Please inform your TA if you find any errors in the solutions.

1. Hasdrubal has designed a rocket. While proving mathematically that it won’t explode, he used the approximation $e^{\frac{1}{3}} \approx 1 + \frac{1}{3} + \frac{1}{3^22!} + \frac{1}{3^3(3!)^2}$. If this approximation is off by more than $\frac{1}{2!(\frac{1}{3})^4}$ then the rocket might blow up. Convince Hasdrubal that it won’t.

**Solution:**

We know that if $f(x) = e^x$ then $f^{(n)}(x) = e^x$. Approximating $e^{\frac{1}{3}}$ by $1 + \frac{1}{3} + \frac{1}{3^22!} + \frac{1}{3^3(3!)^2}$ corresponds to approximating $e^x$ by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$. Taylor’s theorem then tells us that there is a $\xi$ with $0 \leq \xi \leq \frac{1}{3}$
\[
e^{\frac{1}{3}} - (1 + \frac{1}{3} + \frac{1}{3^22!} + \frac{1}{3^3(3!)^2}) = \frac{e^\xi}{4!}\left(\frac{1}{3}\right)^4
\]

Since $e^x$ is an increasing function, we can bound this error by
\[
\frac{e^\xi}{4!}\left(\frac{1}{3}\right)^4 \leq \frac{e^{\frac{1}{3}}}{4!}\left(\frac{1}{3}\right)^4
\]

but this is actually not at all helpful. The whole point of this problem was, after all, to estimate $e^{\frac{1}{3}}!$ We still know that $e \leq 3 < 8$, so that $e^{\frac{1}{3}} < 8^{\frac{1}{3}} = 2$. A final answer is then that
\[
e^{\frac{1}{3}} - (1 + \frac{1}{3} + \frac{1}{3^22!} + \frac{1}{3^3(3!)^2}) \leq \frac{2}{4!}\left(\frac{1}{3}\right)^4
\]

2. Find a bound for $R_n^0 \sin(3x)$ and use this to show that $T_n^0 \sin(3x) \to \sin(3x)$ for all $x$ as $n \to \infty$.

**Solution:** We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = \sin(3x)$ then
\[
R_n^0 \sin(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}
\]

Notice that $f^{(n+1)}(x)$ is one of $3^{n+1}\sin(3x), 3^{n+1}\cos(3x), -3^{n+1}\sin(3x), -3^{n+1}\cos(3x)$. In any of these cases, we know that $|f^{(n+1)}(x)| \leq 3^{n+1}$ for all $x$. So in particular, we have the bound
\[
|R_n^0 \sin(3x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right|
\]
\[
\leq \frac{3^n}{(n+1)!}|x|^{n+1}
\]
\[
= \frac{|3x|^{n+1}}{(n+1)!}
\]

Since $|3x|^{n+1}$ is decreasing in $n$, it follows that $T_n^0 \sin(3x) \to \sin(3x)$ for all $x$ as $n \to \infty$. 
Since for any \( x \), \( \lim_{n \to \infty} \frac{1}{|n+1|} |3x|^{n+1} = 0 \), this shows that for any \( x \), \( R_n^0 \sin(3x) \to 0 \), which implies that \( T_n^0 \sin(3x) = \sin(3x) - R_n^0 \sin(3x) \to \sin(3x) \).

3. Find a bound on \( |R_n \cos(x)|_{x=1} \) and use this information to find a decimal approximation of \( \cos(1) \) with an error of at most .1.

**Solution:** Recall that \( \cos(x) - T_n \cos(x) = R_n \cos(x) \) by definition, so if \( |R_n \cos(x)|_{x=1} \) is less than \( \frac{1}{10} \), then \( \cos(1) \) is within two decimal digits of \( T_n \cos(x)|_{x=1} \). If \( f(x) = \cos(x) \) then \( f^{n+1}(x) \) is \( \pm \cos(x) \) or \( \pm \sin(x) \). In any case, we have \( |f^{n+1}(x)| \leq 1 \). It follows then that

\[
|R_n \cos(x)|_{x=1} \leq \frac{1(1)^{n+1}}{(n + 1)!} = \frac{1}{(n + 1)!}
\]

so if \( n \) is sufficiently large that \( \frac{1}{(n+1)!} \leq \frac{1}{10} \), then \( |R_n \cos(x)|_{x=1} \leq \frac{1}{10} \). This is true, for example, for \( n = 3 \), since \( (3 + 1)! = 24 \). Our approximation is then \( T_3 \cos(x)|_{x=1} = 1 - \frac{1}{2} = .5 \).

4. Find the Taylor series around zero for \( \cosh(2x) = \frac{1}{2} (e^{2x} + e^{-2x}) \).

**Solution:**

\[
\frac{1}{2} (e^{2x} + e^{-2x}) = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} \right)
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + \frac{(-1)^n 2^n x^n}{n!} \right)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n!} 2^n x^n
\]

We can observe that \( 1 + (-1)^n = 0 \) if \( n \) is odd and \( 2 \) if \( n \) is even. We therefore only need to sum over the even positive integers \( n = 2k \)

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \frac{2}{(2k)!} 2^{2k} x^{2k}
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{(2k)!} 2^{2k} x^{2k}
\]

5. Find the degree two Taylor polynomial around 0 of \( \frac{e^x}{1-x} \) without computing any derivatives.
Solution: Recall that $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$ and $\frac{1}{1-x} = 1 + x + x^2 + o(x^2)$. Then

$$\frac{e^x}{1-x} = \left(1 + x + \frac{x^2}{2} + o(x^2)\right) (1 + x + x^2 + o(x^2))$$

$$= \left[1 + x (1 + x + x^2 + o(x^2)) + \frac{x^2}{2} (1 + x + o(x^2)) + x^2 + o(x^2)\right]$$

$$= 1 + x + x^2 + o(x^2) + x^2 + o(x^2) + \frac{x^2}{2} + o(x^2)$$

$$= 1 + 2x + \frac{5}{2} x^2 + o(x^2)$$

Therefore $T_0^\frac{e^x}{1-x} = 1 + 2x + \frac{5}{2} x^2$.

6. Let $f(x) = xe^{3x^2}$. Compute $f^{(2015)}(0)$ and $f^{(2016)}(0)$.

Solution: We start by computing the Taylor series:

$$f(x) = x e^{3x^2}$$

$$= x \sum_{n=0}^{\infty} \frac{1}{n!} (3x^2)^n$$

$$= x \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n+4}$$

The coefficient of $x^n$ in the Taylor series is $\frac{1}{m!} f^{(m)}(0)$. Only even powers of $x$ are showing up, so the coefficient $f^{(2015)}(0) = 0$. As for $f^{(2016)}(0)$, the term with $x^{2016}$ is the term with $n = 1006$. Thus

$$\frac{1}{2016!} f^{(2016)}(0) = \frac{3^{1006}}{1006!}$$

$$f^{(2016)}(0) = \frac{3^{1006}}{1006!} 2016!$$