Please inform your TA if you find any errors in the solutions.

1. Find the general solution to the differential equation
\[
\frac{1}{2x} \frac{dy}{dx} = y + e^{x^2}
\]

**Solution:** We begin by writing the problem in standard form as
\[
\frac{dy}{dx} - 2xy = 2xe^{x^2}
\]

The integrating factor for this problem is \( m(x) = e^{\int -2x \, dx} = e^{-x^2} \). If we multiply through by \( e^{-x^2} \), then the equation becomes separable and we can find the general solution directly.

\[
e^{-x^2} \frac{dy}{dx} - 2xe^{-x^2} y = 2x
\]

\[
\frac{d(e^{-x^2} y)}{dx} = 2x
\]

\[
\int d(e^{-x^2} y) = \int 2x \, dx
\]

\[
e^{-x^2} y = x^2 + C
\]

\[
y(x) = x^2e^{x^2} + Ce^{x^2}
\]

2. Find a particular solution to the differential equation
\[
\frac{1 + x^3}{3x^2} \frac{dy}{dx} = 1 - y(x)
\]

\[
y(1) = 2
\]

**Solution:** We first put the equation into standard form
\[
\frac{dy}{dx} + \frac{3x^2}{1 + x^3} y = \frac{3x^2}{1 + x^3}
\]

The integrating factor for this problem is \( m(x) = (1 + x^3) \) and the solution is

\[
y(x) = \frac{1}{1 + x^3} \left( \int 3x^2 \, dx \right)
\]

\[
= \frac{1}{1 + x^3} \left( x^3 + C \right)
\]

The initial condition \( y(1) = 2 \) gives that \( \frac{1 + C}{2x} = 2 \), so \( C = 3 \) and \( y(x) = \frac{3 + x^3}{1 + x^3} \)
3. A tank starts with 100 liters of water and 1,000 bacteria in it. For now we assume the bacteria do not reproduce. Let \( B(t) \) be the number of bacteria in the tank as a function of time, where \( t \) is in hours. For each of the situations below, write down a first order differential equation satisfied by \( B(t) \), of the form \( B' = f(t, B) \). You do not need to solve it.

(a) A little goblin is pouring bacteria into the tank at a rate of 2015 bacteria per hour.
(b) Like part (a), but we are also draining the tank at a rate of 3 L/hr.
(c) Like part (b), but now the bacteria are reproducing. This is a strain of bacteria which, if left alone, will double its population every hour.

Solution:

(a) \( B' = 2015 \)
(b) \( B' = 2015 - 3 \frac{B}{100 - 3t} \)
(c) \( B' = 2015 - 3 \frac{B}{100 - 3t} + \ln(2)B \)

To get the last part we need to know the exponential growth rate of the bacteria. We know that if left alone, the population obeys \( P(t) = P_02^t \), and so satisfies the differential equation \( P' = \ln(2)P \). Thus the exponential growth rate is \( \ln(2) \).

4. Find an exact solution to the following initial value problem, then use Euler’s method with step size \( \Delta x = 0.1 \) to estimate \( y(2) \)

\[
\frac{dy}{dx} = 2xy + x \\
y(0) = 0
\]

Solution: We begin by putting the differential equation into standard form

\[
\frac{dy}{dx} - 2xy = x
\]

The integrating factor for this problem is \( m(x) = e^{\int -2x \, dx} = e^{-x^2} \). Multiplication turns this into

\[
\frac{d}{dx} \left( e^{-x^2} y \right) = xe^{-x^2}
\]

\[
e^{-x^2} y = \int xe^{-x^2} \, dx
\]

\[
= -\frac{1}{2} e^{-x^2} + C
\]

\[
y(x) = -\frac{1}{2} + Ce^{x^2}
\]

Substituting in the initial condition \( y(0) = 0 \) gives that \( y(x) = \frac{1}{2} e^{x^2} - \frac{1}{2} \).
To approximate \( y(0.2) \) we first need an approximation for \( y(0.1) \).

\[
y(0.1) \approx y(0) + \frac{dy}{dx}(0)\Delta x
\]

where \( \frac{dy}{dx}(0) = 2(0)y(0) + 0 = 0 \). So we have \( \frac{dy}{dx}(0) \approx y(0) + 0(1) = 0 \). We now have

\[
y(0.2) \approx y(0.1) + \frac{dy}{dx}(0.1)\Delta x
\]

where \( \frac{dy}{dx}(0.1) = 2(1)y(0.1) + .1 = 2(0.1) + .1 = .1 \). Then, we have

\[
y(0.2) \approx y(0.1) + \frac{dy}{dx}(0.1)\Delta x
\approx 0 + .1(0.1)
= .01
\]

5. \( y(x) \) is a function satisfying \( y'' = y' + y + x \), \( y(0) = 1 \) and \( y'(0) = 2 \). Approximate \( y(0.2) \) using Euler’s method with step size 0.1. You will need to modify the version of Euler’s method we learned in class to handle a second-order equation like this one. I got \( y(0.2) \approx 1.43 \).

Solution:

\[
y(0) = 1
y(0.1) \approx y(0) + 0.1 \ast y'(0)
= 1 + 0.1 \ast 2
= 1.2
y'(0.1) \approx y'(0) + 0.1 \ast y''(0)
= 2 + 0.1 \ast (2 + 1 + 0)
= 2.3
y(0.2) \approx y(0.1) + 0.1 \ast y'(0.1)
\approx 1.2 + 0.1 \ast 2.3
= 1.43
\]