MATH 222 (002) Fall 2013
Quiz Solutions

Please inform your TA if you find any errors in the quiz solutions.

1. (10 points) Find the Taylor series around zero for \( \cosh(2x) = \frac{1}{2} (e^{2x} + e^{-2x}) \).

Solution:

\[
\frac{1}{2} (e^{2x} + e^{-2x}) = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} \right)
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + \frac{(-1)^n 2^n x^n}{n!} \right)
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n!} 2^n x^n
\]

We can observe that \( 1 + (-1)^n = 0 \) if \( n \) is odd and \( 2 \) if \( n \) is even. We therefore only need to sum over the even positive integers \( n = 2k \)

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \frac{2}{(2k)!} 2^{2k} x^{2k}
= \sum_{k=0}^{\infty} \frac{1}{(2k)!} 2^{2k} x^{2k}
\]

2. (10 points) Is it true that \( \cos(x) - 1 + \frac{x^2}{2} = o(x^3) \)?

Solution: Yes. We know that

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}
= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)
\]
A second way of thinking about this problem is to notice that asking whether \( \cos(x) - P(x) = o(x^3) \) where \( P(x) \) polynomial of degree at most three is the same as asking whether \( P(x) \) is the degree three Taylor polynomial of \( \cos(x) \). We can compute that for \( f(x) = \cos(x) \) we have

\[
\begin{align*}
  f(x) &= \cos(x) & f(0) &= 1 \\
  f'(x) &= -\sin(x) & f'(0) &= 0 \\
  f^{(2)}(x) &= -\cos(x) & f^{(2)}(0) &= -1 \\
  f^{(3)}(x) &= \sin(x) & f^{(3)}(0) &= 0
\end{align*}
\]

so that

\[
T_3^0 \cos(x) = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3
\]

\[
= 1 - \frac{x^2}{2}
\]

and therefore \( \cos(x) - (1 - \frac{x^2}{2}) = o(x^3) \).