quad F = (f - g_0)|f - g_0|^{p-2}.$$

Hint: use Hanner's inequality.

2. (F. Riesz's representation theorem) Assume (X, \mathcal{M}, μ) is a measure space, $p \in (1, \infty)$, and $L \in (L^p(\mu))^*$. Then there is a unique $g \in L^{p'}(\mu)$, $1/p + 1/p' = 1$, such that

$$L(f) = \int_X f \cdot \overline{g} d\mu \quad \text{for any} \quad f \in L^p(\mu).$$

3. Find a sequence of bounded, measurable sets in \mathbb{R} whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function. Is it possible that $f/2$ is a characteristic function?

4. Use approximations by C_0^∞ functions to show that if $f \in L^1(\mathbb{R})$ then

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx = 0.$$

5. Assume that H is a Hilbert space and M is a closed linear subspace of H . Prove that $(M^\perp)^\perp = M$. Show that this statement may fail in $H = L^2(\mathbb{R})$ if M is not assumed to be closed.

6. Assume H is a Hilbert space and $\{u_1, u_2, \dots\} \subseteq H$ is a countable, infinite orthonormal set. Prove that the unit closed ball $B = \{x \in H : \|x\| \leq 1\}$ is not a compact set. More generally, assume $\delta_1, \delta_2, \dots \in [0, \infty)$ is a sequence of positive numbers and define

$$S = \left\{ x = \sum_{i=1}^{\infty} c_i u_i : |c_i| \leq \delta_i, |c_1|^2 + |c_2|^2 + \dots < \infty \right\}.$$

Prove that S is compact if and only if $\sum_{i=1}^{\infty} \delta_i^2 < \infty$.