Please inform your TA if you find any errors in the quiz solutions.

1. (4 points) True or false? Please circle your answer.

- $T_\infty \sin(2x^3) = \sum_{n=0}^{\infty} \frac{2^{2n+1}x^{6n+3}}{(2n+1)!}$
  
  True  False

- $T_{2015} \sum_{n=0}^{2015} 3x^{2n} = \sum_{n=0}^{2015} 3x^{2n}$
  
  True  False

- $f^{(40)}(0) = 0$, where $f(x) = \cos(x^2)$
  
  True  False

- $T_3 \frac{\sin(x^8)+1}{1+x} = 1 - x + x^2 - x^3$
  
  True  False

Solution:

1. False
2. False
3. False

\[ T_\infty \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \]

so

\[ T_\infty \cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!} = 1 - \frac{x^4}{2} + \frac{x^8}{4!} + \ldots. \]

In particular, the powers of \( x \) with non-zero coefficients are exactly the powers which are multiples of four. Since 40 is a multiple of four, we see that \( f^{(40)}(0) \neq 0 \).

4. True

\( T_\infty \sin(x^8) \) has no terms of degree less than 8, so \( T_3 \frac{\sin(x^8)+1}{1+x} = T_3 \frac{1}{1+x} \).

2. (6 points) Show that \( |5 - e^2| \leq \frac{9}{3!}x^3 \). Hint: \( 5 = 1 + \frac{2}{1!} + \frac{(2)^2}{2!} \) is the approximation obtained from the second Taylor polynomial of \( e^x \). Second hint: \( e < 3 \).

Solution: Let \( f(x) = e^x \). We are interested in \( f(2) \) so we should set \( C = 2 \). \( f^{(3)}(x) = e^x \), which is bounded by \( e^2 \) on \([-2, 2]\). Therefore

\[ |5 - e^2| = |T_2 e^x|_{x=2} - e^2| = |R_2 e^x|_{x=2}| \leq \frac{e^2}{3!} 2^3 < \frac{3^2}{3!} 2^3. \]