1. (4 points)
For each of the following, circle true or false:

- $\cos(x) - 1 = o(x)$ True False
- $e^{x^3} - 1 = o(x^3)$ True False
- $(x + x^2)^2 = o(x^2)$ True False
- $\sin(x^2) - x^2 = o(x^5)$ True False

Solution:
1. True
2. False
3. False
4. True

2. (6 points)
Suppose that $y(x)$ is a solution to

$$5 = y''(x) + y(x)$$
$$y(0) = 1 \quad y'(0) = 6.$$

Compute the degree three Taylor polynomial of $y(x)$ around zero.

Solution: Write

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3)$$
$$= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3)$$
$$= a_0 + a_1x + o(x)$$
$$y''(x) = 2a_2 + 6a_3x + o(x).$$

Substituting in, we have

$$5 = y''(x) + y(x)$$
$$= (2a_2 + 6a_3x + o(x)) + (a_0 + a_1x + o(x))$$
$$= (2a_2 + a_0) + (6a_3 + a_1)x + o(x).$$
Equating coefficients, we have

\[ 5 = 2a_2 + a_0 \]  \hspace{1cm} (1)
\[ 0 = 6a_3 + a_1. \]  \hspace{1cm} (2)

We are given that \( a_0 = y(0) = 1 \) and \( a_1 = y'(0) = 6 \). Substituting \( a_0 = 1 \) into (1), we see that \( a_2 = 2 \). Substituting \( a_1 = 6 \) into (2), we see that \( a_3 = -1 \). Consequently, the degree three Taylor polynomial of \( y(x) \) around zero is given by

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 = f(0) + f'(0) + f''(0) + f'''(0) \frac{1}{2} x^2 + \frac{1}{3!} x^3 \]

\[ = 1 + 6x + 2x^2 - x^3. \]