Please inform your TA if you find any errors in the solutions.

1. Determine whether the following series converge. If the series depends on $x$, determine for which values of $x$ it converges:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(c) $\sum_{n=3}^{\infty} \frac{1}{n^4+n-1}$

(d) $\sum_{n=1}^{\infty} \left( \frac{n^3}{n^2} \right)^n$

(e) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

(g) $\sum_{n=1}^{\infty} e^{-(\ln(n))^2}$ (Hints: $a^{bc} = (a^b)^c$ and $e^{-\ln(n)} = \frac{1}{n}$)

Solution:

(a) We can use the integral test for this. Since $\int_{1}^{\infty} \frac{1}{x^3} dx < \infty$ we just have to check that $\frac{1}{x^3}$ is a positive decreasing function. It is clearly positive for $x > 0$, so that is not an issue. To check that it is decreasing, take a derivative. $\frac{d}{dx} \frac{1}{x^3} = -\frac{3}{x^4} < 0$. Since the derivative is negative, the function is decreasing.

(b) This sum diverges. We can see this by applying the $n^{th}$ term test:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{e^n}{n^3} = \infty$$

This limit would have to be zero for the sum to have any hope of converging.

(c) We can do this with a limit comparison test. Call $b_n = \frac{1}{n}$. Then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^3}{n^3 + n - 1} = 1$$

Since the limit is 1, both sequences are positive, and $\sum_{n=0}^{\infty} \frac{1}{n^3} < \infty$, it follows that $\sum_{n=0}^{\infty} \frac{1}{n^4+n-1}$ converges.

(d) This sum converges. We can see this with the root test.

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n^3}{n!} = 0$$

so the series converges.

(e) This series converges for all $x$ and we can see this with the ratio test.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}}$$

$$= \lim_{n \to \infty} \frac{x^2}{(2n+3)(2n+2)} = 0$$

so this sum converges for all $x$. 
(f) We can solve this with the integral test once we check that the function \( \frac{1}{x \ln(x)} \) is positive and decreasing on \((2, \infty)\). It is clearly positive, so we just need to check that it is decreasing.

\[
\frac{d}{dx} \frac{1}{x \ln(x)} = -\frac{\ln(x) + 1}{(x \ln(x))^2} < 0
\]

for \( x \in (2, \infty) \). We can now compare to the integral

\[
\int_{3}^{\infty} \frac{1}{x \ln(x)} \, dx = \lim_{b \to \infty} \int_{3}^{b} \frac{1}{x \ln(x)} \, dx \\
= \lim_{b \to \infty} \left[ \ln \left| x \right| \right]_{x=3}^{x=b} \quad u = \ln(x) \quad du = \frac{1}{x} \, dx \\
= \lim_{b \to \infty} \left[ \ln \left| \ln(x) \right| \right]_{3}^{b} = \infty
\]

so the sum diverges.

(g) This sum converges, which we can see by direct comparison to \( \frac{1}{n^3} \) (or any power of \( n \) that converges). To see this, observe that \( e^{-(\ln(n))^2} = (e^{-\ln(n)})^{\ln(n)} = \frac{1}{n^{\ln(n)}} \) and for \( n \) large, \( e^{-(\ln(n))^2} \) will be strictly less than \( \frac{1}{n^3} \), since \( \ln(n) \to \infty \).

2. Determine whether the following series converge. If the series depends on \( x \), determine for which values of \( x \) it converges.

(a) \( \sum_{n=0}^{\infty} e^{-nx} \)

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^x + 5n} \)

(c) \( \sum_{n=1}^{\infty} \frac{n!}{n^x} \)

(d) \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \)

(e) \( \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n} \right)^n \)

Solution:

(a) Notice that \( e^{-nx} = \left( \frac{1}{e^x} \right)^n \) and we know that this sum converges if and only if \( \left| \frac{1}{e^x} \right| < 1 \) which is if and only if \( e^x > 1 \). So this sum converges exactly for \( x > 0 \).

(b) We can do this by limit comparison. We know that \( \sum_{n=1}^{\infty} \frac{1}{n^x} \) converges, so we just need to find the limit of the ratio of the summands in these two series.

\[
\lim_{n \to \infty} \frac{\frac{1}{n^x}}{\frac{1}{n^x + 5n}} = \lim_{n \to \infty} \frac{n^x + 5n}{n^x} = \lim_{n \to \infty} \frac{n^6 + 5n}{n^6} = 1.
\]

Since this limit is a positive finite number and \( \sum_{n=1}^{\infty} \frac{1}{n^x} \) converges, we know that \( \sum_{n=1}^{\infty} \frac{1}{n^x + 5n} \) converges.
(c) We can do this with the term test. Recall that $\sum_{n=0}^{\infty} a_n$ does not converge if $\lim_{n \to \infty} a_n \neq 0$ or does not exist. But $\lim_{n \to \infty} \frac{n}{e^n} = \infty$, so this sum cannot converge.

(d) This problem can be solved with the integral test. Notice that $\frac{\ln(x)}{x} \geq 0$ for $x \geq 1$ and that $\frac{d}{dx} \frac{\ln(x)}{x} = \frac{1-x \ln(x)}{x^2} < 0$ so long as $1 - x \ln(x) < 0$, which is true for sufficiently large $x$. For example $\ln(x) > 1$ for $x > e$ and clearly $x > 1$ for $x > e$, so on the interval $(3, \infty)$ the function $\frac{\ln(x)}{x}$ is decreasing. We can then apply the integral test.

$$
\int_{3}^{\infty} \frac{\ln(x)}{x} \, dx = \lim_{b \to \infty} \int_{3}^{b} \frac{\ln(x)}{x} \, dx
$$

$$
= \lim_{b \to \infty} \frac{\ln(b)}{2 \ln(3)} - \frac{\ln(3)}{2}
$$

From this, we see that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ does not converge.

(e) This question may have been a little tricky. Although it looks like a good candidate for the root test, the terms here are genuinely not comparable to a geometric series (which is what the root test is checking for). Instead, we can solve this with the term test if we recall that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. From this, we see that $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \neq 0$, so the series cannot converge.

3. Determine whether the following series converge. If the series depends on $x$, determine for which values of $x$ it converges.

(a) $\sum_{k=2}^{\infty} \frac{1}{k^2 - k}$

(b) $\sum_{k=100}^{\infty} \frac{1}{k \ln(k) \ln(\ln(k))}$

(c) $\sum_{k=1}^{\infty} 2^k \ln(x)$

**Solution:**

(a) This can be evaluated exactly. $\sum_{k=2}^{\infty} \frac{1}{k^2 - k} = 1 - \frac{1}{n}$, so $\sum_{k=2}^{\infty} \frac{1}{k^2 - k} = 1$ and therefore converges.

(b) This diverges by the integral test. To see this, we first check that $f(x) = \frac{1}{x \ln(x) \ln(\ln(x))}$ is positive and decreasing on $(100, \infty)$. It is positive because $x$, $\ln(x)$ and $\ln(\ln(x))$ are positive. To see that it is decreasing, either differentiate or note that $x \ln(x) \ln(\ln(x))$ is an increasing function on this region. The $u$ substitution $u = \ln(\ln(x))$ has $du = \frac{dx}{x \ln(x)}$ by the chain rule, so

$$
\int_{100}^{\infty} \frac{dx}{x \ln(x) \ln(\ln(x))} = \lim_{b \to \infty} \int_{\ln(100)}^{\ln(b)} \frac{du}{u}
$$

$$
= \lim_{b \to \infty} \left[ \ln(b) - \ln(\ln(100)) \right] = \infty
$$
(c) This can be done by cases. The expression only makes sense for $x > 0$. If $x = 1$, then $\ln(x) = 0$ and so for each $k$, $2^{k \ln(x)} = 1$. Thus when $x = 0$, the sum diverges. We use the root test to handle the other cases:

$$\lim_{k \to \infty} \sqrt[k]{|2^k|} = \lim_{k \to \infty} 2^{\ln(x)} = 2^{\ln(x)}$$

If $0 < x < 1$ then $\ln(x) < 0$ and therefore $2^{\ln(x)} < 1$, so the series converges. If $x > 1$, then $2^{\ln(x)} > 1$, so the series diverges.