Please inform your TA if you find any errors in the solutions.

1. Find a solution to the initial value problem

\[ \frac{dy}{dx} = e^{yx^3} \]

\[ y(0) = 0 \]

**Solution:** In what follows, the value of the constant of integration may change from line to line.

\[ \frac{dy}{dx} = e^{yx^3} \]

\[ e^{-y} dy = x^3 dx \]

\[ \int e^{-y} dy = \int x^3 dx \]

\[ -e^{-y} = \frac{1}{4} x^4 + C \]

\[ e^{-y} = -\frac{1}{4} x^4 + C \]

\[ y = -\ln(C - \frac{1}{4} x^4) \]

Substituting the initial condition \( 0 = y(0) = -\ln(C) \), we find that \( C = 1 \) and \( y(x) = -\ln(1 - \frac{1}{4} x^4) \).

2. Find a solution to the initial value problem

\[ \frac{dy}{dx} = (1 + y^2) e^x \]

\[ y(0) = 0 \]

**Solution:**

\[ \frac{dy}{dx} = (1 + y^2) e^x \]

\[ \frac{dy}{1 + y^2} = e^x dx \]

\[ \int \frac{dy}{1 + y^2} = \int e^x dx \]

\[ \arctan(y) = e^x + C \]

\[ y = \tan(e^x + C) \]

Substituting in the initial condition, we find that \( 0 = Y(0) = \tan(1 + C) \). A possible choice of \( C \) is \( C = -1 \). Our final answer is then \( y(x) = \tan(e^x - 1) \).
3. Find a solution to the initial value problem
\[
\frac{dy}{dx} = y\sqrt{y^2 - 1}\cos(x)
\]
\[y(0) = 1\]

**Solution:** First, we can observe that one solution to this problem is given by \(y(x) = 1\).
We can find another solution by separating variables.
\[
\frac{dy}{dx} = y\sqrt{y^2 - 1}\cos(x)
\]
\[
\frac{dy}{y\sqrt{y^2 - 1}} = \cos(x)\,dx
\]
\[
\int \frac{dy}{y\sqrt{y^2 - 1}} = \int \cos(x)\,dx
\]
\[
\arcsec(y) = \sin(x) + C
\]
\[
y = \sec(\sin(x) + C)
\]
Substituting in the initial condition \(y(0) = 1\) we find that
\[
1 = y(0) = \sec(C)
\]
So we may take, for example, \(C = 0\). Our final solution is then either of \(y(x) = 1\) or \(y(x) = \sec(\sin(x))\).

4. Find the general solution to the differential equation (for \(x \neq 0\))
\[
x\frac{dy}{dx} = -y + x
\]

**Solution:** We rewrite the equation as
\[
x\frac{dy}{dx} + y = x
\]
and observe that this equation is already in the form
\[
\frac{d(xy)}{dx} = x
\]
which is separable. We solve
\[
\frac{d(xy)}{dx} = x
\]
\[
\int d(xy) = \int x\,dx
\]
\[
xy = \frac{1}{2}x^2 + C
\]
\[
y(x) = \frac{1}{2}x + \frac{C}{x}
\]
5. Find the general solution to the differential equation
\[
\frac{1}{2x} \frac{dy}{dx} = y + e^{x^2}
\]

**Solution:** We begin by writing the problem in standard form as
\[
\frac{dy}{dx} - 2xy = 2xe^{x^2}
\]
The integrating factor for this problem is \(m(x) = e^{\int -2x \, dx} = e^{-x^2}\). If we multiply through by \(e^{-x^2}\), then the equation becomes separable and we can find the general solution directly.

\[
e^{-x^2} \frac{dy}{dx} - 2xe^{-x^2}y = 2x
\]

\[
\frac{d(e^{-x^2}y)}{dx} = 2x
\]

\[
\int d(e^{-x^2}y) = \int 2x \, dx
\]

\[
e^{-x^2}y = x^2 + C
\]

\[
y(x) = x^2e^{x^2} + Ce^{x^2}
\]

6. Find a solution to the initial value problem
\[
\cos(x) \frac{dy}{dx} = 1 - \sin(x)y
\]
\[
y(0) = 1
\]

**Solution:** We begin by writing the equation in standard form
\[
\frac{dy}{dx} + \tan(x)y = \sec(x)
\]
The integrating factor for this problem is \(m(x) = e^{\int \tan(x) \, dx} = e^{-\ln(\cos(x))} = \sec(x)\). Multiplying through by \(m(x)\) makes this equation separable.

\[
\frac{dy}{dx} + \tan(x)y = \sec(x)
\]

\[
\sec(x) \frac{dy}{dx} + \sec(x) \tan(x)y = \sec^2(x)
\]

\[
\frac{d(\sec(x)y)}{dx} = \sec^2(x)
\]

\[
\int d(\sec(x)y) = \int \sec^2(x) \, dx
\]

\[
\sec(x)y = \tan(x) + C
\]

\[
y = \sin(x) + C \cos(x)
\]
Substituting in \(y(0) = 1\) we find that \(C = 1\) and \(y(x) = \sin(x) + \cos(x)\).