Please inform your TA if you find any errors in the solutions.

1. Find a solution to the initial value problem

\[
\frac{dy}{dx} = \sqrt{1 - y^2 \sec^2(x)}
\]

\[
y(0) = 0
\]

Solution:

\[
\frac{dy}{dx} = \sqrt{1 - y^2 \sec^2(x)}
\]

\[
\int \frac{1}{\sqrt{1 - y^2}} dy = \int \sec^2(x) dx
\]

\[
\arcsin(y) = \tan(x) + C
\]

\[
y(x) = \sin(\tan(x) + C)
\]

Using the initial condition, we find that

\[
0 = \sin(0 + C)
\]

so \(C = 0\) gives a solution. Our final answer is then \(\sin(\tan(x))\).

2. Find \(T_{14}^0 e^x - \frac{1}{1-x^5}\).

Solution:

\[
e^x = 1 + x^6 + \frac{x^{12}}{2} + \frac{x^{18}}{3!} + o(x^{18})
\]

\[
= 1 + x^6 + \frac{1}{2} x^{12} + o(x^{14})
\]

\[
\frac{1}{1-x^5} = 1 + x^5 + x^{10} + x^{15} + o(x^{15})
\]

\[
= 1 + x^5 + x^{10} + o(x^{14})
\]

so that

\[
e^x - \frac{1}{1-x^5} = \left(1 + x^6 + \frac{1}{2} x^{12} + o(x^{14}) \right) - \left(1 + x^5 + x^{10} + o(x^{14}) \right)
\]

\[
= -x^5 + x^6 - x^{10} + \frac{1}{2} x^{12} + o(x^{14})
\]

so that \(T_{14}^0 \left( e^x - \frac{1}{1-x^5} \right) = -x^5 + x^6 - x^{10} + \frac{1}{2} x^{12} \).
3. Find \( T_0^\infty x \left( e^x - \frac{1}{1-x} \right) \)

**Solution:**

\[
T_0^\infty x \left( e^x - \frac{1}{1-x} \right) = T_0^\infty x \left( T_0^\infty e^x - T_0^\infty \frac{1}{1-x} \right) \\
= x \left( \sum_{n=0}^\infty \frac{1}{n!} x^n - \sum_{n=0}^\infty x^n \right) \\
= x \left( \sum_{n=0}^\infty \left( \frac{1}{n!} - 1 \right) x^n \right) \\
= \sum_{n=0}^\infty \left( \frac{1}{n!} - 1 \right) x^{n+1}
\]

noticing that the first two terms in this sum are zero, we can rewrite this as \( \sum_{n=2}^\infty \left( \frac{1}{n!} - 1 \right) x^{n+1} \).

4. A 100 litre vat of water begins with an algae concentration of 1,000 organisms per litre. Suppose that the algae naturally reproduce at a rate of five percent per minute and die at a rate of four percent per minute. If the vat is being drained at a rate of one litre per minute, what will the algae concentration be ten minutes from now? You should assume that the algae are uniformly distributed in the vat. Remember to define your variables with units.

**Solution:** We will model the total population of algae in the vat \( P(t) \) and then notice that the concentration at time \( t \) is given by \( \frac{P(t)}{V(t)} \), where \( V(t) \) is the volume of water in the vat. The differential equation for the algae population is

\[
\frac{dP}{dt} = (.05)P(t) - (.04)P(t) - \frac{P(t)}{V(t)}
\]

\[ P(0) = 1,000(100) = 100,000 \]

Since \( V(t) = 100 - t \), this differential equation becomes

\[
\frac{dP}{dt} = \left( .01 - \frac{1}{100 - t} \right) P(t)
\]

\[ P(0) = 100,000 \]

This equation is separable. Recalling that for \( t < 100 \) we have \( \ln(100 - t) = \ln |100 - t| \), it follows that

\[
\frac{dP}{P} = \left( .01 - \frac{1}{100 - t} \right) dt
\]

\[ \ln |P| = .01t + \ln |100 - t| + C \]

\[ P(t) = Ae^{.01t+\ln(100-t)} \]

\[ = A(100 - t)e^{.01t} \]
The initial condition $P(0) = 100,000$ gives us

$$100,000 = 100A$$

so $A = 1000$ and $P(t) = 1000(100 - t)e^{0.01t}$. The concentration in organisms per litre ten minutes from now will be

$$\frac{P(10)}{V(10)} = \frac{1000(100 - 10)e^{(0.01)(10)}}{100 - 10} = 1000e^{1}$$