MATH 222 002 Fall 2013
Section Worksheet 11/26/2013 Solutions

Please inform your TA if you find any errors in the solutions.

1. Determine whether the following series converge:

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

(b) \( \sum_{n=1}^{\infty} e^{\frac{n}{3}} \)

(c) \( \sum_{n=3}^{\infty} \frac{1}{n^3+n-1} \)

(d) \( \sum_{n=1}^{\infty} \left( \frac{n^3}{n!} \right)^n \)

(e) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \)

(f) \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \)

(g) \( \sum_{n=1}^{\infty} e^{-(\ln(n))^2} \) (Hints: \( a^{bc} = (a^b)^c \) and \( e^{-\ln(n)} = \frac{1}{n} \))

Solution:

(a) We can use the integral test for this. Since \( \int_{1}^{\infty} \frac{1}{x^2} \, dx < \infty \) we just have to check that \( \frac{1}{x^2} \) is a positive decreasing function. It is clearly positive for \( x > 0 \), so that is not an issue. To check that it is decreasing, take a derivative. \( \frac{d}{dx} \frac{1}{x^2} = \frac{-3}{x^4} < 0 \). Since the derivative is negative, the function is decreasing.

(b) This sum diverges. We can see this by applying the \( n^{th} \) term test:

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{e^n}{n^3} = \infty
\]

This limit would have to be zero for the sum to have any hope of converging.

(c) We can do this with a limit comparison test. Call \( b_n = \frac{1}{n} \). Then

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^3}{n^3 + n - 1} = 1
\]

Since the limit is 1, both sequences are positive, and \( \sum_{n=0}^{\infty} \frac{1}{n^3} < \infty \), it follows that \( \sum_{n=0}^{\infty} \frac{1}{n^3+1} \) converges.

(d) This sum converges. We can see this with the root test.

\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n^3}{n!} = 0
\]

so the series converges.

(e) This series converges for all \( x \) and we can see this with the ratio test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{x^{2(n+1)+1}}{(2n+1)(2n+2)} = \frac{x^2}{(2n+3)(2n+2)} = 0
\]

so this sum converges for all \( x \).
(f) We can solve this with the integral test once we check that the function \( \frac{1}{x\ln(x)} \) is positive and decreasing on \((2, \infty)\). It is clearly positive, so we just need to check that it is decreasing.

\[
\frac{d}{dx} \frac{1}{x \ln(x)} = -\frac{\ln(x) + 1}{(x \ln(x))^2} < 0
\]

for \( x \in (2, \infty) \). We can now compare to the integral

\[
\int_3^\infty \frac{1}{x \ln(x)} \, dx = \lim_{b \to \infty} \int_3^b \frac{1}{x \ln(x)} \, dx
\]

\[
= \lim_{b \to \infty} \int_{x=3}^{x=b} \frac{1}{u} \, du \\
= \lim_{b \to \infty} [\ln|u|]_{x=3}^{x=b}
\]

\[
= \lim_{b \to \infty} [\ln|\ln(x)|]_3^b = \infty
\]

so the sum diverges.

(g) This sum converges, which we can see by direct comparison to \( \frac{1}{n^3} \) (or any power of \( n \) that converges). To see this, observe that \( e^{-(\ln(n))^2} = (e^{-\ln(n)})^{\ln(n)} = \frac{1}{n^{\ln(n)}} \), and for \( n \) large, \( e^{-(\ln(n))^2} \) will be strictly less than \( \frac{1}{n^3} \), since \( \ln(n) \to \infty \).

2. Let \( \vec{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \), \( \vec{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \) and \( \vec{c} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \). Which of the following expressions are nonsense? Evaluate the sensible ones.

(a) \( 3\vec{a} + \vec{b} \)
(b) \( \vec{a} + \vec{c} \)
(c) \( \vec{a} \cdot \vec{c} \)
(d) \( \vec{a} - 2\vec{b} \)
(e) \( t\vec{a} \) where \( t \) is a real number.
(f) \( \vec{a} \vec{b} \)
(g) \( \vec{a} + 5 \)

Solution:

(a) \( 3\vec{a} + \vec{b} = \begin{pmatrix} 5 \\ 5 \\ 14 \end{pmatrix} \)
(b) nonsense
(c) nonsense
(d) \( \vec{a} - 2\vec{b} = \begin{pmatrix} -3 \\ 7 \end{pmatrix} \)
(e) \( t\vec{a} = \begin{pmatrix} 2t \\ t \\ 5t \end{pmatrix} \)

(f) nonsense

(g) nonsense

3. Let \( \vec{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \) and \( \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \). Find \( s \) and \( t \) so that \( \begin{pmatrix} 3 \\ 5 \end{pmatrix} = s\vec{a} + t\vec{b} \).

Solution: We can rewrite this problem as

\[
\begin{pmatrix} 2s - t \\ 4s + t \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}
\]

We can solve the equation \( 2s - t = 3 \) for \( t \) to find that \( t = 2s - 3 \). Plugging this in to the equation \( 4s + t = 5 \) we find that \( 4s + 2s - 3 = 5 \) so \( 6s = 8 \) and \( s = \frac{4}{3} \). Plugging this back in, we find that \( t = 2\left(\frac{4}{3}\right) - 3 = \frac{-1}{3} \).