Please inform your TA if you find any errors in the quiz solutions.

1. (10 points) Find a solution to the initial value problem

\[ x \frac{dy}{dx} + 2y = \frac{\cos(x)}{x} \]
\[ y(\pi) = 1 \]

Solution: We begin by writing the differential equation in standard form as

\[ \frac{dy}{dx} + \frac{2}{x}y = \frac{\cos(x)}{x^2} \]

The integrating factor for this problem is \( m(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln(x)} = x^2 \). Multiplying through by \( x^2 \) converts this problem to

\[ x^2 \frac{dy}{dx} + 2xy = \cos(x) \]
\[ \frac{d(x^2y)}{dx} = \cos(x) \]
\[ \int d(x^2y) = \int \cos(x) dx \]
\[ x^2y = \sin(x) + C \]
\[ y(x) = \frac{\sin(x)}{x^2} + \frac{C}{x^2} \]

We substitute in the initial condition \( y(\pi) = 1 \) to find that

\[ y(\pi) = \frac{\sin(\pi)}{\pi^2} + \frac{C}{\pi^2} = 1 \]

so \( C = \pi^2 \) and \( y(x) = \frac{\sin(x)}{x^2} + \frac{\pi^2}{x^2} \).

2. (10 points) Find an exact solution to the following initial value problem, then use Euler’s method with step size \( \Delta x = .1 \) to estimate \( y(.2) \)

\[ \frac{dy}{dx} = 2xy + x \]
\[ y(0) = 0 \]
Solution: We begin by putting the differential equation into standard form

\[ \frac{dy}{dx} - 2xy = x \]

The integrating factor for this problem is \( m(x) = e^{\int -2x \, dx} = e^{-x^2} \). Multiplication turns this into

\[
e^{-x^2} \frac{dy}{dx} - 2xe^{-x^2} y = xe^{-x^2}
\]

\[
e^{-x^2} y = \int xe^{-x^2} \, dx
\]

\[
= -\frac{1}{2} e^{-x^2} + C
\]

\[ y(x) = -\frac{1}{2} + Ce^{x^2} \]

Substituting in the initial condition \( y(0) = 0 \) gives that \( y(x) = \frac{1}{2} e^{x^2} - \frac{1}{2} \).

To approximate \( y(.2) \) we first need an approximation for \( y(.1) \).

\[ y(.1) \approx y(0) + \frac{dy}{dx}(0) \Delta x \]

where \( \frac{dy}{dx}(0) = 2(0)y(0) + 0 = 0 \). So we have \( \frac{dy}{dx}(0) \approx y(0) + 0(.1) = 0 \). We now have

\[ y(.2) \approx y(.1) + \frac{dy}{dx}(1) \Delta x \]

where \( \frac{dy}{dx}(1) = 2(.1)y(.1) + .1 = 2(.1)(0) + .1 = .1 \). Then, we have

\[ y(.2) \approx y(.1) + \frac{dy}{dx}(1) \Delta x \]

\[ \approx 0 + .1(.1) \]

\[ = .01 \]